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Affine Surfaces with $AK(S) = \mathbb{C}$

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1. Introduction

In this paper we proceed with our research [BaM1; BaM2] of the smooth surfaces with $\mathbb{C}^+$-actions. We denote by $\mathcal{O}(S)$ the ring of all regular functions on S. Let us recall that the AK invariant $AK(S) \subset \mathcal{O}(S)$ of a surface S is just the subring of the ring $\mathcal{O}(S)$ consisting of those regular functions on S that are invariant under all $\mathbb{C}^+$-actions of S. This invariant can be also described as the subring of $\mathcal{O}(S)$ of all functions that are constants for all locally nilpotent derivations of $\mathcal{O}(S)$ [KKMR; KM; M1].

We would like to give the answer to the following question: What are the surfaces with the trivial invariant AK?

It is quite easy to show (see [M2]) that the complex line $\mathbb{C}$ is the only curve with the trivial invariant. It is also well known that, if $AK(S) = \mathbb{C}$ and $\mathcal{O}(S)$ is a unique factorization domain (UFD), then S is an affine complex plane $\mathbb{C}^2$ [MiS; S]. If we drop the UFD condition then we have many smooth surfaces with trivial invariant—for example, any hypersurface of the form $\{xy = p(z)\} \subset \mathbb{C}^3$, where all roots of $p(z)$ are simple.

Since we did not know any other examples, we had the following working conjecture.

Conjecture. *Any smooth affine surface S with $AK(S) = \mathbb{C}$ is isomorphic to a hypersurface*

$$\{xy = p(z)\} \subset \mathbb{C}^3.$$

It turned out that this conjecture is true only with an additional assumption that S admits a fixed-point–free $\mathbb{C}^+$-action. Also, if we assume that S is a hypersurface with $AK(S) = \mathbb{C}$ then S is indeed isomorphic to a hypersurface defined by the equation $xy = p(z)$.

Surfaces of this kind have been well known since 1989 owing to the following remarkable fact, which was discovered by Danielewski [D] in connection with the generalized Zariski conjecture (see also Fieseler [F]): the surfaces $\{x^n y = p(z)\}$

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with $n > 1$ are not isomorphic to $\{xy = p(z)\}$ (actually, they are pairwise non-isomorphic). Nevertheless, the cylinders over all these surfaces are isomorphic ($S \times \mathbb{C}^n$ is called "the cylinder over surface S"). So it seems natural to introduce a notion of equivalence for the surfaces, where two surfaces are equivalent when cylinders over these surfaces are isomorphic. That is why we also try to consider surfaces with $AK(S) = \mathbb{C}$ up to this equivalence. Though we are far from a complete understanding, we know that there are two classes of surfaces that cannot be mixed by this equivalence relation. The first class consists of the hypersurfaces $\{xy = p(z)\}$ mentioned previously. Here is an example of a surface from the second class:

$$S = \left\{ (x, y, z, u) \in \mathbb{C}^4 : \begin{matrix} xy = (z^2 - 1)z \\ zu = (y^2 - 1)y \\ xu = (z^2 - 1)(y^2 - 1) \end{matrix} \right\}.$$

2. Definitions and Related Notions

If $AK(S) = \mathbb{C}$, then the group of automorphisms of S has a dense orbit. Hence it is natural to compare these surfaces with quasihomogeneous surfaces, which have been investigated by Gizatullin, Danilov, and Bertin [G1; G2; GD; Ber].

DEFINITION. A smooth affine surface S is called *quasihomogeneous* if the group $\mathrm{Aut}(S)$ of all automorphisms of S has an orbit $U = S \setminus N$, where N is a finite set.

We will show that, if $AK(S) = \mathbb{C}$, then indeed S is a quasihomogeneous surface. Therefore, S may be obtained from a smooth rational projective surface $\bar{S}$ by deleting a divisor of special form, which is called a "zigzag" [G1; G2; GD; Ber].

Let us denote by $\mathcal{A}$ the set of all surfaces S with $AK(S) = \mathbb{C}$ and by $\mathcal{H}$ those surfaces that have only three components in the zigzag. We prove in Section 3 that a surface $S \in \mathcal{A}$ is isomorphic to a hypersurface if and only if $S \in \mathcal{H}$ (Theorem 1). In Section 4 we use this fact to prove that:

(1) if $S_1 \in \mathcal{H}$ and $S_2 \in A \setminus \mathcal{H}$, then the cylinders $S_1 \times \mathbb{C}^k$ and $S_2 \times \mathbb{C}^k$ cannot be isomorphic (Theorem 2); and
(2) a surface $S \in \mathcal{A}$ admits a fixed-point–free $\mathbb{C}^+$-action with reduced fibers if and only if $S \in \mathcal{H}$ (Theorem 3).

The following notation will be used in this paper:

$\mathcal{O}(X)$, the ring of regular functions on a variety X;
$K(S)$, canonical divisor of a surface S;
$[D]$, class of linear equivalence of a divisor D;
$\tilde{D}$, proper transform of a divisor D after a blow-up;
D^*, algebraic (total) transform of a divisor D after a blow-up;
(ω), (f), divisors of zeros of a form ω and a function f, respectively;
$\mathrm{Aut}(S)$, automorphism group of a surface S;
$G(S)$, subgroup of $\mathrm{Aut}(S)$, generated by all $\mathbb{C}^+$-actions on a surface S;

$OG(S)$, a general orbit of the group $G(S)$;
$\bar{A}$, a Zariski closure of A (if another meaning is not specified).

"General" means "belonging to a Zariski open subset". A *singular* point of a rational function is a point where the function is not defined.

3. Characterization of Hypersurfaces S with $AK(S) = \mathbb{C}$

Following [Ber; Mi; MiS], by a *line pencil* on a surface S we mean a morphism $\rho\colon S \to C$ into a smooth curve C such that the fiber $\rho^{-1}(z)$ for a general $z \in C$ is isomorphic to $\mathbb{C}$. Then S contains a cylinderlike subset, that is, an open subset that is isomorphic to a direct product of $\mathbb{C}$ and an open subset of C [B, III.4]. The pencils are different if their general fibers do not coincide. Any line pencil ρ over an affine curve C on a surface S corresponds to a $\mathbb{C}^+$-action φ_ρ on S such that the general orbit of φ_ρ coincides with a general fiber of the pencil; moreover, it corresponds to a locally nilpotent derivation (LND) ∂_ρ in the ring $O(S)$ of regular functions on S such that $\partial_\rho f = 0$ if and only if f is φ_ρ-invariant [KM; M1; Mi; Sn]. If there are two different line pencils in S then $\rho(S) = \mathbb{C}$ (indeed, in this case $\rho(S)$ is an affine curve containing the image of a fiber of the second line pencil, and this fiber is isomorphic to $\mathbb{C}$). Since we are looking for the surfaces having many $\mathbb{C}^+$-actions, we shall assume in the sequel that $C \cong \mathbb{C}$.

For a pencil ρ over $\mathbb{C}$, one can find a closure $\bar{S}$ of S such that the extension $\bar{\rho}\colon \bar{S} \to \mathbb{P}^1$ of the map $\rho\colon S \to \mathbb{C}$ is regular and, in the commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\bar{\rho}} \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1, \end{array} \tag{1}$$

the divisor $B = \bar{S} \setminus S$ is connected and has the following properties.

(I) $B = F + D + E$, where:
  (a) $F \cong \mathbb{P}^1$ and $\bar{\rho}(F) = \mathbb{P}^1 - \mathbb{C}$;
  (b) $\bar{\rho}|_D\colon D \to \mathbb{P}^1$ is an isomorphism; and
  (c) $E = \sum E_i + \sum H_i$, where $\bar{\rho}(H_i) \in \mathbb{C} \setminus \rho(S)$ and $\bar{\rho}(E_i) = z_i \in \rho(S)$ are points.

  Moreover, $\overline{\rho^{-1}(z_i)}$ is a union of disjoint smooth rational curves, and each of them intersects B precisely at one point.

(II) B does not contain (-1) curves, except perhaps D.

The structure of fibers is described in [Mi, Lemma 4.4.1]. If there are two different line pencils in S, then $E = \sum E_i$.

Definition. We call a closure $\bar{S}$ a *good ρ-closure* of an affine surface S if it has properties (I) and (II).

Definition. Let $F_z = \rho^{-1}(z) = \sum_{i=1}^{i=m} n_i C_i$, where the C_i are connected (and irreducible, owing to property (I)(c)) components. If $m = 1$ and $n_1 = 1$, then the fiber is called *nonsingular*. The singular fiber is either nonconnected or has $m = 1$ and $n_1 > 1$. If $F_z = \sum_{i=1}^{i=m} C_i$ (i.e., $n_i = 1$), then the fiber is called *reduced*.

PROPOSITION 1. *Let S be a smooth affine surface with a line pencil ρ. Let $\bar{S}$ be a good ρ-closure of S. Let $F_{z_1}, \dots, F_{z_n}$ be all singular fibers of ρ, and let $F_{z_i} = \sum_{j=1}^{j=k_i} n_{i,j} C_{i,j}$ be a sum of irreducible curves $C_{i,j}$ with $C_{i,j} \cong \mathbb{C}$. Then there exists a function $\alpha \in \mathcal{O}(S)$ such that*:

(a) *α is linear along each nonsingular fiber F_z, where $z \neq z_i$ for $i = 1, \dots, n$ (i.e., $\alpha|_{F_z}$ is a nonconstant linear function); and*
(b) *$\alpha|_{C_{i,j}} = \alpha_{i,j} = \text{const}$ for all $1 \leq i \leq n$ and $1 \leq j \leq k_i$.*

Proof. Let ∂_ρ be a nonzero LND corresponding to the line pencil ρ. If there is a nonsingular fiber $F_z = \rho^{-1}(z)$ such that $\partial_\rho(v)|_{F_z} = 0$ for all $v \in \mathcal{O}(S)$, then we may consider another LND $\tilde{\partial}_\rho = \partial_\rho/(\rho - z)$ and repeat this procedure, if needed. Hence we may assume that ∂_ρ does not vanish identically along the nonsingular fibers of ρ.

Since ∂_ρ is a nonzero derivation, there exists a function $v \in \mathcal{O}(S)$ for which $\partial_\rho(v) \neq 0$, that is, the minimal n for which $\partial_\rho^n(v) = 0$ is not smaller than 2. Let us take $u = \partial_\rho^{n-2}(v)$. Since $\partial_\rho^2(u) = 0$, it follows that $\partial_\rho(u) = f(z)$ depends only on $z = \rho(s)$ with $s \in S$. If $f(\tilde{z}) = 0$ ($\tilde{z} \neq z_1, \dots, z_n$), then $u|_{\rho^{-1}(\tilde{z})} = u_0 = \text{const}$, and we consider a new function $(u - u_0)/(\rho - \tilde{z})$.

Repeating this yields a situation in which:

(1) $\partial_\rho u = f(z)$, where f may vanish only at the points z_i, $i = 1, \dots, n$; and
(2) u is a linear function along each fiber $\rho^{-1}(\tilde{z})$, with $\tilde{z} \neq z_i$ for $i = 1, \dots, n$.

We will show that $u = u_i = \text{const}$ along each component $C_{i,j}$ of F_{z_i}, $i = 1, \dots, n$.

Indeed, u is linear along a general fiber, which means that the intersection $(\bar{U}_w, \bar{\rho}^{-1}(z)) = 1$ for the closure $\bar{U}_w$ in $\bar{S}$ of a general level curve $U_w = \{s \in S : u(s) = w\}$ and any z.

If $u|_{C_{i,j}} \neq \text{const}$, then $(\bar{U}_w, C_{i,j}) \geq 1$ and $(\bar{U}_w, \bar{\rho}^{-1}(z_i)) \geq n_{i,j}$. Thus, if $n_{i,j} > 1$ then $(\bar{U}_w, C_{i,j}) = 0$ and $u|_{C_{i,j}} = \text{const}$.

If $n_{i,j} = 1$, then the fiber is nonconnected and $u|_{C_{i,j}} \neq \text{const}$ implies that $\bar{U}_w$ does not intersect $\bar{\rho}^{-1}(z_i) \setminus C_{i,j}$ for a general $w \in \mathbb{C}$. Thus, $u|_{\bar{\rho}^{-1}(z_i) \setminus C_{i,j}}$ must be regular and constant. On the other hand, u has a pole along D and so $u|_{\bar{\rho}^{-1}(z_i) \setminus C_{i,j}} = \infty$. Since u has only regular points, it follows that also $u|_{C_{i,k}} = \infty$ if $k \neq j$. But $u \in \mathcal{O}(S)$, so there are no components with $k \neq j$. Hence $\rho^{-1}(z_i)$ has just one component of multiplicity 1, which contradicts our assumption.

Thus, we may take $\alpha = u$. □

PROPOSITION 2. *Any smooth affine surface S with $AK(S) \cong \mathbb{C}$ is quasihomogeneous.*

Proof. Assume that ϕ and ψ are $\mathbb{C}^+$-actions on S having different orbits. Let ρ and κ be the corresponding line pencils, with ∂_ρ and ∂_κ the corresponding LND. Let $R_z = \rho^{-1}(z)$ and $K_w = \kappa^{-1}(w)$ for general $z, w \in \mathbb{C}$, and let $\bar{R}_z$ and $\bar{K}_w$ be their closures in a good ρ-closure $\bar{S}$ of S. We will now show that $S \setminus OG(S)$ is a finite set.

If a point s is in $S \setminus OG(S)$ and if the fiber $R_{\rho(s)}$ is nonsingular, then $R_{\rho(s)} \subset S \setminus OG(S)$ as well. Indeed, as shown in Proposition 1, we can choose ∂_ρ and ∂_κ in such a way that they do not vanish along nonsingular fibers; that is, there are no fixed points in these fibers.

For the same reason, $R_{\rho(s)}$ does not intersect a general fiber K_w; that is, it is contained in $K_{\kappa(s)}$. But then $\rho \neq \rho(s)$ along a general fiber K_w. Hence $\rho|_{K_w} =$ const, and the fibers of these two actions coincide. Thus, $s \in S \setminus OG(S)$ implies that $s \in R_{z_0} \cap K_{w_0}$ for singular fibers R_{z_0} and K_{w_0}. If $S \setminus OG(S)$ is infinite, then there exists a connected component $C \subset R_{z_0} \cap K_{w_0}$ for singular fibers R_{z_0} and K_{w_0} of ρ and κ, respectively.

Let $\bar{\rho}^{-1}(z_0) = \bar{C} \cup E' \cup \left(\bigcup \bar{C}_i\right)$, where $E' \subset \bar{S} \setminus S$ and the C_i are other components of $\rho^{-1}(z_0)$. Consider $K_w \cong \mathbb{C}$. The intersection $(\bar{K}_w, \bar{R}_z) \geq 1$, so $\bar{K}_w$ intersects $R_\infty = \bar{\rho}^{-1}(\infty)$. Hence, the only puncture of K_w belongs to R_∞, and this means that $\bar{K}_w \cap E' = \emptyset$. Thus, κ has no singular points and must be constant along E'. Since $E' \cap D \neq \emptyset$, we have $\kappa|_{E'} = \kappa|_D$ (see diagram (1) and recall that E' is connected). But $\kappa|_D = \infty$ (if it were not, then κ would be bounded and hence constant along a general fiber R_z).

We conclude that $\kappa|_{E'} = \infty$ and has no singular points. On the other hand, κ is finite and constant along C, which implies that the point $\bar{C} \cap E'$ is singular. The contradiction shows that no such curve C exists and that $S \setminus OG(S)$ is a finite set. Hence S is indeed quasihomogeneous. □

Any good ρ-closure $\bar{S}$ of S may be described by the graph $\Gamma(\bar{S})$ in the following way: The vertices of this graph are in bijection with irreducible components of the divisor $\bar{B} = \bar{S} \setminus S$, and two vertices are connected by an edge if they intersect each other.

Now we shall use the description of quasihomogeneous affine surfaces due to Gizatullin and Bertin [Ber; G1; G2; GD].

Any such surface S is either isomorphic to $\mathbb{C}^2$ or may be obtained by the following blow-up process, described in [G2]. Let $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\bar{\rho}\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a projection onto the second factor. Let $F_0 = \bar{\rho}^{-1}(z_0)$ and $F_1 = \bar{\rho}^{-1}(z_1)$ with $z_0, z_1 \in \mathbb{P}^1$, and let D be a section; that is, $\bar{\rho}|_D\colon D \to \mathbb{P}^1$ is an isomorphism.

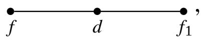
Let $\sigma = \sigma_1 \circ \dots \circ \sigma_n\colon \bar{S} \to S_0$ be the sequence of blow-ups

$$\bar{S} = \bar{S}_n \xrightarrow{\sigma_n} \bar{S}_{n-1} \longrightarrow \cdots \xrightarrow{\sigma_1} S_0,$$

where σ_1 is a blow-up of a point in F_1 and σ_i is a blow-up of a point in $(\sigma_1 \dots \sigma_{i-1})^{-1}(F_1)$. Let $\sigma^{-1}(F_1) = Z \cup A$, where Z is a linear chain of smooth rational curves (zigzag) such that $Z \cap \tilde{D}$ is a point and where $A = \bigcup A_i$ is a union of smooth rational curves A_i such that $A_i \cap A_j = \emptyset$ and $A_i \cap Z$ is a point for each i. Then the quasihomogeneous surface $S = \bar{S} \setminus (Z \cup \tilde{F}_0 \cup \tilde{D})$.

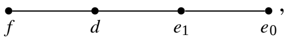
We use G_i to denote all A_i such that $A_i^2 = -1$ and use M_i to denote all A_i with $A_i^2 < -1$. We may assume that the G_i were blown up at the last stage of the process. Then the process consists of the following steps.

Step 0 is an initial step. We start with the divisor, which is described by the following graph:



where vertices f, d, f_1 represent components F_0, D, F_1, respectively.

Step 1 is the blow-up $\sigma_1 \colon \bar{S}_1 \to \bar{S}_0$ of a point $w_1 \in F_1$ into an exceptional component $E \subset \bar{S}_1$. We denote $F_1^* = \tilde{F}_1 + E$ as $E_0 + E_1$, where E_0 and E_1 are two rational curves; the graph of $F_0 \cup D \cup E_1 \cup E_0$ looks like

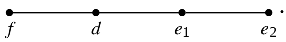


where the vertices f, d, e_1, e_0 represent the components $\tilde{F}_0, \tilde{D}, E_1, E_0$, respectively. Put $Z_1 = E_1 \cup E_0$.

Step 2 is one of the following two procedures.

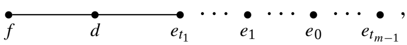
(a) The blow-up $\sigma_2 \colon \bar{S}_2 \to \bar{S}_1$ of a point $w_2 \in Z_1$ into a component $E_2 \subset \bar{S}_2$ in such a way that a graph of $\tilde{F}_0 \cup \tilde{D} \cup \tilde{E}_1 \cup \tilde{E}_0 \cup E_2$ is linear. That is, we blow up either the point $E_1 \cap \tilde{D}$ or the point $E_1 \cap E_0$ or a point in E_0. We put $Z_2 = \tilde{E}_1 \cup \tilde{E}_0 \cup E_2$.

(b) The blow-up of the point $E_0 \cap E_1$ to obtain a curve E_2. Then put $E_0 = M_1$ and $Z_2 = \tilde{E}_1 \cup \tilde{E}_2$. The graph of $\tilde{F}_0 \cup \tilde{D} \cup Z_2$ looks like



There are no other ways to obtain a linear graph.

For a general m, let the graph of $\tilde{F}_0 \cup \tilde{D} \cup Z_{m-1}$ be



(or perhaps without e_0), where a vertex e_{t_i} represents the component E_{t_i} obtained at the step t_i.

Step m is one of the following procedures.

(a) The blow-up $\sigma_m \colon \bar{S}_m \to \bar{S}_{m-1}$ of a point $w_m \in Z_{m-1}$ into a component $E_m \subset \bar{S}_m$ in such a way that the graph of the divisor $\tilde{F}_0 \cup \tilde{D} \cup \tilde{Z}_{m-1} \cup E_m$ is linear. That is, a blown-up point is either $Z_{m-1} \cap \tilde{D}$ or $E_{t_j} \cap E_{t_i}$ with $E_i, E_j \subset Z_{m-1}$, or it is a blow-up of a point in $E_{t_{m-1}}$ (this point may happen to be the intersection $E_{t_{m-1}} \cap M_j$). Put $Z_m = \tilde{Z}_m \cup E_m$.

(b) If $E_{t_{m-1}}$ does not intersect any M_i $(i = 1, \dots, s)$ obtained at a preceding step, denote $E_{t_{m-1}} = M_{s+1}$ and blow up a point in $Z_{m-1} \setminus (E_{t_{m-1}} \setminus (Z_{m-1} \cap E_{t_{m-1}}))$ to obtain a component E_{t_m} in such a way that the graph of $Z_m = E_m \cup \left(\bigcup(\tilde{E}_i)\right)$ $(E_i \neq M_j;\ i = 0, \dots, k-1,\ j = 1, \dots, s+1)$ is linear. If $E_{t_{m-1}}^2 = -1$, then the blown-up point should be an intersection of $E_{t_{m-1}}$ with the adjacent component (since all (-1) curves are added at the last step).

Step $k+1$ is the last step. Let $\alpha_1 \dots \alpha_q$ be different points in Z_k such that each α_i belongs to one component only, $1 \le i \le q$. Let $\tau_1 \dots \tau_q$ be blow-ups of the points $\alpha_1 \dots \alpha_q$ into the curves G_i $(1 \le i \le q)$, respectively, and let $\bar{S}$ be $(\tau_1 \circ \tau_2 \circ \cdots \circ \tau_q)^{-1}(\bar{S}_k)$.

The desired surface $S = \bar{S} \setminus (\tilde{F}_0 \cup \tilde{D} \cup \tilde{Z}_k)$.

REMARK. This description of quasihomogeneous surfaces implies, in particular, that there may be only one singular fiber for a line pencil ρ.

We want to choose the "minimal" way to obtain S by the described process, that is, to obtain a good ρ-closure of S. For this we want to replace $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ by a minimal ruled surface $\mathbb{F}_n$ (see [B]).

In the sequel, for simplicity of notation we will denote $\tilde{Z}_k$, $\tilde{E}_j$ as Z_k, E_j, since this cannot lead to confusion.

PROPOSITION 3. *The surface $S \not\cong \mathbb{C}^2$ obtained by the blow-up process described previously may be obtained by a similar process*: *start with the minimal surface $S_0 = \mathbb{F}_n$ and end with $\bar{S}$ such that $E_j^2 \neq -1$ in $\bar{S}$ for all $E_j \subset Z_k$.*

Proof. We prove the proposition by induction on the number of steps k. We start with the surface $S_0 = \mathbb{F}_n$ and show that, by changing n, we may always eliminate the (-1) components.

Assume that $k = 0$. Since $\rho^{-1}(z_1) \subset S$ is singular (recall that $S \not\cong \mathbb{C}^2$), there are points $\alpha_i \in F_1$ $(1 \leq i \leq q)$ that are blown up at the first (and last) step into the curves G_i. Thus, in $\bar{S}$ this fiber has the form $\tilde{F}_1 + \sum_{i=1}^{i=q} G_i$ (the multiplicities are equal to 1), which implies that the fiber is not connected, $q > 1$, and $(\tilde{F}_1)^2 = -q < -1$.

Assume now that the proposition is true for all $k < k_0$. Let E_j be a component of F_1^* in $\bar{S}_{k_0}$ such that $E_j^2 = -1$. There are two possibilities as follows.

(1) E_j is a result of the blow-up σ_j. The points of this component are not blown up at any later step, since doing so would make $E_j^2 < -1$. Thus, E_j may be contracted back and we may obtain surface S by the same process, omitting the step number j (i.e., as a complement to zigzag obtained by the blow-up process with one less step).

(2) E_j is a proper transform of F_1. In this case we may blow it down after step 1 and obtain the same surface by the same process (with one less step), starting with the surface $S_0 = \mathbb{F}_{n+1}$ or $S_0 = \mathbb{F}_{n-1}$.

By the assumption of the induction, it follows that the proposition is true for k_0. □

DEFINITION. We denote by $\mathcal{A}$ the class of all smooth affine surfaces S with $AK(S) = \mathbb{C}$. Let us denote by $\mathcal{H} \subset \mathcal{A}$ the subset of those surfaces for which $k = 0$ in a good ρ-closure obtained by the described process.

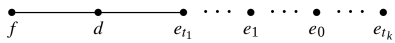
THEOREM 1. *A surface $S \in \mathcal{A}$ is isomorphic to a hypersurface if and only if $S \in \mathcal{H}$.*

Proof. The proof is based on a property of hypersurfaces, which was explained to the authors by V. Lin and M. Zaidenberg. Although this result is classical, we could not find a direct reference. We proceed as follows.

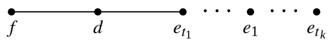
LEMMA 1. *Let $X \subset \mathbb{C}^n$ $(n > 2)$ be a smooth hypersurface. Then the canonical class $K(X)$ of X is trivial* (*i.e., the divisor of zeros of a holomorphic $(n-1)$-form on X is equivalent to zero*).

Proof. By the adjunction formula, the canonical class of a complete intersection in a projective space is a multiple of the linear section [H, p. 188]. Thus, for an affine hypersurface, this class is represented by the divisor with support in the hyperplane section at infinity. □

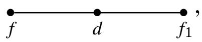
Let $S \in \mathcal{A}$ and $S \neq \mathbb{C}^2$. The graph $\Gamma(\bar{S})$ has the form



or (if $e_0 = M_1$)



or (if $k = 0$)



where the vertices f, d, f_1, e_1, e_0 represent the components $\tilde{F}_0, \tilde{D}, \tilde{F}_1, E_1, E_0$, respectively, and vertex e_{t_i} represents the component E_{t_i} obtained at the step t_i.

DEFINITION. We say that $e_i < e_j$ ($E_i < E_j$) if e_i is on the left of e_j in the graph $\Gamma(S)$. If $E_j = M_s$ and $E_j \cap E_l \neq \emptyset$, then we say that $e_i < e_j$ if $e_i \leq e_l$.

LEMMA 2. *The canonical class* $[K(\bar{S}_k)]$ *of* $\bar{S}_k$ $(k > 0)$ *is the class of the divisor*

$$K(\bar{S}_k) = \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^{k} \varepsilon_i E_i, \tag{2}$$

where

$$\alpha \in \mathbb{Z}; \qquad \varepsilon_i < -1 \quad \textit{if } e_i < e_1; \qquad \varepsilon_i \geq 0 \quad \textit{if } e_i > e_1. \tag{3}$$

Let

$$F_1^k = F_1^* = \sum_{i=0}^{i=k} n_i E_i$$

be the algebraic (*total*) *transform of* F_1 *in* $\bar{S}_k$. *If* $E_0 \neq M_1$, *then*

$$\varepsilon_i < n_i - 1 \quad \textit{if } e_i < e_0; \qquad \varepsilon_i \geq n_i \quad \textit{if } e_i > e_0; \qquad n_1 = n_0 = 1. \tag{4}$$

If $E_0 = M_1$, *then*

$$\begin{gathered} \varepsilon_i < n_i - 1 \quad \textit{if } e_i < e_2; \qquad \varepsilon_i > 0 \quad \textit{if } e_i > e_2 \ (i \neq 0); \\ n_2 = 2, \quad \varepsilon_2 = 0. \end{gathered} \tag{4$'$}$$

Proof. We prove first inequalities (3) by induction on k.

The canonical class of $\mathbb{F}_n$ is $[\alpha F_0 - 2D]$ [B, Prop. III.18]. Consider the first step: the fiber $F_1 \subset \mathbb{F}_n$ is blown up into two rational curves $F_1^* = \tilde{F}_1 + E$. Both curves have self-intersection -1. Two cases are possible.

Case 1: $\tilde{F}_1 \cap \tilde{D} = \emptyset$, $E \cap \tilde{D} \neq \emptyset$. According to the formula for the canonical class of a blow-up [H, Chap. V, Prop. 3.3], the canonical divisor

$$K(\bar{S}_1) = \sigma_1^*(K(\mathbb{F}_n)) + E$$
$$= \alpha\tilde{F}_0 - 2\tilde{D} - 2E + E = \alpha\tilde{F}_0 - 2\tilde{D}_0 - E.$$

In this case we denote $E = E_1$ and $\tilde{F}_1 = E_0$.

Case 2: $\tilde{F}_1 \cap \tilde{D} \neq \emptyset$, $E \cap \tilde{D} = \emptyset$. Then the canonical divisor

$$K(\bar{S}_1) = \sigma_1^*(K(\mathbb{F}_n)) + E$$
$$= \alpha\tilde{F}_0 - 2\tilde{D} + E = (\alpha+1)\tilde{F}_0 - 2\tilde{D} - \tilde{F}_1,$$

since $\tilde{F}_0 \cong E + \tilde{F}_1$. In this case we denote $E = E_0$ and $\tilde{F}_1 = E_1$. Thus, for $k = 1$ the formula is proved.

If $E_0 = M_1$, we check the second step. We have $e_2 > e_1$, $\varepsilon_1 = -1$, $\varepsilon_2 = 0$, and $\varepsilon_0 = 0$.

Assume now that (2) and (3) are proved for all $k < k_0$:

$$K(\bar{S}_{k_0-1}) = \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^{k_0-1} \varepsilon_i E_i.$$

Then

$$K(\bar{S}_{k_0}) = \sigma_{k_0}^*(K(\bar{S}_{k_0-1})) + E_{k_0}$$
$$= \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^{k_0-1} \varepsilon_i E_i + \varepsilon_{k_0} E_{k_0}.$$

Consider the following cases.

(I) At step k_0 we blow up a point w_{k_0} that belongs only to the component E_s and is represented by the vertex on the far right (maximal) or next to maximal (if we decide that the maximal one will be M_j). In this case, e_s is on the right of e_1. By the induction assumption we have $\varepsilon_s \geq 0$, and $\varepsilon_{k_0} = (\varepsilon_s + 1) > 0$.

(II) At step k_0 we blow up the meeting point $E_s \cap E_{s'}$, where $e_s < e_{s'} \leq e_1$. Then $\varepsilon_s < -1$, $\varepsilon_{s'} \leq -1$, and $\varepsilon_{k_0} = \varepsilon_s + \varepsilon_{s'} + 1 < -1 - 1 + 1 < -1$.

(III) At step k_0 we blow up the meeting point $E_s \cap E_{s'}$, where $e_s > e_{s'} \geq e_1$ (it may be that $e_{s'} > e_1$ and $E_s = M_j$). Then $\varepsilon_s \geq 0$, $\varepsilon_{s'} \geq -1$, and $\varepsilon_{k_0} = \varepsilon_s + \varepsilon_{s'} + 1 \geq -1 + 1 \geq 0$.

(IV) At step k_0 we blow up the meeting point $E_s \cap \tilde{D}$. Then $e_s \leq e_1$ and $\varepsilon_{k_0} = \varepsilon_s - 2 + 1 \leq -1 - 1 < -1$.

Since the graph $\Gamma(S)$ is linear, we have exhausted all the possibilities.

Now let us prove the inequalities (4) and (4′). For $k = 1$ we have $F_1^1 = E_1 + E_0$ and $K(\bar{S}_1) = \alpha\tilde{F}_0 - 2\tilde{D} - E_1$; therefore, $\varepsilon_1 < n_1 - 1$. In case $E_0 = M_1$, we check $k = 2$: this yields $e_1 < e_2$, $\varepsilon_2 = 0$, and $n_2 = 2$.

We prove (4) for any k by induction. Assume that it is proved for all $k < k_0$. Then in $\bar{S}_{k_0}$ we have

$$F_1^{k_0} = \sigma_{k_0}^*(F_1^{k_0-1}) = \sum_{i=0}^{i=k_0-1} n_i E_i + n_{k_0} E_{k_0},$$

where $n_{k_0} = n_s + n_r$ if E_{k_0} appears as a blow-up of the intersection $E_s \cap E_r$ and where $n_{k_0} = n_s$ if E_{k_0} is the result of a blow-up of either $D \cap E_s$ or of a point of the maximal (or adjacent) component E_s only.

Using the inequalities (4) for $k < k_0$, we obtain the following relations:

$n_{k_0} = n_s \leq \varepsilon_s < \varepsilon_s + 1 = \varepsilon_{k_0}$ if E_s is the maximal (or adjacent) component and $s \neq 0$;
$n_{k_0} = n_0 = 1 \leq 1 = \varepsilon_{k_0}$ if $E_s = E_0$;
$n_{k_0} = n_s + n_r \leq \varepsilon_s + \varepsilon_r < \varepsilon_s + \varepsilon_r + 1 = \varepsilon_{k_0}$ if $e_0 < e_s < e_r$;
$n_{k_0} = n_0 + n_r = 1 + n_r \leq 0 + \varepsilon_r + 1 = \varepsilon_{k_0}$ if $e_0 = e_s < e_r$;
$n_{k_0} = n_s + n_0 = 1 + n_s > 1 + \varepsilon_s + 1 = \varepsilon_{k_0} + 1$ if $e_s < e_r = e_0$;
$n_{k_0} = n_s + n_r > \varepsilon_s + 1 + \varepsilon_r + 1 = \varepsilon_{k_0} + 1$ if $e_s < e_r < e_0$;
$n_{k_0} = n_s > \varepsilon_s + 1 = \varepsilon_{k_0} + 2 > \varepsilon_{k_0} + 1$ if E_s is the minimal component.

Assume now that $E_0 = M_1$. Since $E_2 < M_s$ for all s, the inequalities (4) still hold for $e_s < e_2$ (the process is the same in this interval). Any component $E_s > E_2$, $s \neq 0$, is obtained from E_2 by sequence of blow-ups. Since $\varepsilon_2 = 0$ and since we add positive integer each time, we can obtain only positive values for e_s; hence, this part of (4′) is evident. □

LEMMA 3. *Denote the transform of F_1 in $\bar{S}$ by*

$$F_1^{k+1} = F_1^* = \sum_{E_i \subset Z_k} n_i E_i + \sum_{i=1}^{i=q} g_i G_i + \sum_{i=1}^{i=t} m_i M_i,$$

where sums include (respectively) all the components $E_i \subset Z_k$, G_i, and M_i and where $n_1 = 1$, $g_i > 0$, $n_i > 0$, and $m_i > 0$.

Then $[K(S)] = 0$ if and only if the divisor $K(\bar{S})$ is equivalent to a linear combination

$$\sum_{E_i \subset Z_k} \alpha_i E_i + f\tilde{F}_0 + d\tilde{D} + m\left(\sum_{i=1}^{i=q} g_i G_i + \sum_{i=1}^{i=t} m_i M_i\right) \tag{5}$$

for some $m \in \mathbb{Z}$.

Proof.

$$\begin{aligned} K(\bar{S}) &= K(\bar{S}_k)^* + \sum G_i \\ &= \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=1}^{k} \varepsilon_i E_i + \sum_{i=1}^{q} \delta_i G_i, \end{aligned} \tag{6}$$

where $\delta_i = \varepsilon_s + 1$ for each G_i intersecting E_s and where all M_j are included in the first sum.

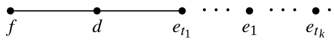
If $[K(S)] = 0$, then $K(S)$ is the divisor of a rational function h that has zeros and poles in S only along components G_i and M_i. But then h does not vanish and has no poles in any fiber F_z, $z \neq z_1$. Since general fiber is isomorphic to $\mathbb{C}$, it follows that h is constant along each fiber, that is, $h(s) = (\rho(s) - z_1)^m$. But then $\delta_i = mg_i$ and $\varepsilon_i = mm_i$. □

DEFINITION. We call component E_s *essential* if there is a component G_{i_s} of the fiber $F_1^* \subset \bar{S}$ such that $G_{i_s} \cap E_s \neq \emptyset$.

REMARK. We see from Lemma 3 that $[K(S)] = 0$ implies $\varepsilon_s + 1 = mn_s$ for any essential component E_s. At least one essential component should exist, since the fiber contains at least one (-1) curve.

LEMMA 4. *If* $k > 0$, *then* $[K(S)] \neq 0$.

Proof. Consider the graph



Assume that $[K(S)] = 0$; that is, $\varepsilon_s + 1 = mn_s$ for an essential component and $mm_i = \varepsilon_i$. Several cases are possible regarding the place of essential components in the graph.

(I) $E_0 \neq M_1$ and there is an essential component E_s such that $e_s \geq e_0$. Then, according to Lemma 2, $n_s \leq \varepsilon_s + 1 = mn_s$ and so $m \geq 1$.

(II) $E_0 \neq M_1$ and there is an essential component E_s such that $e_1 < e_s < e_0$. Then, according to Lemma 2, $n_s > \varepsilon_s + 1 = mn_s > 0$ and hence $1 > m > 0$.

(III) $E_0 \neq M_1$ and there is an essential component E_s such that $e_s \leq e_1$. Then, according to Lemma 2, $0 \geq \varepsilon_s + 1 = mn_s$ and $m \leq 0$.

(IV) $E_0 = M_1$; since $\varepsilon_0 = 0$, it follows that $m = 0$.

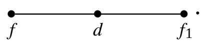
We may thus have only one of these cases.

Let us assume that $e_s \leq e_1$ for any essential component E_s and that $E_0 \neq M_1$. Let $t_0 = \max\{t : e_t > e_1,\ t \geq 0\}$. By construction, $(E_{t'_0})^2 = -1$ in $\bar{S}_k$ (it is the result of a blow-up). Hence it should contain a point that is blown up at the last $(k+1)$ step. But then $E_{t'_0}$ is essential, which is impossible in this case (since $e_1 < e_{t'_0}$).

The case $e_s \geq e_0$, $E_0 \neq M_1$, for all essential components can be treated analogously, since the last component to the left of E_0 also must be essential.

Case (II) is impossible, since $m \in \mathbb{Z}$. In case (IV), $m = 0$ and thus $\varepsilon_s = -1$ for any essential component E_s. By Lemma 2, there is only one such component E_1. But then $Z_k = E_1 \cup E_2$ and $E_2^2 = -1$, which is impossible.

Therefore, (5) can be true only if the graph has three components:



□

LEMMA 5. *If* $k = 0$, *then S is a hypersurface.*

Proof. Let $\rho\colon S \to \mathbb{C}$ be a line pencil in S, let $\bar{\rho}$ be its extension to a good ρ-closure $\bar{S}$ of S, and let φ_ρ and ∂_ρ be the corresponding $\mathbb{C}^+$-action and LND respectively. Let $\rho^{-1}(0)$ be the only singular fiber. All the multiplicities are 1 in this case, so the fiber cannot be connected. Let $u \in O(S)$ be a function such that:

(1) $\partial_\rho u = \rho^n$;
(2) u is a linear function along each fiber $\rho^{-1}(z)$, $z \neq 0$; and
(3) $u = u_i =$ const along each component G_i of $\rho^{-1}(0)$, $i = 1, \dots, q$.

Such a function exists, by Proposition 1. We will show that we can choose u such that $u_i \neq u_j$ when $i \neq j$ and such that the rational extension $\bar{u}$ of u to $\bar{S}$ is finite and nonconstant along $\tilde{F}_1$. Indeed, u is linear along a general fiber, which means that the intersection $(\bar{U}_w, \bar{F}_z) = 1$ for the closure of a general level curve $U_w = \{s \in S : u(s) = w\}$ and the closure $\bar{F}_z$ of a general fiber $F_z = \{s \in S : \rho(s) = z\}$.

There are three possibilities, as follows.

I. $\bar{u}|_{\tilde{F}_1} = u_0 \in \mathbb{C}$ and $u_0 \neq u_1 = \bar{u}|_{G_1}$. Then the intersection $G_1 \cap \tilde{F}_1 = \alpha_1$ is a singular point, and a general level curve passes through α_1. Another singular point $\alpha_2 = D \cap \tilde{F}_1$, since $\bar{u}|_D = \infty$. Thus, a general level curve U_w must pass through α_2 as well. But this contradicts $(\bar{U}_w, \bar{F}_z) = 1$.

Thus, $\bar{u}|_{\tilde{F}_1} = u_0 \in \mathbb{C}$ implies $u_0 = u_1 = u_2 = \dots = u_q$, and we can consider a new function $(u - u_0)/\rho$ instead of u (because $F_1^* = \tilde{F}_1 + \sum G_i$, i.e., ρ has a simple zero along each component).

II. $\bar{u}$ has a pole along $\tilde{F}_1$. Then each point $\alpha_i = \tilde{F}_1 \cap G_i$ $(i = 1, \dots, q)$ should be a singular point of $\bar{u}$, and $\bar{U}_w$ should pass through each α_i. From $(\bar{U}_w, \bar{F}_z) = 1$ it follows that there is only one component G_1, and the fiber $\rho^{-1}(0)$ is connected in this case.

Then $S \simeq \mathbb{C}^2$ (see e.g. [S]) and is evidently isomorphic to a hypersurface.

III. $\bar{u}$ is not constant along $\tilde{F}_1$. Because $(\bar{U}_w, \tilde{F}_1) = 1$ for a general w, it takes every value only once along $\tilde{F}_1$. From $G_i \cap G_j = \emptyset$, it follows that $u_i \neq u_j$ for $i \neq j$ and $i, j = 1, \dots, s$.

Now consider a polynomial $p(u) = (u - u_1)\dots(u - u_q)$ and $\bar{v} = p(\bar{u})/\rho$. Since $\bar{u}$ is finite along $\tilde{F}_1$, $\bar{v}$ is regular and finite at all points of S and has a simple pole along $\tilde{F}_1$.

Let $A_j = H_j + \bar{G}_j$ be the divisor $\bar{u} = u_j$. Since $(\bar{U}_w, \tilde{F}_1) = 1$ for a general w, we have $(A_j, \tilde{F}_1) = 1$ and $(H_j, \tilde{F}_1) = (A_j, \tilde{F}_1) - (\bar{G}_j, \tilde{F}_1) = 0$. Thus, $\tilde{F}_1$ does not intersect zeros of function $\bar{v}$. In particular, the intersection points $s_j = \bar{G}_j \cap \tilde{F}_1$ are not singular for $\bar{v}$; the restriction $\bar{v}|_{\bar{G}_j}$ has simple poles in s_j and is linear along each G_i, $i = 1, \dots, q$ (i.e., it takes every value $z \in \mathbb{P}^1$ at precisely one point of $\bar{G}_i$).

The restriction of $\bar{v}$ on S we denote by v, $v \in O(S)$. We define a regular map $\phi\colon S \to \mathbb{C}^3$ as $\phi(s) = (\rho(s), v(s), u(s))$. We want to show that ϕ is an isomorphism of S onto a hypersurface

$$S' = \{(x, y, t) \in \mathbb{C}^3 \mid xy = p(t)\} \subset \mathbb{C}^3.$$

(A) ϕ is an embedding. Indeed, the functions ρ and u divide points in $\big(S \setminus \big(\bigcup G_i\big)\big)$, since ρ divides fibers of a line pencil and u is linear along each fiber $\rho^{-1}(z)$, $z \neq 0$.

The values $u|_{G_i} = u_i$ provide the distinction between the components G_i of $\rho^{-1}(0)$, since $u_i \neq u_j$ when $i \neq j$. The function v is linear along each G_i, so its values are different in the different points of each G_i.

(B) ϕ is onto. Let $s' \in S'$ and $s' = (x', y', t')$. If $x' \neq 0$, then in the fiber $\rho^{-1}(x')$ there is a point such that $u(s) = t'$. (Indeed, $\rho^{-1}(x') \cong \mathbb{C}$ and $u|_{\rho^{-1}(x')}$ is linear.) Now, $v(s) = p(u)/\rho = p(t')/x' = y'$, so $\phi(s) = s'$.

If $x' = 0$, then $p(t') = 0$ and so $t = u_j$ for some $1 \leq j \leq q$. The function v is linear along the component G_j, so there is a point $s \in G_j$ such that $v(s) = y'$. Then $\phi(s) = (0, y', u_j) = (0, y', t') = s'$. □

Proof of Theorem 1 (cont.). Any surface $S \in \mathcal{H}$ is a hypersurface by Lemma 5. If $S \in \mathcal{A}$ but $S \notin \mathcal{H}$, then (by Lemma 4) $[K(S)] \neq 0$ and (by Lemma 1) S cannot be isomorphic to a hypersurface. □

An example of a surface $S \in \mathcal{A} \setminus \mathcal{H}$ was given in Section 1: $S \subset \mathbb{C}^4$ is defined by the system of equations

$$\begin{cases} xy = (z^2-1)z, \\ zu = (y^2-1)y, \\ xu = (y^2-1)(z^2-1). \end{cases}$$

We will show that this surface is not isomorphic to a hypersurface. On the other hand, there are two locally nilpotent derivations defined in the ring $O(S)$, namely:

$$\begin{cases} \partial_1 x = 0, \\ \partial_1 z = x^2, \\ \partial_1 y = (3z^2-1)x, \\ \partial_1 u = 2z(y^2-1)x + 2y(z^2-1)(3z^2-1); \end{cases}$$

$$\begin{cases} \partial_2 u = 0, \\ \partial_2 y = u^2, \\ \partial_2 z = (3y^2-1)u, \\ \partial_2 x = 2y(z^2-1)u + 2z(y^2-1)(3y^2-1). \end{cases}$$

It follows that $AK(S) = \mathbb{C}$.

Corollary to Lemma 1. *The surface $S \subset \mathbb{C}^4$ defined by equations*

$$\begin{cases} xy = (z^2-1)z, \\ zu = (y^2-1)y, \\ xu = (y^2-1)(z^2-1) \end{cases}$$

is not isomorphic to a hypersurface.

Proof. Consider the 2-form $w = (dx \wedge dz)/x$. It is regular in the Zariski open subset $U_0 = \{(x, y, z, u) \in S \mid x \neq 0\}$, where (x, z) are the local coordinates.

The fiber $\{x = 0\}$ consists of four components:

$$G_1 = \{x = 0, z = 1\}, \qquad G_2 = \{x = 0, z = -1\},$$

$$G_3 = \{x = 0, z = 0, y = 1\}, \quad G_4 = \{x = 0, z = 0, y = -1\}.$$

We consider the respective Zariski open neighborhoods U_1, U_2, U_3, U_4 of these components as follows:

$U_1 = \{(x, y, z, u) \in S \mid z \neq 0,\ z \neq -1\}$ with local coordinates $\varphi_1 = (z-1)/x$ and $\psi_1 = x$;
$U_2 = \{(x, y, z, u) \in S \mid z \neq 0,\ z \neq 1\}$ with local coordinates $\varphi_2 = (z+1)/x$ and $\psi_2 = x$;
$U_3 = \{(x, y, z, u) \in S \mid z^2 \neq 1,\ y \neq 0,\ y \neq -1\}$ with local coordinates $\varphi_3 = (y-1)/z$ and $\psi_3 = z$;
$U_4 = \{(x, y, z, u) \in S \mid z^2 \neq 1,\ y \neq 0,\ y \neq 1\}$ with local coordinates $\varphi_4 = (y+1)/z$ and $\psi_4 = z$.

Rewriting ω in these coordinates, we obtain:

$$\omega = \frac{dx \wedge dz}{x} \qquad \text{in } U_0,$$
$$\omega = d\psi_1 \wedge d\varphi_1 \qquad \text{in } U_1,$$
$$\omega = d\psi_2 \wedge d\varphi_2 \qquad \text{in } U_2,$$
$$\omega = -\frac{\psi_3 d\varphi_3 \wedge d\psi_3}{\varphi_3\psi_3 + 1} \quad \text{in } U_3,$$
$$\omega = -\frac{\psi_4 d\varphi_4 \wedge d\psi_4}{\varphi_4\psi_4 - 1} \quad \text{in } U_4.$$

Since $\varphi_3\psi_3 + 1 = y \neq 0$ in U_3 and $\varphi_4\psi_4 - 1 = y \neq 0$ in U_4, this form is holomorphic everywhere on S. However, $\omega|_{G_3} = \omega|_{G_4} = 0$ and the divisor $(\omega) = G_3 + G_4$ is not equivalent to zero on S, by Lemma 3. Therefore, by Lemma 1, the surface S cannot be isomorphic to a hypersurface. □

4. Corollaries for Cylinders and $\mathbb{C}^+$-Actions

THEOREM 2. *Let S_1 and S_2 be smooth affine surfaces such that $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{A} \setminus \mathcal{H}$. Then $S_1 \times \mathbb{C}^k \not\simeq S_2 \times \mathbb{C}^k$ for any $k \in \mathbb{N}$.*

Proof. Assume, to the contrary, that $S_1 \times \mathbb{C}^k \simeq S_2 \times \mathbb{C}^k = W$.

Since $S_1 \in \mathcal{H}$, by Theorem 1 it is isomorphic to a hypersurface $S \subset \mathbb{C}^3$, and $W \simeq S \times \mathbb{C}^k$ is a hypersurface in $\mathbb{C}^{k+3}$ as well. Hence the canonical classes of W and S_2 are trivial. But then, by Lemma 5, S_2 is a hypersurface and, owing to Theorem 1, $S_2 \in \mathcal{H}$. □

THEOREM 3. *A surface $S \in \mathcal{A}$ admits a fixed-point $\mathbb{C}^+$-action with all the fibers reduced if and only if $S \in \mathcal{H}$.*

Proof. Let $S \in \mathcal{A}$ and let φ_ρ be a fixed-point–free $\mathbb{C}^+$-action. Let ρ be a corresponding line pencil and let $\rho^{-1}(0)$ consist of q components $G_1, \dots, G_q$. Consider another surface $S_q = \{xy = (z-1)\dots(z-q)\} \subset \mathbb{C}^3$. This surface is smooth, affine, and has two $\mathbb{C}^+$-actions:

$$\varphi_x^\lambda(x, y, z) = \left(x, \frac{(z + \lambda x - 1)\dots(z + \lambda x - q)}{x}, z + \lambda x\right);$$

$$\varphi_y^\lambda(x, y, z) = \left(\frac{(z + \lambda y - 1)\dots(z + \lambda y - q)}{y}, y, z + \lambda y\right).$$

Thus, $S_q \in \mathcal{A}$. The actions φ_x^λ and φ_y^λ have no fixed points, because the corresponding LNDs,

$$\partial_x : \partial_x(x) = 0,\ \partial_x(z) = x,\ \partial_x(y) = p'(z)$$

and

$$\partial_y : \partial_y(y) = 0,\ \partial_y(z) = y,\ \partial_y(x) = p'(z),$$

never vanish.

The fibers of the line pencil ρ_x in S_q corresponding to ∂_x are the curves $\{x = \text{const}\}$. All of them are connected except the fiber $x = 0$, which has q connected components. The fibers of the line pencil ρ in S have precisely the same structure.

By the theorem of Daniliewski and Fieseler [D; F], the cylinders $S \times \mathbb{C} \simeq S_q \times \mathbb{C}$. But S_q is a hypersurface and so $S_q \in \mathcal{H}$, by Theorem 1. By Theorem 2, we also have $S \in \mathcal{H}$. Therefore, if S admits a fixed-point–free $\mathbb{C}^+$-action then $S \in \mathcal{H}$.

Now assume that $S \in \mathcal{H}$. As shown in Lemma 5, S is isomorphic to the surface

$$S' = \{(x, y, z) \in \mathbb{C}^3 \mid xy = p(t)\} \subset \mathbb{C}^3.$$

Since S is smooth, all the roots $t_1, \dots, t_q$ of $p(t)$ are simple. That is why the LND ∂, defined as

$$\partial : \partial(x) = 0,\ \partial(t) = x,\ \partial(y) = p'(t),$$

does not vanish on S'. But then the $\mathbb{C}^+$-action defined by ∂ has no fixed points. □

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