Michigan Math. J. 48 (2000)

Stratified Spaces Formed by Totally Positive Varieties

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This paper is dedicated to William Fulton

1. Introduction and Main Results

In 1984, Björner [4] showed that every interval in the Bruhat order of a Coxeter group W is the "face poset" of some stratified space in which each closed stratum (resp., the boundary of each stratum) has the homology of a ball (resp., of a sphere). Passing to the Euler characteristic, this result implies Verma's formula [19; 20] for the Möbius function of the Bruhat order—namely, $\mu(u,v) = (-1)^{\ell(v)-\ell(u)}$ $(u \le v)$, where ℓ denotes the length function. (This is, in turn, equivalent to saying that each Bruhat interval contains equally many elements of even and odd length.)

In fact, Björner proved a stronger result: Every interval in the Bruhat order is the face poset of a regular cell complex (i.e., closed strata actually *are* balls). However, the construction of such complex in [4] was entirely "synthetic" (essentially, a succession of cell attachments; cf. [5, 4.7.23]). Furthermore, it was based on the existence of a combinatorial shelling, which by itself easily implies Verma's formula, bypassing all geometry. A question posed in [4] asked for a natural geometric construction of a stratified space with the desired properties.

In this paper, we propose such a construction for the case where W is the Weyl group of a semisimple group G. In the type-A case, where W is the symmetric group and G the special linear group, we prove that our stratified spaces indeed have the required homological properties. The spaces we construct are links of cells in the Bruhat decomposition of the *totally nonnegative* part of the unipotent radical of G.

In the remainder of Section 1, we present the details of this construction and state our main results and conjectures. The rest of the paper is devoted to proofs.

Let G be a semisimple, simply connected algebraic group defined and split over $\mathbb{R}$. Let B and B_- be two opposite Borel subgroups of G, so that $H = B_- \cap B$ is an $\mathbb{R}$-split maximal torus in G; we denote by N and N_- the unipotent radicals of B and B_-, respectively.

For the type A_{n-1}: the group G is the real special linear group $\mathrm{SL}(n,\mathbb{R})$; H, B, and B_- are the subgroups of diagonal, upper triangular, and lower triangular matrices, respectively; N and N_- are the subgroups of B and B_- that consist of matrices whose diagonal entries are equal to 1.

Received January 27, 2000. Revision received April 6, 2000.
The first author was supported in part by NSF grant no. DMS-9700927.

We denote by Y the set of all *totally nonnegative* elements in N. In the case of the special linear group, Y consists of the upper-triangular unipotent matrices all of whose minors are nonnegative. The general definition was first suggested by Lusztig (see [17] and references therein). In our current notation, Lusztig defined Y as the multiplicative submonoid of N generated by the elements $\exp(te_i)$, $t \geq 0$, where the e_i are the Chevalley generators of the Lie algebra of N. An alternative description in terms of nonnegativity of certain "generalized minors" was given in [10] (cf. our Proposition 2.9).

Let W be the Weyl group of G. The length of an element $w \in W$ is denoted by $\ell(w)$. The group W is partially ordered by the Bruhat order, defined geometrically by

$$u \leq v \iff B_-uB_- \subseteq \overline{B_-vB_-}.$$

The Bruhat decomposition $G = \bigcup_{w \in W} B_-wB_-$ induces the partition of Y into mutually disjoint *totally positive varieties* $Y_w^\circ = Y \cap B_-wB_-$, $w \in W$ (this terminology is borrowed from [9]).

We denote $Y_w = \overline{Y_w^\circ}$. The varieties Y_w° were first studied by Lusztig in [16], where, in particular, the following basic properties were obtained.

PROPOSITION 1.1 [16]. *Each totally positive variety Y_w° is a cell*; *more precisely, Y_w° is homeomorphic to $\mathbb{R}^{\ell(w)}$. Furthermore, $Y_w = \bigcup_{u \leq w} Y_u^\circ$.*

EXAMPLE: $G = \mathrm{SL}(3, \mathbb{R})$. In this case,

$$Y = \left\{ x = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} : x_{12} \geq 0,\ x_{23} \geq 0,\ x_{13} \geq 0,\ \begin{vmatrix} x_{12} & x_{13} \\ 1 & x_{23} \end{vmatrix} \geq 0 \right\}.$$

Thus the set Y is described in the coordinates (x_{12}, x_{23}, x_{13}) as the closure of one of the pieces into which the plane $x_{13} = 0$ and the hyperbolic paraboloid $x_{12}x_{23} = x_{13}$ partition the 3-space—namely, the piece containing the point $(1, 1, \frac{1}{2})$. The semialgebraic set Y decomposes naturally into six algebraic strata: the origin, two rays (the positive semi-axes for x_{12} and x_{23}), two 2-dimensional pieces connecting them, and the 3-dimensional interior. These are the six Bruhat strata Y_w° for $w \in W = \mathcal{S}_3$ (the symmetric group). Figure 1 shows a planar cross-section of this stratification (or, equivalently, the link of the 0-dimensional cell). The adjacency of the strata Y_w° is indeed described by the Bruhat order on $\mathcal{S}_3$, in agreement with Proposition 1.1.

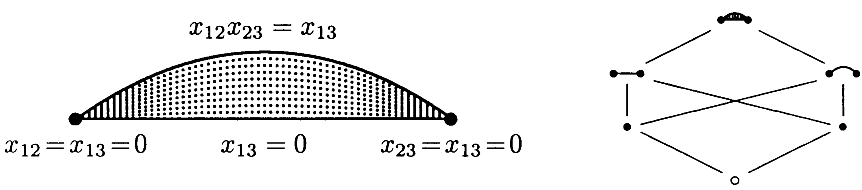

Figure 1 Totally nonnegative varieties Y_w° in the special case $G = \mathrm{SL}(3, \mathbb{R})$

For $u, v \in W$ $(u \leq v)$, the *Bruhat interval* $[u, v]$ is defined by $[u, v] = \{u \leq w \leq v\}$, with the partial order inherited from W. Similarly, $(u, v] \overset{\text{def}}{=} \{u < w \leq v\}$.

In view of Proposition 1.1, it is natural to suggest that the geometric model for a Bruhat interval $[u, v]$ (or $(u, v]$) is provided by the *link*

$$\mathrm{Lk}(u, v) = \mathrm{lk}(Y_u^\circ, Y_v)$$

of the cell Y_u° inside the subcomplex $Y_v \subset Y$.

The following are our main results.

THEOREM 1.2. *For any $u \leq v$, the link $\mathrm{Lk}(u, v)$ is well-defined as a stratified space. The strata $S_{u,v,w} = \mathrm{Lk}(u, v) \cap Y_w^\circ$ are labeled by the elements $w \in (u, v]$, and each stratum $S_{u,v,w}$ is an open smooth manifold of dimension $\ell(w) - \ell(u) - 1$. The closures and boundaries of the strata $S_{u,v,w}$ are given by*

$$\overline{S_{u,v,w}} = \bigcup_{u<w'\leq w} S_{u,v,w'}, \qquad \partial\overline{S_{u,v,w}} = \bigcup_{u<w'<w} S_{u,v,w'}. \tag{1.1}$$

Here, by a stratified space we mean a decomposition of a semialgebraic set into disjoint smooth submanifolds labeled by the elements of a partially ordered set, as described for example in [11, Sec. 1.1]. Although the stratifications we consider seem to satisfy Whitney's regularity conditions (cf. [11, Sec. 1.2]), we will not need to verify these conditions to justify our constructions.

THEOREM 1.3 (Type A only). *All the strata $S_{u,v,w}$ are orientable.*

THEOREM 1.4 (Type A only). *Each closed stratum $\overline{S_{u,v,w}}$ is contractible. Moreover, the contraction can be chosen so that it restricts to a contraction of the open stratum $S_{u,v,w}$.*

These theorems ensure that the stratified spaces $\mathrm{Lk}(u, v)$ have the desired homological properties, as we will now explain. Let $\mathrm{H}_i(X)$ (resp., $\mathrm{H}_i(X, X')$) denote, as usual, the ordinary ith homology group CW complex X (resp., pair of CW complexes $X' \subset X$). The corresponding Euler characteristics are denoted by $\chi(X)$ and $\chi(X, X')$, respectively.

COROLLARY 1.5 (Type A only). *For any $u < w \leq v$, we have*

$$\mathrm{H}_i(\overline{S_{u,v,w}}, \partial\overline{S_{u,v,w}}) = \begin{cases} \mathbb{Z} & \text{if } i = \ell(w) - \ell(u) - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Consequently, $\chi(\overline{S_{u,v,w}}, \partial\overline{S_{u,v,w}}) = (-1)^{\ell(w)-\ell(u)-1}$.

Proof. We will need the Lefschetz duality isomorphism [8, Ex. 18.3]:

$$\mathrm{H}_i(X, A) \simeq \mathrm{H}^{n-i}(X \setminus A; \mathbb{Z}), \quad i > 0,$$

where X is a compact topological space and A its closed subset such that $X \setminus A$ is a smooth orientable n-dimensional manifold. Take $X = \overline{S_{u,v,w}}$ and $A = \partial\overline{S_{u,v,w}}$. Then Theorems 1.2 and 1.3 ensure that the above conditions are satisfied, with

$n = \ell(w) - \ell(u) - 1$. Hence $\mathrm{H}_i(\overline{S_{u,v,w}}, \partial\overline{S_{u,v,w}}) = \mathrm{H}^{\ell(w)-\ell(u)-1-i}(S_{u,v,w})$. Since $S_{u,v,w}$ is contractible by Theorem 1.4, (1.2) follows. □

COROLLARY 1.6 (Verma's formula for the type A). $\sum_{u\le w\le v}(-1)^{l(w)} = 0$.

Thus every Bruhat interval is an *Eulerian poset* [18].

Proof. The additivity of the Euler characteristic [7, V.5.7], which applies in view of Theorem 1.2, gives

$$\chi(\mathrm{Lk}(u,v)) = \sum_{u<w\le v} \chi(\overline{S_{u,v,w}}, \partial\overline{S_{u,v,w}}).$$

In the last identity, the left-hand side is equal to 1 by Theorem 1.4, while the right-hand side is equal to $\sum_{u<w\le v}(-1)^{l(w)-l(u)-1}$ by Corollary 1.5. Simplifying, we obtain the desired formula. □

For the type A, we prove the following refinement of Theorem 1.2. Let us define the stratified space $Y_{[u,v]}$ by $Y_{[u,v]} = \bigcup_{w\in[u,v]} Y_w^\circ$. Note that $\mathrm{Lk}(u,v) = \mathrm{lk}(Y_u^\circ, Y_v) = \mathrm{lk}(Y_u^\circ, Y_{[u,v]})$.

THEOREM 1.7 (Type A only). *The stratified space $Y_{[u,v]}$ has the structure of the direct product of the cell Y_u° and the cone over the link* $\mathrm{Lk}(u,v)$. *More precisely, there exists an isomorphism of stratified spaces $Y_{[u,v]}$ and $Y_u^\circ \times \mathrm{Cone}(\mathrm{Lk}(u,v))$ whose restriction to each stratum is a diffeomorphism.*

REMARK 1.8. Theorem 1.3 can be deduced from Theorem 1.7 as follows. By (1.1), the stratum $S_{u,v,w}$ coincides with the interior of $\mathrm{Lk}(u,w)$. Thus Theorem 1.7 asserts, in particular, that the cell Y_w° is a direct product of the cell Y_u° and the interior of the cone over $\mathrm{Lk}(u,w)$. Both cells Y_w° and Y_u° are evidently orientable. Therefore (see e.g. [12, Ex. 3.2.24]), the interior of the cone over $\mathrm{Lk}(u,w)$ is orientable and so is the interior of $\mathrm{Lk}(u,w)$. □

CONJECTURE 1.9. *Theorems 1.4 and 1.7* (*hence Theorem 1.3 and Corollary 1.5*) *hold for any semisimple algebraic group G.*

We believe that Conjecture 1.9 can be strengthened as follows.

CONJECTURE 1.10. *Each stratum $S_{u,v,w}$* (*resp., its closure and its boundary*) *is homeomorphic to an affine space* (*resp., a closed ball and a sphere*) *of dimension* $\ell(v)-\ell(u)-1$ (*resp.,* $\ell(v)-\ell(u)-1$ *and* $\ell(v)-\ell(u)-2$). *Thus* $\mathrm{Lk}(u,v)$ *is a regular cell complex.*

Assuming Conjecture 1.10 holds, each stratified link $\mathrm{Lk}(u,v)$ provides a geometric realization of the "generalized synthetic Schubert variety" whose existence was hypothesized by Björner [4].

We hope to extend the construction of the spaces $\mathrm{Lk}(u,v)$ to an arbitrary simply laced Coxeter group, and possibly further, so that the analogs of all statements formulated above would still hold. (Note that Björner's original result applies to intervals in *any* Coxeter group.)

It should be mentioned that one of our "hidden motivations" has been the desire to better understand the combinatorics of Kazhdan–Lusztig polynomials. It was already pointed out in their original paper [14] that Verma's formula is equivalent to the assertion that the constant term of any Kazhdan–Lusztig polynomial is 1.

The remainder of this paper is organized as follows. Sections 2–3 introduce some useful Lie-theoretic machinery; in particular, we define a projection onto a cell Y_u° that plays a crucial role in subsequent proofs. In Section 4, we prove Theorem 1.2. Section 5 contains the proofs of Theorems 1.4 and 1.7. These proofs are based on a technical lemma (Lemma 5.4), which is proved in Section 6 for the special case of $G = \mathrm{SL}(n)$; this is the only "type-specific" ingredient of our proofs.

Acknowledgments. The authors are grateful to Saugata Basu, Ilia Itenberg, Viatcheslav Kharlamov, Boris Shapiro, Eugenii Shustin, Viktor Vassiliev, and Andrei Zelevinsky for valuable advice.

2. Preliminaries

This section introduces necessary technical background; throughout it, we do not claim any originality. The notation used here is consistent with [9]. In particular, we denote by

$$G_0 = B_-B = N_-HN$$

the set of elements of $x \in G$ that have a Gaussian decomposition; for the latter, we use the notation $x = [x]_-[x]_0[x]_+$.

We think of the Weyl group W as the quotient of the normalizer of H modulo H, and we identify each element $w \in W$ with a fixed representative in G.

Lemma 2.1. *For $w \in W$, we have $w^{-1}B_-w \subset G_0$ and $w^{-1}Bw \subset G_0$. Moreover, $w^{-1}N_-w \subset N_-N$ and $w^{-1}Nw \subset N_-N$.*

Proof. Since $B = HN$, $B_- = HN_-$, and w normalizes H, it suffices to prove the last statement. It is well known (cf. [9, Prop. 2.12] or [2, (5.3)]) that any $x \in N$ is uniquely factored as $x = x_1x_2$ with $x_1 \in N \cap wN_-w^{-1}$ and $x_2 \in N \cap wNw^{-1}$. Hence $N \subset wN_-w^{-1} \cdot wNw^{-1} = wN_-Nw^{-1}$, as desired. □

Lemma 2.2. *If $z \in w^{-1}B_-w$ then $[z]_-, [z]_+ \in w^{-1}N_-w$. Analogously, if $z \in w^{-1}Bw$ then $[z]_-, [z]_+ \in w^{-1}Nw$.*

Proof. It is enough to show that $z \in w^{-1}Bw$ implies $[z]_+ \in w^{-1}Nw$. Just as in the proof of Lemma 2.1, we can write $z = hw^{-1}x_1x_2w$, where $h \in H$, $x_1 \in N \cap wN_-w^{-1}$, and $x_2 \in N \cap wNw^{-1}$. Then $z = (hw^{-1}x_1w)(w^{-1}x_2w)$, where the factors belong to B_- and N, respectively. Thus $[z]_+ = w^{-1}x_2w \in w^{-1}Nw$, as desired. □

We define the subgroups

$$N_-(w) = w^{-1}Bw \cap N_- = w^{-1}Nw \cap N_-,$$
$$N(w) = w^{-1}Bw \cap N = w^{-1}Nw \cap N,$$

and the set

$$N^w = B_- w B_- \cap N.$$

LEMMA 2.3. *For any $w \in W$ and $x_w \in N^w$, there exists a unique $y \in N_-(w)$ satisfying $x_w = [wy]_+$. Specifically, $y = w^{-1}[x_w w^{-1}]_+ w$.*

Proof. Immediate from [9, Prop. 2.10 and 2.17]. □

LEMMA 2.4. *Let $x_w \in N^w$, $b_- \in B_-$, and $x_w b_- \in G_0$. Then $[x_w b_-]_+ \in N^w$.*

Proof. $[x_w b_-]_+ \in B_- x_w b_- \subset B_- \cdot B_- w B_- \cdot b_- = B_- w B_-$. □

The following statement, though obvious, is quite useful.

LEMMA 2.5. *If $x \in G_0$ and $y \in G$, then $[[x]_+ y]_+ = [xy]_+$, provided one of the two sides is well-defined*.

LEMMA 2.6. *For any $x_w, \tilde{x}_w \in N^w$, there exists a unique $n_1 \in N_-(w)$ satisfying $x_w = [\tilde{x}_w n_1]_+$. Specifically, $n_1 = w^{-1}([\tilde{x}_w w^{-1}]_+)^{-1}[x_w w^{-1}]_+ w$.*

Proof. Uniqueness follows from the uniqueness part of Lemma 2.3, together with Lemma 2.5 and the fact that $N_-(w)$ is a group. In more detail: Assume that $x_w = [\tilde{x}_w n_1]_+ = [\tilde{x}_w n_1']_+$, where $n_1 \neq n_1'$ and $n_1, n_1' \in N_-(w)$. Let y be as in Lemma 2.3. Then $x_w = [\tilde{x}_w n_1 \cdot n_1^{-1} n_1']_+ = [x_w n_1^{-1} n_1']_+ = [wyn_1^{-1} n_1']_+$, where $y \neq yn_1^{-1}n_1' \in N_-(w)$, a contradiction.

With the notation $y = w^{-1}[x_w w^{-1}]_+ w$ and $\tilde{y} = w^{-1}[\tilde{x}_w w^{-1}]_+ w$, it remains to check that $n_1 = \tilde{y}^{-1} y$ satisfies $x_w = [\tilde{x}_w n_1]_+$. Indeed, $x_w = [wy]_+ = [w\tilde{y}n_1]_+ = [x_w n_1]_+$. □

We now turn to total nonnegativity. Let us first recall Lusztig's original definition [16], whereby the set Y of totally nonnegative elements in N is defined as the multiplicative monoid generated by the elements

$$x_i(t) = \exp(te_i), \tag{2.1}$$

where $t \geq 0$ and the e_i are the Chevalley generators of the Lie algebra of N. One of the first results in [16] is the following description of the Bruhat stratum $Y_w^\circ = Y \cap B_- w B_-$.

PROPOSITION 2.7 [16]. *Let $(a_1, \dots, a_l)$ be a reduced word for $w \in W$. Then the map*

$$(t_1, \dots, t_l) \mapsto x_{a_1}(t_1) \cdots x_{a_l}(t_l) \tag{2.2}$$

is a bijection between $\mathbb{R}_{>0}^l$ and Y_w°.

(It is clear from this description that Y_w° is indeed a cell of dimension $l = \ell(w)$; cf. Proposition 1.1.) One is tempted to use the parameterizations (2.2) to prove our main theorems. Unfortunately, this approach encounters substantial difficulties, due chiefly to the fact that the relationship between parameterizations of adjacent cells is generally quite complicated. In what follows, we make very little use of Proposition 2.7.

PROPOSITION 2.8 [17, 6.3]. *The cell Y_w° is a connected component in N^w (in the ordinary topology).*

For $u \in W$, we denote $Y_{\geq u} = \bigcup_{v \geq u} Y_v^\circ$.

To state the next result, we will need the notion of a generalized minor of an element $x \in G$ (see [9, Sec. 14]). Generalized minors are certain regular functions on G that can be defined as suitably normalized matrix coefficients corresponding to pairs of extremal weights in some fundamental representation of G. In the case of type A, this notion coincides with the ordinary notion of a minor of a square matrix.

PROPOSITION 2.9 [10, Thm. 3.1]. *An element $x \in G$ is totally nonnegative, in the sense of Lusztig* [16], *if and only if all its generalized minors are nonnegative.*

LEMMA 2.10. *If a generalized minor does not vanish at some point $x \in Y_u^\circ$, then it vanishes nowhere in Y_u° and, moreover, nowhere in $Y_{\geq u}$.*

Proof. For the type A, this is an immediate corollary of [1, Prop. 5.2.2]. The general case can be deduced from (highly nontrivial) [3, Prop. 7.4]. According to the latter, for any generalized minor Δ and any sequence of indices $a = (a_1, \ldots, a_m)$, the function $P_a(t_1, \ldots, t_m) = \Delta(x_{a_1}(t_1) \cdots x_{a_m}(t_m))$ (cf. (2.2)) is either identically zero or a polynomial with positive integer coefficients. (The type-A version of this statement is well known; see e.g. [1, Thm. 2.4.4].) Since Δ does not vanish at some point in Y_u°, we know that P_a is a nonzero polynomial for any reduced word a for u. For $v \geq u$, any reduced word b for v contains some reduced word a for u as a subword (see [13, 5.10]). Hence P_a is a specialization of P_b obtained by setting some of the variables equal to zero. Then $P_a \neq 0$ implies $P_b \neq 0$. On the other hand, P_b is a polynomial with positive coefficients, so $P_b(t_1, t_2, \ldots) \neq 0$ for any $t_1, t_2, \ldots > 0$ or, equivalently, $\Delta(x) \neq 0$ for any $x \in Y_v^\circ$. □

LEMMA 2.11. *For any $u \in W$, we have $B_-uB_- \subset G_0u$. In particular, $N^u \subset G_0u$.*

Proof. This follows from Lemma 2.1. □

COROLLARY 2.12. $Y_{\geq u} \subset G_0u$.

Proof. By [9, Cor. 2.5], the set G_0u is defined by several inequalities of the form $\Delta \neq 0$, where Δ is a generalized minor. Since $Y_u^\circ \subset G_0u$ (by Lemma 2.11), none of these minors vanishes on Y_u° and hence none vanishes anywhere on $Y_{\geq u}$, by Lemma 2.10. □

THEOREM 2.13 ([6, Cor. 1.2]; cf. [15, Sec. 1.2]). *For $u, v \in W$, the intersection $B_-vB_- \cap BuB_-$ is non-empty if and only if $u \le v$.*

COROLLARY 2.14. $G_0 u \subset \bigcup_{v \ge u} B_-vB_-$.

Proof. Let $x \in G_0 u$ and $x \in B_-vB_-$, and let $v \in W$. Then, by Theorem 2.13,

$$B_-vB_- \cap B_-Bu \ne \emptyset \implies B_-vB_- \cap Bu \ne \emptyset \implies u \le v,$$

as desired. □

COROLLARY 2.15. $Y_{\ge u} = Y \cap G_0 u$.

Proof. The inclusion $Y_{\ge u} \subset Y \cap G_0 u$ is Corollary 2.12. The opposite inclusion is immediate from Corollary 2.14. □

LEMMA 2.16. $Y_{\ge u} Y \subset Y_{\ge u}$.

Proof. By [16, Lemma 2.14], for any $w_1, w_2 \in W$ we have $Y^\circ_{w_1} Y^\circ_{w_2} = Y^\circ_{w_3}$ for some $w_3 \in W$. Moreover, it is clear from the proof of this statement in [16] that $w_3 \ge w_1$, and the lemma follows. □

EXAMPLE: $G = \mathrm{SL}(3, \mathbb{R})$. Let $u = s_1$, the transposition of 1 and 2 in the symmetric group $W = \mathcal{S}_3$. Then, using the notation $x = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}$ for the elements $x \in N$, we have

$$N(u) = \{x_{12} = 0\}, \quad N^u = \left\{ \begin{matrix} x_{12} \ne 0 & x_{13} = 0 \\ & x_{23} = 0 \end{matrix} \right\}, \quad N \cap G_0 u = \{x_{12} \ne 0\}$$

and

$$Y^\circ_u = \left\{ \begin{matrix} x_{12} > 0 & x_{13} = 0 \\ & x_{23} = 0 \end{matrix} \right\}, \quad Y_{\ge u} = \left\{ \begin{matrix} x_{12} > 0 & x_{13} \ge 0 \\ & x_{23} \ge 0 \end{matrix} \quad \begin{vmatrix} x_{12} & x_{13} \\ 1 & x_{23} \end{vmatrix} \ge 0 \right\}.$$

3. Projecting on a Cell

In this section, we introduce a projection $\pi_u \colon Y_{\ge u} \to Y^\circ_u$ that will later be used to construct and study the links $\mathrm{Lk}(u, v) = \mathrm{lk}(Y^\circ_u, Y_v)$. This projection can be viewed as the totally positive version of the projection of an affine open neighborhood of a Schubert cell onto the cell itself, which arises from the direct product decomposition described by Kazhdan and Lusztig in [15, Secs. 1.3–1.4].

Let us fix an element $u \in W$.

LEMMA 3.1. *If $x \in G_0 u \cap G_0$ (in particular, if $x \in Y_{\ge u}$—cf. Corollary 2.12), then $u^{-1}[xu^{-1}]_+ u \in G_0$ and $u[u^{-1}[xu^{-1}]_+ u]_- \in G_0$.*

Proof. The first statement follows from Lemma 2.1. Proof of the second: For some $b_- \in B_-$ and $b \in B$, we have $u[u^{-1}[xu^{-1}]_+ u]_- = uu^{-1}b_- xu^{-1}ub = b_- xb \in G_0$. □

LEMMA 3.2. *The map $(x_u, x^u) \mapsto x = x_u x^u$ is a bijection*

$$N^u \times N(u) \to N \cap G_0 u.$$

The inverse map $x \mapsto (x_u, x^u)$ is given by

$$x_u = [u[u^{-1}[xu^{-1}]_+u]_-]_+ \tag{3.1}$$

and

$$x^u = [u^{-1}[xu^{-1}]_+u]_+. \tag{3.2}$$

Furthermore, if $x \in N \cap G_0u$ is totally nonnegative (i.e., if $x \in Y_{\geq u}$—cf. Corollary 2.15), then x_u is totally nonnegative (i.e., $x_u \in Y_u^\circ$).

Proof. Assume that $x_u \in N^u$, $x^u \in N(u)$, and $x = x_u x^u$. Then $x = x_u x^u \in G_0 u \cdot u^{-1}Nu = G_0 u$ (by Lemma 2.11 and the definition of $N(u)$), as claimed.

Let us prove that the map in question is a surjection. Let $x \in N \cap G_0 u$, and let x_u and x^u be given by (3.1) and (3.2); note that the right-hand sides of these formulas are well-defined (by Lemma 3.1). Thus $x_u = [u[y]_-]_+$ and $x^u = [y]_+$, where $y = u^{-1}[xu^{-1}]_+u$. Then $x_u \in B_-u[y]_- \subset B_-uB_-$ and $x^u \in N(u)$ (by Lemma 2.2). Furthermore, $x_u x^u = [u[y]_-]_+[y]_+ = [uy]_+$ (since $y \in N_-N$ by Lemma 2.1) and therefore $x_u x^u = [uy]_+ = [[xu^{-1}]_+u]_+ = x$ (by Lemma 2.5).

Let us now prove injectivity. Again, suppose $x_u \in N^u$, $x^u \in N(u)$, and $x = x_u x^u$. We will show that x_u and x^u can be recovered from x via (3.1)–(3.2). Since $x \in N \cap G_0 u$, the right-hand sides of (3.1)–(3.2) are well-defined (by Lemma 3.1). Then

$$\begin{aligned}
[u[u^{-1}[xu^{-1}]_+u]_-]_+ &= [u[u^{-1}[x_u x^u u^{-1}]_+u]_-]_+ \\
&= [u[u^{-1}[x_u u^{-1}]_+ u x^u u^{-1} u]_-]_+ && (\text{since } ux^uu^{-1} \in N) \\
&= [u[u^{-1}[x_u u^{-1}]_+ u]_-]_+ \\
&= [u \cdot u^{-1}[x_u u^{-1}]_+ u]_+ && (\text{by Lemma 2.3}) \\
&= x_u && (\text{by Lemma 2.5}),
\end{aligned}$$

proving (3.1).

Let us prove (3.2). Denote $A = u^{-1}[xu^{-1}]_+u$. We have

$$x = [x]_+ = [u(u^{-1}[xu^{-1}]_+u)]_+ = [uA]_+ = [u[A]_-]_+[A]_+$$

(by Lemma 2.1). On the other hand, we already proved that $x_u = [u[A]_-]_+$. Thus $x = x_u[A]_+$ (i.e., $x^u = [A]_+$), as desired.

It remains to prove that x_u is totally nonnegative whenever x is. Assume that $x \in Y_w^\circ \subset Y_{\geq u}$. Consider a path that connects x with a point $x_0 \in Y_u^\circ$ and stays inside Y_w° (such a path exists because Y_w° is connected and its boundary contains Y_u°; see Proposition 1.1). The image of this path under the projection $N \cap G_0 u \to N^u$ connects x_u with x_0. Since $x_0 \in Y_u^\circ$, Proposition 2.8 implies that $x_u \in Y_u^\circ$. $\square$

In view of Lemma 3.2, the formula

$$\pi_u(x) = [u[u^{-1}[xu^{-1}]_+u]_-]_+ \tag{3.3}$$

defines a continuous projection $\pi_u \colon Y_{\geq u} \to Y_u^\circ$. (The map π_u is a projection since $x = x \cdot 1$ gives the factorization in question for $x \in Y_u^\circ$.)

Example: $G = \mathrm{SL}(3, \mathbb{R})$, $u = s_1$. For $x \in Y_{\geq u}$ (or, more generally, $x \in N \cap G_0 u$), the factorization $x = x_u x^u$ is given by

$$\begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x_{13} - x_{12}x_{23} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

The fiber of the projection $\pi_u : x \mapsto x_u$ over a point $x_u = \left[\begin{smallmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \in Y_u^\circ$ $(a > 0)$ is therefore

$$\pi_u^{-1}(x_u) = Y \cap \{x_{12} = a\} = \left\{ \begin{bmatrix} 1 & a & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} : a x_{23} \geq x_{13} \geq 0 \right\}.$$

4. Transversals and Links. Proof of Theorem 1.2

Our next goal is to prove that the restriction of the projection π_u onto $Y_{[u,v]}$ is globally trivialized along Y_u°.

LEMMA 4.1. *For any $\tilde{x} \in N \cap G_0 u$ and any $x_u \in N^u$, there exists unique $n_- \in N_-(u)$ such that the element $x' = [\tilde{x} n_-]_+$ is well-defined and belongs to $x_u N(u)$.*

If, moreover, $\tilde{x}$ and x_u are totally nonnegative, then x' is also totally nonnegative. We thus obtain a cell-preserving projection

$$\begin{aligned} \rho_{x_u} : Y_{\geq u} &\to \pi_u^{-1}(x_u) \\ \tilde{x} &\mapsto x' \end{aligned} \tag{4.1}$$

(*see Figure 2*).

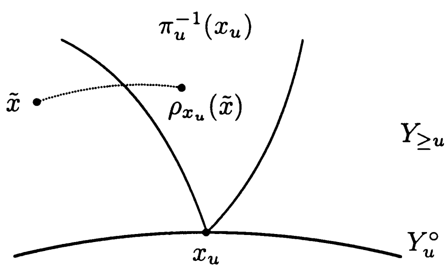

Figure 2 The projection ρ_{x_u}

Proof. Let $\tilde{x} = \tilde{x}_u \tilde{x}^u$, where $\tilde{x}_u \in N^u$ and $\tilde{x}^u \in N(u)$, as in Lemma 3.2. Let $n_1 \in N_-(u)$ be such that $x_u = [\tilde{x}_u n_1]_+$ (such n_1 exists and is unique by Lemma 2.6). Set

$$n_- = [(\tilde{x}^u)^{-1} n_1]_-. \tag{4.2}$$

Since both $\tilde{x}^u$ and n_1 belong to $u^{-1}Nu$, the element n_- is well defined in view of Lemma 2.1, and belongs to $N_-(u)$ by Lemma 2.2. Let us prove that the element n_- defined by (4.2) has the desired properties; that is, $x' = [\tilde{x} n_-]_+$ is well-defined and belongs to $x_u N(u)$, as shown in the following diagram:

$$\begin{array}{ccc} \tilde{x} & \rightsquigarrow & x' = [\tilde{x} n_-]_+ \\ {\scriptstyle \pi_u}\downarrow & & \downarrow \\ \tilde{x}_u & \rightsquigarrow & x_u = [\tilde{x}_u n_1]_+ . \end{array} \tag{4.3}$$

Denote $z = \tilde{x}^u n_-$. Once again, $z \in u^{-1} N u \subset N_- N$, and (4.2) implies $[z]_- = [\tilde{x}^u n_-]_- = [\tilde{x}^u [(\tilde{x}^u)^{-1} n_1]_-]_- = n_1$. Then $\tilde{x} n_- = \tilde{x}_u z = \tilde{x}_u n_1 [z]_+ \in G_0$ (because $[\tilde{x}_u n_1]_+ = x_u$), so x' is indeed well-defined. Furthermore, $x' = x_u [z]_+ \in x_u N(u)$ (by Lemma 2.2), as desired.

Uniqueness is proved by a similar argument. Suppose that $n_- \in N_-(u)$ is such that $x' = [\tilde{x} n_-]_+ \in x_u N(u)$. As before, denote $z = \tilde{x}^u n_-$. Then $z = [z]_- [z]_+$ and $x' = [\tilde{x}_u z]_+ = [\tilde{x}_u [z]_-]_+ \cdot [z]_+$. Since $[\tilde{x}_u [z]_-]_+ \in N^u$ (by Lemma 2.4) and $[z]_+ \in N(u)$, it follows from Lemma 3.2 that $[\tilde{x}_u [z]_-]_+ = x_u = [\tilde{x}_u n_1]_+$. Hence, by Lemmas 2.2 and 2.6, $n_1 = [z]_- = [\tilde{x}^u n_-]_-$, implying (4.2).

It remains to prove the second part of the lemma. In view of Lemma 2.6, a path connecting $\tilde{x}_u$ and x_u within Y_u° gives a continuous deformation of the identity $1 \in G$ into n_1 within $N_-(u)$, which gives rise (via (4.2)) to a continuous deformation of 1 into n_- and, finally, to a path connecting $\tilde{x}$ and $x' = [\tilde{x} n_-]_+$ within the Bruhat cell containing $\tilde{x}$. Hence x' is totally nonnegative by Proposition 2.8. $\square$

EXAMPLE: $G = \mathrm{SL}(3, \mathbb{R})$, $u = s_1$. For

$$\tilde{x} = \begin{bmatrix} 1 & \tilde{x}_{12} & \tilde{x}_{13} \\ 0 & 1 & \tilde{x}_{23} \\ 0 & 0 & 1 \end{bmatrix} \in Y_{\geq u} \quad \text{and} \quad x_u = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in Y_u^\circ,$$

computations give

$$n_- = \begin{bmatrix} 1 & 0 & 0 \\ a^{-1} - \tilde{x}_{12}^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$x' = \rho_{x_u}(\tilde{x}) = [\tilde{x} n_-]_+ = \begin{bmatrix} 1 & a & \dfrac{a\tilde{x}_{13}}{\tilde{x}_{12}} \\ 0 & 1 & \dfrac{\tilde{x}_{12}\tilde{x}_{23} - \tilde{x}_{13}}{a} + \dfrac{\tilde{x}_{13}}{\tilde{x}_{12}} \\ 0 & 0 & 1 \end{bmatrix}.$$

Total nonnegativity of x' does indeed follow from total nonnegativity of $\tilde{x}$ and x_u.

We denote by w_o the element of maximal length in W.

THEOREM 4.2. (1) *For $x_u \in Y_u^\circ$, the set $x_u N(u)$ is a smooth submanifold in $N \cap G_0 u$ diffeomorphic to the affine space $\mathbb{R}^{\ell(w_\mathrm{o}) - \ell(u)}$. Furthermore, $x_u N(u)$ is transversal to every Bruhat stratum N^w, $w \geq u$ (and hence to every stratum $Y_w^\circ \subset Y_{\geq u}$).*

(2) *For $x_u, \tilde{x}_u \in Y_u^\circ$, the map ρ_{x_u} described in Lemma 4.1 establishes a diffeomorphism between $\tilde{x}_u N(u)$ and $x_u N(u)$. This diffeomorphism respects total nonnegativity and the Bruhat stratification; more precisely, it restricts to a stratified diffeomorphism between the fibers $\pi_u^{-1}(\tilde{x}_u)$ and $\pi_u^{-1}(x_u)$.*

Proof. The map $n_+ \mapsto x_u n_+$ establishes a diffeomorphism between $N(u) \cong \mathbb{R}^{\ell(w_o)-\ell(u)}$ and $x_u N(u)$. To prove transversality, consider a point $x \in x_u N(u) \cap N^w$. It will be enough to show that $x_u N(u)$ is transversal to the smooth submanifold $[xN_-(u)]_+$ of dimension $\ell(u)$ in N^w. Assume the contrary—in other words, that there exists a common tangent vector v to $[xN_-(u)]_+$ and $x_u N(u)$ at the point x. Let us evaluate the differential D of the projection $N \cap G_0 u \to N^u$ at the vector v. On the one hand, the projection is constant on $x_u N(u)$ and hence $D(v) = 0$. On the other hand, in view of (4.2), the restriction of the projection onto $[xN_-(u)]_+$ is a diffeomorphism and hence $D(v) \neq 0$—a contradiction.

Let us prove the second part of the theorem. From (4.2) and (4.3), we have $x' = [\tilde{x}[(\tilde{x}^u)^{-1} n_1]_-]_+$, where $\tilde{x}^u$ is given by (3.2) and n_1 by Lemma 2.6. The resulting map $\tilde{x}_u N(u) \to x_u N(u)$ is rational and therefore differentiable on its domain. Its inverse is again a map of the same kind, with the roles of x_u and $\tilde{x}_u$ reversed. Hence these maps are diffeomorphisms. Furthermore, they preserve the Bruhat stratification (in view of Lemma 2.4) and total nonnegativity (by the second part of Lemma 4.1). □

Recall the notation $Y_{[u,v]} = \bigcup_{w \in [u,v]} Y_w^\circ$ and $Y_{\geq u} = \bigcup_{w \geq u} Y_w^\circ$.

COROLLARY 4.3. *For $u, v \in W$, $(u \leq v)$ and any $x_u \in Y_u^\circ$, we have the diffeomorphism of stratified spaces,*

$$Y_{[u,v]} \cong Y_u^\circ \times (\pi_u^{-1}(x_u) \cap Y_{[u,v]}).$$

In particular, $Y_{\geq u} \cong Y_u^\circ \times \pi_u^{-1}(x_u)$.

Proof of Theorem 1.2. Corollary 4.3 shows that the link of Y_u° in $Y_{[u,v]}$ is well-defined (up to a stratified diffeomorphism); it is explicitly given by

$$\mathrm{Lk}(u, v) = (\pi_u^{-1}(x_u) \cap Y_{[u,v]}) \cap S_\varepsilon(x_u),$$

where x_u is an arbitrary point on Y_u° and $S_\varepsilon(x_u)$ is a small sphere centered at x_u. The first two statements of Theorem 1.2 follow right away. The equalities (1.1) follow from the analogous property for the Bruhat stratification of Y (cf. Proposition 1.1), combined with Corollary 4.3. □

5. Proofs of Theorems 1.4 and 1.7

Recall that the elements $x_i(t)$ are defined by (2.1). For the type A_{n-1}, $x_i(t)$ is the $n \times n$ matrix that differs from the identity matrix in a single entry (equal to t) located in row i and column $i+1$.

DEFINITION 5.1. We define the regular map $\mathrm{str}\colon N \to \mathbb{C}$ by the conditions $\mathrm{str}(x_i(t)) = t$ and $\mathrm{str}(xy) = \mathrm{str}(x) + \mathrm{str}(y)$. In particular, in the case of type A, we have $\mathrm{str}(x) = \sum_i x_{i,i+1}$, the sum of the matrix elements immediately above the main diagonal.

DEFINITION 5.2. For $\tau > 0$, let $d(\tau) \in H$ be uniquely defined by the conditions $(d(\tau))^{\alpha_i} = \tau$ for all simple roots α_i. Then $d(\tau) x_i(a) d(\tau)^{-1} = x_i(\tau a)$ for any i. For the type A_{n-1},

$$d(\tau) = \tau^{-(n-1)/2} \begin{bmatrix} \tau^{n-1} & 0 & \cdots & 0 \\ 0 & \tau^{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{5.1}$$

and the automorphism $x \mapsto d(\tau)xd(\tau)^{-1}$ of the group N multiplies each matrix entry x_{ij} of x by τ^{j-i}.

Note that the automorphism $x \mapsto d(\tau)xd(\tau)^{-1}$ preserves the cells N^w and the subgroups $N(w)$; it also preserves total nonnegativity.

DEFINITION 5.3. For $u \in W$ and $x_u \in Y_u^\circ$, we define the vector field ψ on $\pi_u^{-1}(x_u)$ by

$$\psi(x) = \frac{d}{d\tau}(\rho_{x_u}(d(\tau)xd(\tau)^{-1}))\bigg|_{\tau=1} \tag{5.2}$$

(recall that ρ_{x_u} is defined by (4.1)).

LEMMA 5.4 (Type A only). *The vector field ψ vanishes nowhere on $\pi_u^{-1}(x_u)$ except at the point x_u. The directional derivative $\nabla_\psi \operatorname{str}(x)$ is positive at every point $x \neq x_u$.*

The proof of this lemma is given in Section 6.

Proof of Theorem 1.7. In view of Corollary 4.3, it remains to show that the fiber $\pi_u^{-1}(x_u) \cap Y_{[u,v]}$ has the structure of the cone over the link $\mathrm{Lk}(u, v)$.

The vector field ψ can be extended (by the same formula (5.2), with $x_u = \pi_u(x)$) to the open subset $N \cap G_0 u$ of N. Furthermore, $\psi(x)$ is given by rational functions in the affine coordinates of x; therefore, the theorem of uniqueness and existence of solutions applies to this extension of ψ (and hence to ψ itself). Since ψ is tangent to each stratum of the preimage $\pi^{-1}(x_u)$ (all these strata are smooth by part (1) of Theorem 4.2), it follows that every trajectory of ψ is contained in a single stratum.

The intersection $Y \cap \{\operatorname{str} \le c\}$ is compact for any $c > 0$. Lemma 5.4 then implies that, for every $x_0 \in \pi^{-1}(x_u)$, the solution $x(t)$ of the Cauchy problem $\dot{x} = \psi(x)$ and $x(0) = x_0$ with $t < 0$ exists for $t \in (-\infty, 0]$. (Otherwise, the trajectory $T_- = \{x(t) : t \le 0\}$ would hit the boundary of the stratum containing x_0.) The trajectory T_- must have limit points; let $x_{\lim}$ be one of them. The function $s\colon t \mapsto \operatorname{str}(x(t))$, $t < 0$, is increasing (by Lemma 5.4) and bounded from below. Therefore $\lim_{t\to-\infty} \dot{s}(t) = 0$, implying that $\nabla_\psi \operatorname{str}(x_{\lim}) = 0$. By Lemma 5.4, this means that $x_{\lim} = x_u$. Thus every trajectory of ψ originates at the point x_u (at $t = -\infty$). A similar argument shows that $\lim_{t\to t^+} \operatorname{str}(x(t)) = +\infty$, where t^+ denotes the upper limit of the maximal domain of definition of $x(t)$ (so $t^+ \in [0, \infty]$). We conclude that the function str increases from $\operatorname{str}(x_u)$ to ∞ along each trajectory of ψ, except for the trajectory $x(t) = x_u$. Thus every nontrivial trajectory $T \subset Y_{[u,v]}$ intersects the set

$$L_\varepsilon(u, v) = L_{\varepsilon,x_u}(u, v) = \pi_u^{-1}(x_u) \cap Y_{[u,v]} \cap \{x : \operatorname{str}(x) = \operatorname{str}(x_u) + \varepsilon\} \tag{5.3}$$

at exactly one point; see Figure 3. Therefore, $\pi^{-1}(x_u) \cap Y_{[u,v]}$ is diffeomorphic to the cone $\mathrm{Cone}(L_\varepsilon(u,v))$. (In particular, $\pi^{-1}(x_u) \cong \mathrm{Cone}(L_\varepsilon(u,w_o))$.) This implies that $L_\varepsilon(u,v)$ is isomorphic (as a stratified space) to the link $\mathrm{Lk}(u,v)$. Theorem 1.7 is proved. □

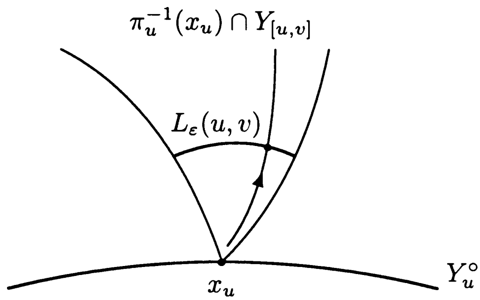

Figure 3 Embedding of the link into $\pi^{-1}(x_u) \cap Y_{[u,v]}$

Proof of Theorem 1.4. For $x \in \pi_u^{-1}(x_u) \cap Y_{[u,v]}$, let $\lambda_{u,v}(x)$ denote the unique point of intersection of the link $L_\varepsilon(u,v)$ with the trajectory of ψ that passes through x (see the sentence containing (5.3)).

Fix $z \in Y_{\geq v}$, and define $R_{u,v}\colon L_\varepsilon(u,v) \times [0,1] \to L_\varepsilon(u,v)$ by

$$R_{u,v}(x,\tau) = \lambda_{u,v}(\rho_{x_u}(\pi_v(d(\tau)zd(\tau)^{-1}d(1-\tau)xd(1-\tau)^{-1}))). \tag{5.4}$$

(Note that $d(\tau)zd(\tau)^{-1} \in Y_{\geq v}$ and therefore $d(\tau)zd(\tau)^{-1}d(1-\tau)xd(1-\tau)^{-1} \in Y_{\geq v}$, by Lemma 2.16; hence the right-hand side of (5.4) is well-defined.)

Let us show that the map $R_{u,v}$ is a deformation retraction of $L_\varepsilon(u,v)$ into a point. First, $R_{u,v}$ is continuous with respect to τ. Second, $\lim_{\tau\to 0} d(\tau)xd(\tau)^{-1} = 1$ and $\lim_{\tau\to 1} d(\tau)zd(\tau)^{-1} = z$. Therefore $R_{u,v}(x,0) = x$. On the other hand, $R_{u,v}(x,1) = \lambda_{u,v}(\rho_{x_u}(\pi_v(z)))$, a point independent of x.

It remains only to observe that $\overline{S_{u,v,w}} \cong \overline{S_{u,w,w}} \cong L_\varepsilon(u,w)$. □

6. Proof of Lemma 5.4

Our first goal is an explicit formula for the vector field ψ.

Throughout this section, we use the following notation. For $x \in Y_{\geq u}$, denote

$$\begin{aligned} x_u &= \pi_u(x) = [u[u^{-1}[xu^{-1}]_+u]_-]_+, \\ A &= u^{-1}[xu^{-1}]_+u, \\ y &= [A]_-, \\ x^u &= [A]_+ = x_u^{-1}x. \end{aligned} \tag{6.1}$$

Thus $x = x_u x^u$, $x_u = [uy]_+$ (agreeing with Lemma 2.3), and $A = yx^u$.

Let us fix an arbitrary totally nonnegative element $d \in H$. We will need some basic properties of the cell-preserving automorphism $x \mapsto dxd^{-1}$.

LEMMA 6.1. *The automorphism $x \mapsto dxd^{-1}$ of $Y_{\geq u}$ commutes with the maps $x \mapsto x_u$ and $x \mapsto x^u$:*

$$(dxd^{-1})_u = dx_u d^{-1}, \qquad (dxd^{-1})^u = dx^u d^{-1}.$$

Proof. In view of Lemma 3.2, the statement follows from the factorization $dxd^{-1} = dx_u d^{-1} \cdot dx^u d^{-1}$, where the two factors on the right belong to Y_u° and $N(u)$, respectively. □

LEMMA 6.2. *The automorphism $x_u \mapsto dx_u d^{-1}$ of Y_u° commutes with the map $x_u \mapsto y$ (cf. Lemma 2.3). In other words,*

$$dx_u d^{-1} = [u \cdot dyd^{-1}]_+.$$

The unique element $n_1 \in N_-(u)$ such that $x_u = [dx_u d^{-1} n_1]_+$ (cf. Lemma 2.6) is given by $n_1 = dy^{-1}d^{-1}y$.

Proof. First part: $dx_u d^{-1} = d[uy]_+ d^{-1} = [duyd^{-1}]_+ = [udyd^{-1}]_+$. The second part is then a special case of Lemma 2.6. □

LEMMA 6.3. $\rho_{x_u}(dxd^{-1}) = x\big([dA^{-1}d^{-1}A]_+\big)^{-1}$.

Proof. Applying (4.1) to $\tilde{x} = dxd^{-1}$ and using Lemmas 6.1 and 6.2 along with (6.1), we obtain

$$\begin{aligned}\rho_{x_u}(dxd^{-1}) &= [dxd^{-1}[d(x^u)^{-1}d^{-1}dy^{-1}d^{-1}y]_-]_+ = [dxd^{-1}[dA^{-1}d^{-1}A]_-]_+ \\ &= [dxd^{-1} \cdot dA^{-1}d^{-1}A]_+\big([dA^{-1}d^{-1}A]_+\big)^{-1} \\ &= [xA^{-1}d^{-1}A]_+\big([dA^{-1}d^{-1}A]_+\big)^{-1}.\end{aligned}$$

It remains to show that $[xA^{-1}d^{-1}A]_+ = x$. This is done as follows:

$$\begin{aligned}[xA^{-1}d^{-1}A]_+ &= [x_u y^{-1}d^{-1}A]_+ = [[uy]_+ y^{-1}d^{-1}A]_+ = [uyy^{-1}d^{-1}A]_+ \\ &= [ud^{-1}yx^u]_+ = [ud^{-1}y]_+ x^u = [uy]_+ x^u = x_u x^u = x. \qquad \square\end{aligned}$$

Let $\mathfrak{g}$, $\mathfrak{n}$, and $\mathfrak{b}_-$ denote the Lie algebras of groups G, N, and B_-, respectively. Let $\pi_\mathfrak{n}$ denote the projection $\mathfrak{g} \to \mathfrak{n}$ along $\mathfrak{b}_-$. (In the case $\mathfrak{g} = \mathfrak{sl}_n$, $\pi_\mathfrak{n}$ replaces all lower triangular entries of a traceless matrix by zeros.)

LEMMA 6.4. *Let $f(\tau) = f_-(\tau)f_+(\tau)$, where $f_-(\tau) \in B_-$ and $f_+(\tau) \in N$ for all $\tau > 0$. Assume that $f(1) = 1$. Then $f'_+(1) = \pi_\mathfrak{n} f'(1)$.*

Proof. The equality $f(1) = 1$ implies $f_-(1) = f_+(1) = 1$. Then $f'(1) = f'_-(1)f_+(1) + f_-(1)f'_+(1) = f'_-(1) + f'_+(1)$. Since $f'_-(1) \in \mathfrak{b}_-$ and $f'_+(1) \in \mathfrak{n}$, we are done. □

PROPOSITION 6.5. $\psi(x) = x\pi_\mathfrak{n}(A^{-1}d'(1)A)$.

Proof. By Lemma 6.3, $\rho_{x_u}(d(\tau)xd(\tau)^{-1}) = x\big([d(\tau)A^{-1}d(\tau)^{-1}A]_+\big)^{-1}$. Hence

$$\psi(x) = \frac{d}{d\tau}\big(\rho_{x_u}(d(\tau)xd(\tau)^{-1})\big)\Big|_{\tau=1} = x\frac{d}{d\tau}\big([d(\tau)A^{-1}d(\tau)^{-1}A]_+\big)^{-1}\Big|_{\tau=1}.$$

Applying Lemma 6.4 and observing that $d(1)=1$, we obtain:

$$\begin{aligned}\frac{d}{d\tau}\big([d(\tau)A^{-1}d(\tau)^{-1}A]_+\big)^{-1}\Big|_{\tau=1} &= -\frac{d}{d\tau}[d(\tau)A^{-1}d(\tau)^{-1}A]_+\Big|_{\tau=1}\\ &= -\pi_{\mathfrak n}\frac{d}{d\tau}(d(\tau)A^{-1}d(\tau)^{-1}A)\Big|_{\tau=1}\\ &= -\pi_{\mathfrak n}(d'(1)-A^{-1}d'(1)A)=\pi_{\mathfrak n}(A^{-1}d'(1)A),\end{aligned}$$

implying the claim. □

In the rest of this section, we consider only the case of the type A_{n-1}. Thus W is the symmetric group $\mathcal{S}_n$. We treat the elements of W as bijective maps $\{1,\dots,n\}\to\{1,\dots,n\}$ and choose permutation matrices as their representatives in $G=\mathrm{GL}(n)$.

LEMMA 6.6.

(1) *If* $1\le i\le j\le n$ *and* $u(j)\le u(i)\le u(j+1)$, *then* $(A^{-1})_{j,i}\cdot A_{i,j+1}\ge 0$.
(2) *If* $1\le j<i\le n$ *and* $u(j)\le u(i)\le u(j+1)$, *then* $(A^{-1})_{j,i}\cdot A_{i,j+1}\le 0$.
(3) *Otherwise,* $(A^{-1})_{j,i}\cdot A_{i,j+1}=0$.

Proof. For a matrix $a\in u^{-1}Nu$, the matrix element a_{ij} vanishes unless $u(i)\le u(j)$. It follows that $(A^{-1})_{j,i}\cdot A_{i,j+1}=0$ unless $u(j)\le u(i)\le u(j+1)$, proving part (3) of the lemma. In order to prove parts (1) and (2), set $s=u(i)$, $p=u(j)$, and $q=u(j+1)$; thus $p\le s\le q$.

In what follows, we denote by $z^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ the determinant of the submatrix of a matrix z formed by the rows $i_1,\dots,i_r$ and the columns $j_1,\dots,j_r$ (in that order). Using the definition of A and that $x\in N$, we obtain

$$\begin{aligned}A_{i,j+1}&=([xu^{-1}]_+)_{s,q}=([xu^{-1}]_+)^{1,\dots,s}_{1,\dots,s-1,q}\\ &=\frac{(xu^{-1})^{1,\dots,s}_{1,\dots,s-1,q}}{(xu^{-1})^{1,\dots,s}_{1,\dots,s}}=\frac{x^{1,\dots,s}_{u^{-1}(1),\dots,u^{-1}(s-1),u^{-1}(q)}}{x^{1,\dots,s}_{u^{-1}(1),\dots,u^{-1}(s)}}.\end{aligned}$$

Because x is totally nonnegative, the sign of $A_{i,j+1}$ is either zero or $(-1)^{n_1+n_2}$, where

$$\begin{aligned}n_1&=\mathrm{card}\{l:l\le s-1,\ u^{-1}(l)>i\},\\ n_2&=\mathrm{card}\{l:l\le s-1,\ u^{-1}(l)>j+1\}.\end{aligned}$$

The sign of $(A^{-1})_{j,i}$ can be determined in a similar fashion. Using the notation $\hat{i}$ to indicate that the index i is being removed, we obtain:

$$\begin{aligned}(A^{-1})_{j,i}&=(([xu^{-1}]_+)^{-1})_{p,s}=(-1)^{(s-p)}([xu^{-1}]_+)^{1,\dots,\hat{s},\dots,n}_{1,\dots,\hat{p},\dots,n}\\ &=(-1)^{(s-p)}([xu^{-1}]_+)^{1,\dots,s-1}_{1,\dots,\hat{p},\dots,s}=(-1)^{(s-p)}\frac{(xu^{-1})^{1,\dots,s-1}_{1,\dots,\hat{p},\dots,s}}{(xu^{-1})^{1,\dots,s-1}_{1,\dots,s-1}}\\ &=(-1)^{(s-p)}\frac{x^{1,\dots,s-1}_{u^{-1}(1),\dots,\widehat{u^{-1}(p)},\dots,u^{-1}(s)}}{x^{1,\dots,s-1}_{u^{-1}(1),\dots,u^{-1}(s-1)}}.\end{aligned}$$

Hence the sign of $(A^{-1})_{j,i}$ is either zero or $(-1)^{s-p+n_1+n_3+n_4}$, where n_1 is the same as before and

$$n_3 = \operatorname{card}\{l : l \le p-1,\ u^{-1}(l) > j\},$$
$$n_4 = \operatorname{card}\{l : p < l \le s,\ u^{-1}(l) < j\}.$$

Since

$$n_3 + s - p - n_4 = \operatorname{card}\{l : l \le s,\ u^{-1}(l) > j\},$$

we conclude that the sign of $A_{i,j+1}(A^{-1})_{j,i}$ is either zero or $(-1)^{\operatorname{card}\{l:l=s,\, u^{-1}(l)>j\}}$, matching the claim of the lemma. $\square$

Let us extend the notation $\operatorname{str}(x) = \sum_i x_{i,i+1}$ to arbitrary matrices x.

Lemma 6.7. *Let* $\nu = \operatorname{diag}(n, n-1, \ldots, 1)$. *Then* $\operatorname{str}(A^{-1}\nu A) \ge 0$.

Proof. For $k = 1, \ldots, n$, let $\nu_k = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$ denote the diagonal matrix whose first k diagonal entries are equal to 1, and all other entries vanish. The equality $A^{-1}A = 1$ implies

$$\sum_{i=1}^{j} (A^{-1})_{j,i} \cdot A_{i,j+1} + \sum_{i=j+1}^{n} (A^{-1})_{j,i} \cdot A_{i,j+1} = 0,$$

where (by Lemma 6.6) all terms in the first sum are nonnegative while all terms in the second sum are nonpositive. Then $(A^{-1}\nu_k A)_{j,j+1} = \sum_{i=1}^{k} (A^{-1})_{j,i} \cdot A_{i,j+1} \ge 0$, implying $\operatorname{str}(A^{-1}\nu_k A) \ge 0$. Since $\nu = \sum_{k=1}^{n} \nu_k$, the lemma follows. $\square$

We are now prepared to complete the proof of Lemma 5.4. First, let us note that $d(\tau)x_u d(\tau)^{-1} \in Y_u^\circ$; hence $\rho_{x_u}(d(\tau)x_u d(\tau)^{-1}) \in Y_u^\circ \cap \pi_u^{-1}(x_u) = \{x_u\}$. Thus $\psi(x_u) = 0$.

Let $x \in \pi^{-1}(x_u)$, $x \ne x_u$. Since $d'(1) = \nu + \lambda$ for some scalar matrix λ, Proposition 6.5 yields $\psi(x) = x\pi_{\mathfrak{n}}(A^{-1}\nu A)$. Therefore,

$$\operatorname{str}(\psi(x)) = \operatorname{str}(x) + \operatorname{str}(\pi_{\mathfrak{n}}(A^{-1}\nu A)) = \operatorname{str}(x) + \operatorname{str}(A^{-1}\nu A)$$

(here we identify the tangent vector $\psi(x)$ with the corresponding traceless matrix). Since $\operatorname{str}(x) > 0$, Lemma 6.7 implies that $\operatorname{str}(\psi(x)) > 0$; in particular, $\psi(x) \ne 0$. Then $\operatorname{str}(x + \psi(x)d\tau) = \operatorname{str}(x) + \operatorname{str}(\psi(x))d\tau$; hence $\nabla_\psi \operatorname{str}(x) = \operatorname{str}(\psi(x)) > 0$, as desired. $\square$

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. 122 (1996), 49–149.

[2] A. Berenstein and A. Zelevinsky, *Total positivity in Schubert varieties*, Comment. Math. Helv. 72 (1997), 128–166.

[3] ———, *Tensor product multiplicities, canonical bases, and totally positive varieties*, Invent. Math. (to appear).

[4] A. Björner, *Posets, regular CW complexes and Bruhat order*, European J. Combin. 5 (1984), 7–16.

[5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented matroids,* Encyclopedia Math. Appl., 46, Cambridge Univ. Press, Cambridge, U.K., 1993.
[6] V. V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells,* Invent. Math. 79 (1985), 499–511.
[7] A. Dold, *Lectures on algebraic topology,* Springer-Verlag, Berlin, 1995.
[8] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern geometry—Methods and applications. Part III. Introduction to homology theory,* Grad. Texts in Math., 124, Springer-Verlag, New York, 1990.
[9] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity,* J. Amer. Math. Soc. 12 (1999), 335–380.
[10] ———, *Totally nonnegative and oscillatory elements in semisimple groups,* Proc. Amer. Math. Soc. (to appear).
[11] M. Goresky and R. MacPherson, *Stratified Morse theory,* Ergeb. Math. Grenzgeb. (3), 14, Springer-Verlag, Berlin, 1988.
[12] V. Guillemin and A. Pollack, *Differential topology,* Prentice-Hall, Englewood Cliffs, NJ, 1974.
[13] J. E. Humphreys, *Reflection groups and Coxeter groups,* Cambridge Stud. Adv. Math., 29, Cambridge Univ. Press, Cambridge, U.K., 1994.
[14] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras,* Invent. Math. 53 (1979), 165–184.
[15] ———, *Schubert varieties and Poincaré duality,* Proc. Sympos. Pure Math., 36, pp. 185–203, Amer. Math. Soc., Providence, RI, 1980.
[16] G. Lusztig, *Total positivity in reductive groups,* Progr. Math., 123, pp. 531–568, Birkhäuser, Boston, 1994.
[17] ———, *Introduction to total positivity,* de Gruyter Exp. Math., 26, pp. 133–145, de Gruyter, Berlin, 1998.
[18] R. P. Stanley, *A survey of Eulerian posets,* NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 440, pp. 301–333, Kluwer, Dordrecht, 1994.
[19] D.-N. Verma, *Möbius inversion for the Bruhat order on a Weyl group,* Ann. Sci. École Norm. Sup. (4) 4 (1971), 393–398.
[20] ———, *A strengthening of the exchange property of Coxeter groups,* preprint, 1972.

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