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Poincaré Duality and Equivariant (Co)homology

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To Bill Fulton for his 60th birthday

Introduction

Let X be a compact complex algebraic variety of pure dimension n whose Betti numbers vanish in all odd degrees. Then the cohomology ring $H^*(X)$ with complex coefficients is a commutative, positively graded algebra, of finite dimension as a complex vector space. It is well known that the dualizing module (in the sense of commutative algebra, see e.g. [8]) of $H^*(X)$ is the homology $H_*(X)$; moreover, $H^*(X)$ is Gorenstein if and only if X satisfies Poincaré duality. This holds if X is smooth or, more generally, rationally smooth; that is, the local cohomology at any point is the same as the local cohomology of complex affine n-space (see [17] for other characterizations).

We shall generalize these observations to the richer setting of equivariant homology and cohomology, with applications to Coxeter groups. Assume that a d-dimensional torus T acts on X with isolated fixed points (examples include rationally smooth projective varieties where a complex reductive group acts with finitely many orbits, Schubert varieties, and varieties of complete flags fixed by a given linear transformation). Then the equivariant cohomology ring $H_T^*(X)$ with complex coefficients is positively graded, commutative and reduced; it is a free module of finite rank over the equivariant cohomology ring of the point. The latter is a polynomial ring in d variables. Thus, the ring $H_T^*(X)$ is Cohen–Macaulay. We show that several topological invariants of the T-variety X can be read off that ring.

Specifically, restriction to the T-fixed point set $H_T^*(X) \to H_T^*(X^T)$ is the normalization of $H_T^*(X)$. It follows that the complex affine algebraic variety $V(X)$ associated to $H_T^*(X)$ is a finite union of copies of the Lie algebra of T, glued along rational hyperplanes (Proposition 2). The dualizing module of $H_T^*(X)$ turns out to be the equivariant Borel–Moore homology $H_*^T(X)$ (Proposition 1); it admits a more concrete description in terms of regular differential forms on $V(X)$ (Proposition 3). On the other hand, the conductor of $H_T^*(X)$ in its normalization $H_T^*(X^T)$ is closely related to equivariant cohomology with support in X^T, and also to equivariant multiplicities; the latter are uniquely determined by the abstract ring $H_T^*(X)$, up to a common scalar multiple (Section 3).

These considerations yield the following linear inequalities for the Betti numbers of a variety X as above, if all equivariant multiplicities are nonzero:

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$$b_q(X) + b_{q-1}(X) + \cdots + b_0(X) \\ \leq b_{2n-q}(X) + b_{2n-q+1}(X) + \cdots + b_{2n}(X) \qquad (1)$$

for $q = 0, 1, \ldots, n-1$ (Theorem 2; recall that the Betti numbers are assumed to vanish in all odd degrees). It follows easily that

$$2b_2(X) + 4b_4(X) + \cdots + 2nb_{2n}(X) \geq n\chi(X), \qquad (2)$$

where $\chi(X)$ denotes the Euler characteristic. Moreover, equality in (2) is equivalent to $b_q(X) = b_{2n-q}(X)$ for $q = 0, 1, \ldots, n-1$ and, in turn, to Poincaré duality for X (Theorem 1).

The assumptions of Theorem 2 are satisfied if X is the disjoint union of locally closed T-stable subvarieties ("cells") that are isomorphic to complex affine spaces. Moreover, the ratio

$$a(X) = \frac{b_2(X) + 2b_4(X) + \cdots + nb_{2n}(X)}{b_0(X) + b_2(X) + \cdots + b_{2n}(X)}$$

is just the average dimension of cells. In this setting, (2) translates into the inequality

$$a(X) \geq \frac{1}{2} \dim(X),$$

with equality if and only if X satisfies Poincaré duality.

The latter result was discovered by Carrell and Peterson (see [4]) for Schubert varieties in the flag variety of a Kac–Moody group (these have a natural decomposition into Schubert cells). Finding an explanation and generalization of this result in terms of equivariant cohomology was the main motivation for the present article.

When applied to Schubert varieties, the sharper inequalities (1) yield the following purely combinatorial statement on the repartition of lengths of elements in a Bruhat interval $[1, w]$ of a crystallographic Coxeter group W:

$$\#\{x \in [1, w],\ \ell(x) \leq q\} \leq \#\{x \in [1, w],\ \ell(x) \geq \ell(w) - q\}$$

for $1 \leq q < \frac{1}{2}\ell(w)$ (Corollary 2; it extends to arbitrary Coxeter groups).

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1. Equivariant Homology and Cohomology

Throughout this article, we consider a complex algebraic variety X of pure (complex) dimension n, endowed with an algebraic action of a torus $T \cong (\mathbb{C}^*)^d$ of dimension d. We denote by $\mathfrak{t} \cong \mathbb{C}^d$ the Lie algebra of T.

In this situation, we review the definitions and some properties of equivariant cohomology (see e.g. [12]) and of equivariant Borel–Moore homology (see [9, Chap. 19] for Borel–Moore homology and [6; 11] for its equivariant version); both will be considered with complex coefficients. For any positive integer m, consider the space

$$E_{T,m} = (\mathbb{C}^{m+1} - 0)^d,$$

where T acts by $(t_1, \dots, t_d) \cdot (v_1, \dots, v_d) = (t_1 v_1, \dots, t_d v_d)$. This action is free, and the quotient

$$p_m \colon E_{T,m} = (\mathbb{C}^{m+1} - 0)^d \to (\mathbb{P}^m)^d = B_{T,m}$$

is a principal T-bundle. The maps $p_m \colon E_{T,m} \to B_{T,m}$ define a direct system for the obvious inclusions $B_{T,m} \subset B_{T,m+1}$; the direct limit $p \colon E_T \to B_T$ is a universal principal T-bundle, with the $E_{T,m}$ as algebraic approximations.

For a T-variety X, let $X \times^T E_T$ be the quotient of $X \times E_T$ by the diagonal T-action; then we have a map

$$p_X \colon X \times^T E_T \to E_T/T = B_T,$$

a fibration with fiber X. The cohomology ring of $X \times^T E_T$ is the equivariant cohomology ring of X, denoted by $H_T^*(X)$. It is a graded algebra over the equivariant cohomology ring of the point, $H_T^*(\mathrm{pt}) = H^*(B_T)$.

Each character χ of T defines a line bundle on B_T, whence an element $c(\chi)$ of $H^2(B_T)$. The map $\chi \mapsto c(\chi)$ extends to an isomorphism of the symmetric algebra over $\mathbb{C}$ of the character group of T, onto $H^*(B_T)$; this isomorphism doubles degrees. Assigning to each character its differential at the identity element, we identify the character group with a discrete subgroup of $\mathfrak{t}^*$. This identifies $H^*(BT)$ to the ring of polynomial functions $\mathbb{C}[\mathfrak{t}]$, where the nonzero linear forms have degree 2. Restriction to a fiber of p_X defines a map $H_T^*(X) \to H^*(X)$ that vanishes on $\mathfrak{t}^* H_T^*(X)$.

One may check that, for a fixed degree q, we have $H_T^q(X) = H^q(X \times^T E_{T,m})$ when $m \geq q/2$. The qth equivariant Borel–Moore homology group is defined similarly, as the Borel–Moore homology group $H_{q+2md}(X \times^T E_{T,m})$ for $m \geq n - q/2$. This group is independent of m; it will be denoted by $H_q^T(X)$. (Specifically, for $m' \geq m \geq n - q/2$, the Gysin map $H_{q+2m'd}(X \times^T E_{T,m'}) \to H_{q+2md}(X \times^T E_{T,m})$ is an isomorphism.) The space

$$H_*^T(X) = \bigoplus_{q \in \mathbb{Z}} H_q^T(X)$$

is a graded $H_T^*(X)$-module via the cap product

$$H_T^p(X) \times H_q^T(X) \to H_{q-p}^T(X), \quad (\alpha, \beta) \mapsto \alpha \cap \beta.$$

In particular, $H_*^T(X)$ is a graded $\mathbb{C}[\mathfrak{t}]$-module, where $\mathfrak{t}^*$ acts with degree -2.

Any closed T-stable subvariety Y of X defines a class $[Y]_T \in H_{2\dim(Y)}^T(X)$. This yields the equivariant Poincaré duality map

$$H_T^q(X) \to H_{2n-q}^T(X), \quad \alpha \mapsto \alpha \cap [X]_T.$$

This map is an isomorphism if X is rationally smooth. In particular, $H_*^T(\mathrm{pt})$ is isomorphic to $\mathbb{C}[\mathfrak{t}]$ (with the opposite grading.) The Gysin maps

$$H_{q+2md}(X \times_T E_{T,m}) \to H_q(X)$$

fit into a map $H_*^T(X) \to H_*(X)$ that vanishes on $\mathfrak{t}^* H_*^T(X)$ and sends each $[Y]_T$ to $[Y]$.

The following version of the localization theorem [10; 12] will be our main tool.

LEMMA 1. *Let T' be a subtorus of T, and let $i\colon X^{T'} \to X$ be the inclusion of the fixed point set. Then both $\mathbb{C}[\mathfrak{t}]$-linear maps*

$$i^*\colon H_T^*(X) \to H_T^*(X^{T'}) \quad \text{and} \quad i_*\colon H_*^T(X^{T'}) \to H_*^T(X)$$

become isomorphisms after inverting finitely many characters of T that restrict nontrivially to T'.

In particular, let χ be an indivisible character of T. Then $\ker(\chi)$ is a subtorus of codimension 1; the maps $H_T^*(X) \to H_T^*(X^{\ker(\chi)})$ and $H_*^T(X^{\ker(\chi)}) \to H_*^T(X)$ are isomorphisms at the generic point of the hyperplane $(\chi = 0)$ of $\mathfrak{t}$. The union of the subsets $X^{\ker(\chi)}$ is the union of all T-orbits of dimension ≤ 1.

The indivisible character χ will be called *singular* if $X^{\ker(\chi)} \ne X^T$. Note that the kernels and cokernels of the maps $H_T^*(X) \to H_T^*(X^T)$ and $H_*^T(X^T) \to H_*^T(X)$ have support in the union of singular hyperplanes and of a subset of codimension at least 2 in $\mathfrak{t}$.

For compact X, the map $p_X\colon X \times^T E_T \to B_T$ is proper and yields a $\mathbb{C}[\mathfrak{t}]$-linear map $p_{X*}\colon H_*^T(X) \to H_*^T(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}]$. In turn, this defines a $\mathbb{C}[\mathfrak{t}]$-linear map

$$\int_X \colon H_*^T(X) \to \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]), \quad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha)).$$

This map is the equivariant version of the usual map from homology to the dual of cohomology. The latter is an isomorphism, but $\int_X$ may be trivial. In fact, it follows from the localization theorem that $\int_X$ is nonzero if and only if X contains T-fixed points.

We shall see that $\int_X$ is an isomorphism if X is *equivariantly formal* in the sense of [10], that is, if the cohomology spectral sequence associated with the fibration $p_X\colon X \times^T E_T \to B_T$ collapses. Equivalently, the $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$ is free and the map $H_T^*(X)/\mathfrak{t}^* H_T^*(X) \to H^*(X)$ is an isomorphism. First of all, we record the following well-known lemma.

LEMMA 2. *Consider the following conditions for a T-variety X.*

(i) *X is equivariantly formal.*

(ii) *The $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$ is free.*

(iii) *The Betti numbers of X vanish in odd degrees.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

If X^T is finite, then all these conditions are equivalent; moreover, they hold for rationally smooth X.

If X is equivariantly formal as a T-variety, then it is as a T'-variety for any subtorus T' of T, and the natural map $\mathbb{C}[\mathfrak{t}'] \otimes_{\mathbb{C}[\mathfrak{t}]} H_T^*(X) \to H_{T'}^*(X)$ is an isomorphism. Moreover, $X^{T'}$ is equivariantly formal as a T-variety and as a (T/T')-variety, and the $\mathbb{C}[\mathfrak{t}]$-algebra $H_T^*(X^{T'})$ is isomorphic to $\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}/\mathfrak{t}']} H_{T/T'}^*(X^{T'})$.

Proof. (i) $\Rightarrow$ (ii) $\Leftarrow$ (iii) are obvious and (ii) $\Rightarrow$ (i) follows, for example, from the Eilenberg–Moore spectral sequence [12, p. 38].

Assuming that X^T is finite, we check that (ii) $\Rightarrow$ (iii). Recall that the restriction $H_T^*(X) \to H_T^*(X^T) = \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} H^*(X^T)$ becomes an isomorphism after inverting finitely many nonzero elements of $\mathfrak{t}^*$. Since the $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$ is free and X^T is finite, it follows that $H_T^q(X) = 0$ for all odd q, whence (iii). If X^T is finite and X is rationally smooth, then X is equivariantly formal by [10, Thm. 14.1].

If the T-variety X is equivariantly formal, then the Eilenberg–Moore spectral sequence yields the isomorphism $\mathbb{C}[\mathfrak{t}'] \otimes_{\mathbb{C}[\mathfrak{t}]} H_T^*(X) \cong H_{T'}^*(X)$; it follows that the T'-variety X is equivariantly formal. Choose another subtorus T'' such that the product map $T' \times T'' \to T$ is an isomorphism. Then

$$X^{T'} \times_T E_T \cong B_{T'} \times (X^{T'} \times_{T''} E_{T''}).$$

Thus, with obvious notation, $H_T^*(X^{T'}) \cong \mathbb{C}[\mathfrak{t}'] \otimes_{\mathbb{C}} H_{T''}^*(X^{T'})$. This implies the latter isomorphism of the lemma. The $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X^{T'})$ becomes free after inverting finitely many elements of $\mathfrak{t}^*$ that restrict nontrivially to $\mathfrak{t}'$; that is, this module is locally free in a neighborhood of $\mathfrak{t}'$ in $\mathfrak{t}$. It follows that the $\mathbb{C}[\mathfrak{t}'']$-module $H_{T''}^*(X^{T'})$ is locally free at 0 and thus free because it is positively graded. $\square$

Proposition 1. *Let X be a compact, equivariantly formal T-variety. Then the $\mathbb{C}[\mathfrak{t}]$-module $H_*^T(X)$ is free, and the map $H_*^T(X)/\mathfrak{t}^* H_*^T(X) \to H_*(X)$ is an isomorphism. Moreover, $\int_X : H_*^T(X) \to \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}])$ is an isomorphism as well.*

Proof. The main ingredient is the following lemma.

Lemma 3. *Let E be a compact topological space and let $p: E \to B$ be a fibration, where B is an orientable topological manifold with orientation class $u_B \in H^r(B)$. Let $i: F \to E$ be the inclusion of a fiber of p, and let $i^!: H_*(E) \to H_*(F)(-r)$ be the corresponding Gysin map. Then, for any $\alpha \in H^q(E)$ and $\beta \in H_{q+r}(E)$, we have*

$$\langle i^*\alpha, i^!\beta\rangle_F = \langle u_B, p_*(\alpha \cap \beta)\rangle_B,$$

where $\langle \cdot, \cdot \rangle_F$ denotes the pairing between $H^*(F)$ and $H_*(F)$.

If moreover $i^*: H^*(E) \to H^*(F)$ is surjective, then so is $i^!: H_*(E) \to H_*(F)(-r)$.

Proof. For the first assertion, note that

$$\begin{aligned}\langle i^*\alpha, i^!\beta\rangle_F &= \langle \alpha, i_* i^!\beta\rangle_E = \langle \alpha, p^*u_B \cap \beta\rangle_E \\ &= \langle p^*u_B, \alpha \cap \beta\rangle_E = \langle u_B, p_*(\alpha \cap \beta)\rangle_B.\end{aligned}$$

Assume that i^* is surjective but that $i^!$ is not. Then there exists a homogeneous nonzero $\gamma \in H^*(F)$ such that $\langle \gamma, i^!\beta\rangle_F = 0$ for all $\beta \in H_*(E)$. Let $\alpha \in H^*(E)$ be a homogeneous element such that $i^*\alpha = \gamma$. Then we have

$$0 = \langle i^*\alpha, i^!\beta\rangle_F = \langle \alpha, p^*u_B \cap \beta\rangle_E = \langle \alpha \cup p^*u_B, \beta\rangle_E.$$

Thus, $\alpha \cup p^* u_B = 0$. Let M be a subspace of $H^*(E)$ such that the restriction $i^*\colon M \to H^*(F)$ is an isomorphism. Let $\alpha_1 = 1, \alpha_2, \dots, \alpha_N$ be a homogeneous basis of $H^*(B)$. By the Leray–Hirsch theorem, we can write $\alpha = \sum_{j=1}^N m_j \cup p^*\alpha_j$ with uniquely defined homogeneous $m_1, \dots, m_N$ in M. Thus, $\sum_{j=1}^N m_j \cup p^*(\alpha_j \cup u_B) = 0$; that is, $m_1 = 0$. Moreover, $i^* p^* \alpha_j = 0$ for $j \geq 2$, because the degrees of these β_j are at least 1. Then we have $i^*\alpha = \sum_{j=2}^N i^* m_j \cup i^* p^* \alpha_j = 0$, a contradiction. □

We return to the proof of Proposition 1. The first assertion is a consequence of Lemma 3 together with the Leray–Hirsch theorem, applied to the fibration $X \times_T E_{T,m} \to B_{T,m}$ for sufficiently large m.

For the remaining assertions, note that the $\mathbb{C}[\mathfrak{t}]$-module $\mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}])$ is free because X is equivariantly formal. By the graded Nakayama lemma, it suffices to check that the map

$$H_*^T(X) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]/\mathfrak{t}^*\mathbb{C}[\mathfrak{t}] \to \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]/\mathfrak{t}^*\mathbb{C}[\mathfrak{t}]$$

is an isomorphism. But

$$\mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]/\mathfrak{t}^*\mathbb{C}[\mathfrak{t}] \cong \mathrm{Hom}_{\mathbb{C}}(H_T^*(X)/\mathfrak{t}^* H_T^*(X), \mathbb{C})$$

is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(H^*(X), \mathbb{C})$, because X is equivariantly formal; and the map

$$H_*^T(X) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]/\mathfrak{t}^*\mathbb{C}[\mathfrak{t}] = H_*^T(X)/\mathfrak{t}^* H_*^T(X) \to H_*(X)$$

is also an isomorphism. Thus, it suffices to check that the diagram

$$\begin{tikzcd}
H_*^T(X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]) \arrow[d] \\
H_*(X) \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(H^*(X), \mathbb{C})
\end{tikzcd}$$

commutes. In a fixed degree q and for large m, this amounts to the commutativity of the diagram

$$\begin{tikzcd}
H_{q+2md}(X \times^T E_{T,m}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{C}}(H^q(X \times^T E_{T,m}), \mathbb{C}) \arrow[d] \\
H_q(X) \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(H^q(X), \mathbb{C}),
\end{tikzcd}$$

where the top horizontal map sends β to the map $(\alpha \mapsto \langle p_{X*}(\alpha \cap \beta), u_{B_{T,m}} \rangle)$. But this follows again from Lemma 3. □

2. Equivariant Homology and Regular Differential Forms

We assume from now on that X is a compact, equivariantly formal T-variety with isolated fixed points. By Lemma 1, it follows that $H^q(X) = 0 = H_T^q(X)$ for all odd q. Thus, the algebras $H^*(X)$ and $H_T^*(X)$ are commutative. We shall obtain geometric interpretations of $H_T^*(X)$ and $H_*^T(X)$.

Let r be the number of T-fixed points in X; then the algebra $H_T^*(X^T)$ identifies to $\mathbb{C}[\mathfrak{t}]^r = \mathbb{C}[\mathfrak{t}] \times \cdots \times \mathbb{C}[\mathfrak{t}]$ (r factors). By Lemmas 2 and 3, the $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$ is free of rank r and the inclusion $i\colon X^T \to X$ defines an injective $\mathbb{C}[\mathfrak{t}]$-algebra homomorphism

$$i^*\colon H_T^*(X) \to \mathbb{C}[\mathfrak{t}]^r, \quad \alpha \mapsto (\alpha_x)_{x\in X^T}$$

that becomes an isomorphism after inverting all singular characters. As a consequence, the algebra $H_T^*(X)$ is finitely generated, Cohen–Macaulay of dimension d, and reduced. Let $V(X)$ be the corresponding complex affine algebraic variety (defined over the rationals).

Proposition 2. *The map $i^*\colon H_T^*(X) \to \mathbb{C}[\mathfrak{t}]^r$ is the normalization. In other words, the normalization of $V(X)$ is a union of disjoint copies $\mathfrak{t}_x$ of $\mathfrak{t}$, indexed by the T-fixed points.*

Moreover, the set $V(X)$ is obtained as follows: for any singular character χ and for any T-fixed points x and y in the same connected component of $X^{\ker(\chi)}$, we identify the hyperplanes $(\chi = 0)$ in $\mathfrak{t}_x$ and $\mathfrak{t}_y$.

Proof. The algebra $\mathbb{C}[\mathfrak{t}]^r$ is integrally closed in its total ring of fractions; it is a finite module over $\mathbb{C}[\mathfrak{t}]$ and hence over $H_T^*(X)$. Moreover, $H_T^*(X)$ and $\mathbb{C}[\mathfrak{t}]^r$ have the same total ring of fractions, by the localization theorem. Thus, i^* is the normalization.

For the second assertion, consider first the case where X is connected and $T \cong \mathbb{C}^*$. Then the $\mathbb{C}[\mathfrak{t}]$-algebra structure of $H_T^*(X)$ yields a finite flat map $V(X) \to \mathbb{A}^1$. The fiber at 0 is the spectrum of $H^*(X)$, whereas the other fibers consist of r distinct points. Since X is connected, the set-theoretical fiber at 0 is a unique point. Thus, $V(X)$ is a union of r affine lines with the origins identified.

The general case follows by induction on d, using Lemmas 1 and 2. □

Note that this description of $V(X)$ as a set does not determine $H_T^*(X)$ uniquely. For example, if X is connected and $T \cong \mathbb{C}^*$, then the set $V(X)$ depends only on the number of fixed points.

We now turn to a description of $H_*^T(X)$ in terms of regular differential forms. These were defined in [18] for curves and in [16] for arbitrary schemes. This definition simplifies as follows in the present setting.

Define the space $\omega_{\mathfrak{t}}$ of regular differential forms on $\mathfrak{t}$ as the set of all polynomial differential forms of degree d on that affine space. Then $\omega_{\mathfrak{t}}$ is a free module of rank 1 over $\mathbb{C}[\mathfrak{t}]$; tensoring with its quotient field $\mathbb{C}(\mathfrak{t})$, we obtain the space of rational differential forms. Now a *regular differential form* on $V(X)$ is an r-tuple $(\omega_x)_{x\in X^T}$ of rational differential forms on $\mathfrak{t}$ such that the form $\sum_{x\in X^T} \alpha_x \omega_x$ is regular for all $\alpha \in H_T^*(X)$.

By the localization theorem, the latter condition is equivalent to the following: For any character χ of T and for any connected component Y of $X^{\ker(\chi)}$, the form $\sum_{x\in Y^T} \alpha_x \omega_x$ has no pole along the hyperplane $(\chi = 0)$. In particular, the poles of the ω_x are contained in the finite union of the singular hyperplanes.

The set of all regular differential forms on $V(X)$ is stable under multiplication by any element of $i^*H_T^*(X)$: it is a graded $H_T^*(X)$-module that we denote by $\omega_{V(X)}$.

PROPOSITION 3. *With notation as before, $\omega_{V(X)}$ is the dualizing module of the graded Cohen–Macaulay ring $H_T^*(X)$. Moreover, the graded $H_T^*(X)$-module $H_*^T(X)$ is isomorphic to $\omega_{V(X)}(2d)$ with the opposite grading.*

Under this isomorphism, the image of the class $[x]_T$ of any T-fixed point has value a generator of $\omega_{\mathfrak{t}}$ on $\mathfrak{t}_x$ and 0 on the other $\mathfrak{t}_y$. If X is irreducible, then the image of the fundamental class $[X]_T$ generates the space of homogeneous elements of minimal degree in $\omega_{V(X)}$.

Proof. We have

$$H_*^T(X) \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]) \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \omega_{\mathfrak{t}})(2d)$$

because $\omega_{\mathfrak{t}} \cong \mathbb{C}[\mathfrak{t}](-2d)$. Moreover, $\mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \omega_{\mathfrak{t}})$ is the dualizing module of $H_T^*(X)$ (see e.g. [8, Thm. 21.15]. And $\mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \omega_{\mathfrak{t}})$ is mapped injectively to

$$\mathrm{Hom}_{\mathbb{C}(\mathfrak{t})}(H_T^*(X) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t}), \omega_{\mathfrak{t}} \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t})) \cong \omega_{\mathfrak{t}}^r \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t}).$$

The image, by definition, is $\omega_{V(X)}$.

The assertion on the images of classes of T-fixed points is obvious. For the latter assertion, observe that $[X]_T$ is nonzero, since the same holds for $[X]$. Thus, $[X]_T$ generates the space of homogeneous elements of maximal degree in $H_*^T(X) \cong \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} H_*(X)$. □

Let Y be another compact, equivariantly formal T-space with isolated fixed points, and let $f: X \to Y$ be an equivariant morphism. Then f defines a ring homomorphism

$$f^*: H_T^*(Y) \to H_T^*(X)$$

together with a $H_T^*(X)$-linear map

$$f_*: H_*^T(X) \to H_*^T(Y).$$

This yields a finite morphism $V(X) \to V(Y)$, whence a trace map

$$\mathrm{Tr}: \omega_{V(X)} \to \omega_{V(Y)}.$$

By [8, Thm. 21.15], we can view $\omega_{V(X)}$ as $\mathrm{Hom}_{H_T^*(Y)}(H_T^*(X), \omega_{V(Y)})$; then the trace map becomes evaluation at 1.

PROPOSITION 4. *With notation as before,* Tr *identifies with f_*. Moreover, we have*

$$\mathrm{Tr}((\omega_x)_{x\in X^T}) = \Big(\sum_{x\in X^T,\, f(x)=y} \omega_x\Big)_{y\in Y^T}.$$

Proof. By Proposition 3, both assertions hold for the inclusion $i: X^T \to X$. Using functoriality of the trace map, we reduce to the case where X and Y are finite sets; then the statements are obvious. □

This description of $H_*^T(X)$ becomes much more precise if X contains only finitely many T-orbits of dimension 1; equivalently, the fixed point set of any subtorus of codimension 1 contains only finitely many T-orbits. By [10, Thm. 7.1], the image of $i^*\colon H_T^*(X) \to \mathbb{C}[\mathfrak{t}]^r$ consists then of all r-tuples $(f_x)_{x\in X^T}$ of polynomial functions on $\mathfrak{t}$ such that: $f_x - f_y$ is divisible by χ whenever x, y are fixed points in the closure of a 1-dimensional orbit where T acts through the character χ. Here we obtain the following dual statement.

COROLLARY 1. *Let X be a compact, equivariantly formal T-variety containing only finitely many orbits of dimension ≤ 1. Then $H_*^T(X)$ consists of all tuples $(\omega_x)_{x\in X^T}$ of rational differential forms on $\mathfrak{t}$ satisfying the following conditions.*

(i) *The poles of each ω_x are contained in the union of singular hyperplanes, and their order is at most* 1.

(ii) *For any singular character χ and for any connected component Y of $X^{\ker(\chi)}$, the sum of residues of the ω_x $(x \in Y^T)$ along the hyperplane $(\chi = 0)$ is zero.*

Proof. By Lemmas 1 and 2, we may assume that $d = 1$ and that X is connected of dimension 1. Then the normalization of X is a disjoint union of copies of the complex projective line $\mathbb{P}^1$. Since the cohomology of X vanishes in degree 1, the Mayer–Vietoris exact sequence implies that each irreducible component C of X contains 2 fixed points; moreover, the union of all other components is either disconnected or meets C in a unique fixed point. In other words, X is a tree of curves homeomorphic to $\mathbb{P}^1$. Now the statement follows easily from an explicit description of $H_*^{\mathbb{C}^*}(\mathbb{P}^1)$ together with induction on the number of irreducible components. □

This result will be applied to Schubert varieties in Section 4.

3. Equivariant Multiplicities and the Conductor

We still assume that X is a compact, equivariantly formal T-variety, with a finite T-fixed point set that we denote F. By the localization theorem, we can assign to each $x \in F$ a rational function $e_T(x, X)$ on $\mathfrak{t}$ (the *T-equivariant multiplicity of X at x*) such that

$$[X]_T = \sum_{x\in F} e_T(x, X)[x]_T$$

in $H_*^T(X)\otimes_{\mathbb{C}[\mathfrak{t}]}\mathbb{C}(\mathfrak{t})$. Then $e_T(x, X)$ is either zero or a homogeneous rational function of degree $-2n$ whose denominator is a product of singular characters. (This definition makes sense, more generally, for an isolated fixed point in a T-variety; see [3; 7].)

For irreducible X, note that the equivariant multiplicities depend only on the algebra $H_T^*(X)$ up to multiplication by a common nonzero complex number (this follows from Proposition 3).

The equivariant multiplicity is related with the *equivariant Euler class* of [2], as follows. Let $x \in F$ and let $H_{T,x}^*(X)$ denote equivariant cohomology of X with support in $\{x\}$. By the localization theorem again, the restriction map

$$H^*_{T,x}(X) \to H^*_T(x) \cong \mathbb{C}[\mathfrak{t}], \quad \alpha \mapsto \alpha_x$$

is an isomorphism after inverting finitely many nontrivial characters of T. Thus, we may choose a non-$\mathbb{C}[\mathfrak{t}]$-torsion $\alpha \in H^*_{T,x}(X)$. Then we have $\alpha_x \neq 0$ and

$$\int_X \alpha \cap [X]_T = e_T(x, X)\alpha_x.$$

Therefore, the equivariant Euler class $\mathrm{Eu}_T(x, X)$ is the inverse of $e_T(x, X)$ if the latter is nonzero. This holds, for example, if X is rationally smooth at x; then $\mathrm{Eu}_T(x, X)$ is a scalar multiple of a product of singular characters (see [2, Sec. 2]).

Note that equivariant multiplicity may well be zero in our setting. Consider, for example, the action of $T = \mathbb{C}^*$ on $\mathbb{P}^4$ defined by its linear action on $\mathbb{A}^5$ with weights 0, 1, -1, 1, -1, and let $z_0, \dots, z_4$ be the corresponding homogeneous coordinates; then the T-fixed points are the coordinate points $x_0, \dots, x_4$. The subvariety $X \subset \mathbb{P}^4$ defined by $z_1z_2 - z_3z_4 = 0$ is T-stable and equivariantly formal; moreover, $X^T = \{x_0, \dots, x_4\}$ and $e_T(x_0, X) = 0$.

Let now $\mathfrak{c}$ be the *conductor* of $H^*_T(X)$ into $H^*_T(F)$; that is,

$$\mathfrak{c} = \{\alpha \in H^*_T(X) \mid i^*\alpha \cup \beta \in i^*H^*_T(X) \ \forall \beta \in H^*_T(F)\}.$$

In other words, $i^*\mathfrak{c}$ is the greatest ideal of $H^*_T(F)$ contained in $i^*H^*_T(X)$. Thus, we have

$$i^*\mathfrak{c} = \prod_{x\in F} \mathfrak{c}_x$$

where the $\mathfrak{c}_x$ are ideals of $\mathbb{C}[\mathfrak{t}]$. Note that the map

$$\mathrm{Hom}_{H^*_T(X)}(H^*_T(F), H^*_T(X)) \to \mathfrak{c}, \quad u \mapsto u(1)$$

is an isomorphism.

We construct elements of $\mathfrak{c}$ as follows. Denote by $H^*_{T,F}(X)$ the equivariant cohomology with support in F and by $r : H^*_{T,F}(X) \to H^*_T(X)$ the natural map. Set

$$\mathfrak{d} = r(H^*_{T,F}(X)) = \sum_{x\in F} r(H^*_{T,x}(X)),$$

a $\mathbb{C}[\mathfrak{t}]$-submodule of $H^*_T(X)$. Note that $i^*\mathfrak{d} = \prod_{x\in F} \mathfrak{d}_x$, where $\mathfrak{d}_x$ denotes the image of the natural map $H^*_{T,x}(X) \to H^*_T(x) \cong \mathbb{C}[\mathfrak{t}]$. Moreover, each $\mathfrak{d}_x$ is an ideal of $\mathbb{C}[\mathfrak{t}]$. As a consequence, $\mathfrak{d}$ is contained in $\mathfrak{c}$; in other words, each $\mathfrak{d}_x$ is contained in $\mathfrak{c}_x$.

In fact, $\mathfrak{c}_x$ and $\mathfrak{d}_x$ are closely related to each other and to the equivariant multiplicity at x, as shown by the following proposition.

PROPOSITION 5. (i) *Each ideal $\mathfrak{c}_x$ is generated by a monomial in the singular characters and satisfies $e_T(x, X)\mathfrak{c}_x \subseteq \mathbb{C}[\mathfrak{t}]$.*

(ii) *If X is rationally smooth at x, then $\mathfrak{d}_x$ equals $\mathfrak{c}_x$ and is generated by $\mathrm{Eu}_T(x, X)$.*

(iii) *If $X^{\ker(\chi)}$ is rationally smooth at x for all singular characters χ of T, then the rational function $e_T(x, X)$ is nonzero and its denominator generates the ideal*

$\mathfrak{c}_x$. *Moreover, the support of the* $\mathbb{C}[\mathfrak{t}]$*-module* $\mathfrak{c}_x/\mathfrak{d}_x$ *has codimension at least* 2 *in* $\mathfrak{t}$.

Proof. (i) Let $(\alpha_1, \dots, \alpha_r)$ be a basis of the free $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$. Write $F = \{x_1, \dots, x_r\}$ and

$$i^*\alpha_k = (a_{k1}, \dots, a_{kr}) \in \mathbb{C}[\mathfrak{t}]^r$$

for $1 \le k \le r$, and let $f = (f_1, \dots, f_r) \in \mathbb{C}[\mathfrak{t}]^r$. Then $f \in i^*\mathfrak{c}$ if and only if, for $1 \le j \le r$, there exists a $\beta_j \in H_T^*(X)$ such that $(\beta_j)_{x_j} = f_j$ and $(\beta_j)_{x_k} = 0$ for all $k \neq j$. Writing $\beta_j = \sum_{k=1}^r f_{jk}\alpha_k$ with $f_{jk} \in \mathbb{C}[\mathfrak{t}]$, the latter condition translates into the following system of linear equalities:

$$\sum_{k=1}^r a_{kl} f_{jk} = \begin{cases} f_j & \text{if } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

Solving this system yields $\Delta f_{jk} = (-1)^{j+k}\Delta_{jk} f_j$ for $1 \le j, k \le r$, where Δ denotes the determinant of the matrix (a_{kl}) and Δ_{jk} its principal (j, k)-minor. Therefore, $f_j \in \mathfrak{c}_j$ if and only if f_j is divisible by all $\Delta/(\Delta, \Delta_{jk})$ for $k = 1, \dots, r$, where (Δ, Δ_{jk}) denotes the greatest common divisor of these polynomial functions. This shows that $\mathfrak{c}_j$ is generated by the least common multiple of the $\Delta/(\Delta, \Delta_{jk})$ $(1 \le k \le r)$. On the other hand, $\mathfrak{c}_j$ contains a monomial in the singular characters, by the localization theorem. This proves the first assertion.

Let $f \in \mathfrak{c}_x$. Then there exists an $\alpha \in H_T^*(X)$ such that $\alpha_x = f$ and $\alpha_y = 0$ for all $y \in F$ with $y \neq x$. Now $e_T(x, X)f = \int_X \alpha \cap [X]_T$ is in $\mathbb{C}[\mathfrak{t}]$.

(ii) By [2, 2.3], the $\mathbb{C}[\mathfrak{t}]$-module $H_{T,x}^*(X)$ is freely generated by a homogeneous element α of degree $2n$. Moreover, the image of α in $H_T^*(x) = \mathbb{C}[\mathfrak{t}]$ equals $\mathrm{Eu}_T(x, X)$, the inverse of $e_T(x, X)$. Thus $\mathrm{Eu}_T(x, X)$ generates $\mathfrak{d}_x$, and $\mathfrak{c}_x$ is contained in $\mathrm{Eu}_T(x, X)\mathbb{C}[\mathfrak{t}] = \mathfrak{d}_x$, whence $\mathfrak{d}_x = \mathfrak{c}_x$.

(iii) By the localization theorem, $e_T(x, X)$ is the product of $e_T(x, X^{\ker(\chi)})$ (a constant multiple of a power of the singular character χ) with a rational function defined along the hyperplane $(\chi = 0)$. It follows that the denominator of $e_T(x, X)$ is the product of the denominators of the $e_T(x, X^{\ker(\chi)})$, where χ runs over the singular characters up to multiple. Now the assertion follows from (ii) together with the localization theorem. $\square$

Next we obtain sufficient conditions for equality $\mathfrak{c}_x = \mathfrak{d}_x$ to hold (we do not know any example where $\mathfrak{c}_x \neq \mathfrak{d}_x$).

Recall that x is called *attractive* if all weights of T in the Zariski tangent space of X at x are contained in an open half-space. Equivalently, there exist an open affine T-stable neighborhood X_x and a one-parameter subgroup λ of T such that $\lim_{t\to 0} \lambda(t)y = x$ for all $y \in X_x$. Then such a neighborhhod X_x is unique and, setting $\dot{X}_x = X_x - \{x\}$, the quotient $\dot{X}_x/\lambda(\mathbb{C}^*)$ is a projective T-variety that we denote by $\mathbb{P}(X_x)$ (see e.g. [3]). Finally, the rational function $e_T(x, X)$ is defined at λ (identified with its differential at 1), and its value is a positive rational number; in particular, $e_T(x, X)$ is nonzero.

PROPOSITION 6. *Assume that x is attractive, $X^{\ker(\chi)}$ is rationally smooth at x for all singular characters χ, and $\mathbb{P}(X_x)$ is equivariantly formal. Then* $\mathfrak{c}_x = \mathfrak{d}_x$.

Proof. Let $f \in \mathfrak{c}_x$ be homogeneous of degree q. Then there exists a unique $\alpha \in H_T^q(X)$ such that $\alpha_x = f$ and $\alpha_y = 0$ for all $y \in F$ with $y \neq x$. We check that α is in the image of the natural map $H_{T,x}^q(X) \to H_T^q(X)$ or, equivalently, that f is in the image of the composition

$$H_{T,x}^q(X) \to H_T^q(X) \to H_T^q(x).$$

By excision, the latter identifies with the image of the composition

$$H_{T,x}^q(X_x) \to H_T^q(X_x) \to H_T^q(x).$$

Moreover, the map $H_T^*(X_x) \to H_T^*(x)$ is an isomorphism, because x is attractive. Since the sequence

$$H_{T,x}^q(X_x) \to H_T^q(X_x) \to H_T^q(\dot{X}_x)$$

is exact, we have to check that α maps to zero in $H_T^*(\dot{X}_x)$.

Note that the T-fixed points in $\mathbb{P}(X_x)$ are the $\mathbb{P}(X_x^{\ker(\chi)})$, where χ runs over all singular characters. Since $\mathbb{P}(X_x)$ is equivariantly formal, the restriction map

$$H_T^*(\dot{X}_x) \to \prod_\chi H_T^*\big((\dot{X}_x)^{\ker(\chi)}\big)$$

is injective. Now we conclude by Proposition 5(ii). □

4. Poincaré Duality and Betti Numbers

Combining the results of the previous sections, we obtain the following criterion for Poincaré duality.

THEOREM 1. *For a compact, equivariantly formal T-variety X of dimension n with isolated fixed points, the following conditions are equivalent.*

(i) *X satisfies Poincaré duality.*

(ii) *The algebra $H_T^*(X)$ is Gorenstein.*

(iii) *The Betti numbers of X satisfy $b_q(X) = b_{2n-q}(X)$ for $0 \leq q \leq n$, and all equivariant multiplicities are nonzero.*

If one of these conditions holds, then all equivariant multiplicities are in fact inverses of polynomial functions.

Proof. (i) ⇔ (ii) By Proposition 3, the algebra $H_T^*(X)$ is Gorenstein if and only if the $H_T^*(X)$-module $H_*^T(X)$ is freely generated by $[X]_T$. But this amounts to Poincaré duality for X, by Proposition 1 and the graded Nakayama lemma.

(ii) ⇒ (iii) The algebra $H^*(X)$ is Gorenstein as the quotient of the Gorenstein algebra $H_T^*(X)$ by the ideal $\mathfrak{t}^* H_T^*(X)$ generated by a regular sequence. It follows that $b_q(X) = b_{2n-q}(X)$ for all $q \in \mathbb{Z}$ (see e.g. [8, p. 551]). Morever, the proof of (i) ⇔ (ii) shows that the dualizing module $\omega_{V(X)}$ is freely generated by $[X]_T$, a homogeneous element of degree $2(d-n)$. Thus, the conductor $\mathfrak{c}$ satisfies

$$\begin{aligned}\mathfrak{c} \cong \operatorname{Hom}_{H_T^*(X)}(H_T^*(F), H_T^*(X)) &\cong \operatorname{Hom}_{H_T^*(X)}(H_T^*(F), \omega_{V(X)}) \otimes_{H_T^*(X)} \omega_{V(X)}^* \\ &\cong \omega_{V(F)} \otimes_{H_T^*(X)} \omega_{V(X)}^*,\end{aligned}$$

where $\omega^*_{V(X)}$ denotes the inverse of the canonical module; the third isomorphism follows from [8, Thm. 21.15]. But the dualizing module $\omega_{V(F)}$ of $H^*_T(X) \cong \mathbb{C}[\mathfrak{t}]^r$ is freely generated in degree $2d$. Therefore, the $\mathbb{C}[\mathfrak{t}]^r$-module $i^*\mathfrak{c}$ is freely generated in degree $2n$ by (say) $(f_x)_{x\in F}$. For a fixed $x \in F$, there exists $\alpha \in H^{2n}_T(X)$ such that $\alpha_x = f_x$ and $\alpha_y = 0$ for all $y \in F$ with $y \neq x$. Thus,

$$e_T(x, X)\alpha_x = \int_X \alpha(x) \cap [X]_T$$

is in $\mathbb{C}[\mathfrak{t}]$. But α_x and $e_T(x, X)$ are homogeneous of opposite degrees, so that $e_T(x, X)$ is the inverse of a polynomial.

(iii) $\Rightarrow$ (ii) We claim that the equivariant Poincaré duality map

$$\bigcap [X]_T : H^q_T(X) \to H^T_{2n-q}(X)$$

is injective for all $q \in \mathbb{Z}$. Let $\alpha \in H^*_T(X)$ be such that $\alpha \cap [X]_T = 0$. Then

$$\int_X (\alpha \cup \beta) \cap [X]_T = 0$$

for all $\beta \in H^*_T(X)$. Thus, we have

$$\sum_{x\in F} e_T(x, X)\alpha_x\beta_x = 0$$

in $\mathbb{C}(\mathfrak{t})$. By the localization theorem, this equality holds for all sequences $(\beta_x)_{x\in F}$ in $\mathbb{C}(\mathfrak{t})$. Since no $e_T(x, X)$ vanishes, we must have $\alpha_x = 0$ for all $x \in F$ and $\alpha = 0$. This proves our claim.

On the other hand, the assumption on Betti numbers combined with the isomorphisms

$$H^*_T(X) \cong \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} H^*(X) \quad \text{and} \quad H^T_*(X) \cong \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} H^T_*(X)$$

implies that the dimension of $H^q_T(X)$ equals that of $H^T_{2n-q}(X)$ for all $q \in \mathbb{Z}$. Thus, the equivariant Poincaré duality map is an isomorphism, and the same holds for the usual one. $\square$

We now come to our main result.

THEOREM 2. *Let X be a compact, equivariantly formal T-variety of dimension n with isolated fixed points. If all equivariant multiplicities are nonzero* (*e.g., if all fixed points are attractive*), *then the following inequalities hold for the Betti numbers*:

$$b_q(X) + b_{q-1}(X) + \cdots + b_0(X) \leq b_{2n-q}(X) + b_{2n-q+1}(X) + \cdots + b_{2n}(X)$$

for $0 \leq q \leq n-1$;

$$2b_2(X) + 4b_4(X) + \cdots + 2nb_{2n}(X) \geq n\chi(X),$$

where $\chi(X) = b_0(X) + b_2(X) + \cdots + b_{2n}(X)$ is the Euler characteristic. Moreover, X satisfies Poincaré duality if and only if

$$2b_2(X) + 4b_4(X) + \cdots + 2nb_{2n}(X) = n\chi(X).$$

Proof. Since $e_T(x, X)$ is a nonzero rational function for all $x \in F$, we may choose a 1-dimensional subtorus T' of T such that $X^{T'} = F$ and each $e_{T'}(x, X)$ is nonzero as well. As in the proof of Theorem 1, it follows that the map

$$\bigcap [X]_{T'}\colon H^q_{T'}(X) \to H^{T'}_{2n-q}(X)$$

is injective for all $q \in \mathbb{Z}$. Moreover, since X is equivariantly formal as a T'-variety (by Lemma 2), we have $H^*_{T'}(X) \cong \mathbb{C}[t] \otimes_{\mathbb{C}} H^*(X)$ and $H^{T'}_*(X) \cong \mathbb{C}[t] \otimes_{\mathbb{C}} H_*(X)$ as graded vector spaces, where t is an indeterminate of degree 2. It follows that

$$\dim H^q_{T'}(X) = \sum_{j \geq 0} b_{q-2j}(X), \qquad \dim H^{T'}_{2n-q}(X) = \sum_{j \geq 0} b_{2n-q+2j}(X).$$

Together with vanishing of Betti numbers in odd degrees, this implies the first inequalities. Summing them up for $q = 0, \ldots, n-1$, we obtain

$$nb_0(X) + \cdots + 2b_{n-2}(X) + b_{n-1}(X) \leq b_{n+1}(X) + 2b_{n+2}(X) + \cdots + nb_{2n}(X),$$

which is equivalent to the second inequality.

If X satisfies Poincaré duality, then $b_q(X) = b_{2n-q}(X)$ for all $q \in \mathbb{Z}$, whence

$$2b_2(X) + 4b_4(X) + \cdots + 2nb_{2n}(X) = n\chi(X).$$

Conversely, if the latter equality holds then we have

$$b_q(X) + b_{q-1}(X) + \cdots + b_0(X) = b_{2n-q}(X) + b_{2n-q+1}(X) + \cdots + b_{2n}(X)$$

for $0 \leq q \leq n-1$, by the foregoing arguments. This in turn implies $b_q(X) = b_{2n-q}(X)$. Thus, X satisfies Poincaré duality by Theorem 1. □

Next let (W, S) be a Coxeter system with length function ℓ and Bruhat order $\leq$ (cf. [13]). We assume that W is crystallographic, that is, the product of any two distinct elements of S has order 2, 3, 4, 6, or ∞. Equivalently, W is the Weyl group of a complex Kac–Moody Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$, the reflection representation [19].

To each $w \in W$ is associated the Schubert variety $X(w)$, a complex projective variety of dimension $\ell(w)$. The maximal torus T of the Kac–Moody group associated to $\mathfrak{g}$ acts on $X(w)$ with isolated fixed points, indexed by the Bruhat interval

$$[1, w] = \{x \in W,\ x \leq w\}.$$

Each such fixed point is attractive, and $X(w)$ is the disjoint union of Schubert cells $X^0(x)$ $(x \in [1, w])$, where $X^0(x)$ is T-stable and isomorphic to a complex affine space of dimension $\ell(x)$. Thus, $X(w)$ satisfies our assumptions.

The T-equivariant cohomology ring of $X(w)$ is determined in [1] and [14]; see also [11, Sec. 4]. An alternative description follows readily from [10, Thm. 7.1], because $X(w)$ contains only finitely many T-orbit closures of dimension 1. Each such curve is uniquely determined by its T-fixed points x and sx, where $x \in W$, s is a reflection of W, and $x, sx \leq w$; moreover, T acts on that curve through a character χ such that $(\chi = 0)$ is the hyperplane fixed by s [4, Thm. F]. Thus, the image of the restriction map $i^*\colon H_T^*(X(w)) \to H_T^*(X(w)^T)$ is the set of all tuples $(f_x)_{x\in[1,w]}$ in $\mathbb{C}[\mathfrak{t}]$ such that $f_x - f_{s_\alpha x}$ is divisible by α whenever (a) $s_\alpha \in W$ is a reflection with hyperplane $(\alpha = 0)$ and (b) $x, s_\alpha x \le w$.

Similarly, the equivariant homology $H_*^T(X(w))$ is determined by Corollary 1: it consists of all tuples $(\omega_x)_{x\in[1,w]}$ of rational differential forms on $\mathfrak{t}$ with at most simple poles on reflection hyperplanes, satisfying

$$\mathrm{Res}_{\alpha=0}(\omega_x + \omega_{s_\alpha x}) = 0$$

whenever s_α is a reflection and $x, s_\alpha x \le w$. Theorem 2 yields the following.

Corollary 2. *For any Bruhat interval* $[1, w]$ *in a crystallographic Coxeter group W, we have*

$$\#\{x \in [1,w],\ \ell(x) \le q\} \le \#\{x \in [1,w],\ \ell(x) \ge \ell(w) - q\}$$

for $1 \le q < \frac{1}{2}\ell(w)$.

Moreover, the second inequality in Theorem 2 yields the inequality $a(w) \ge \frac{1}{2}\ell(w)$ for the average length $a(w)$ of elements of $[1, w]$, with equality if and only if $X(w)$ satisfies Poincaré duality. This statement is due to Carrell and Peterson [4], together with equivalence of Poincaré duality and rational smoothness for Schubert varieties. The latter result can be recovered from Theorem 1 combined with the characterization of rational smoothness in terms of equivariant multiplicities (see [2; 3; 15]).

Note finally that Corollary 2 actually holds for an arbitrary Coxeter group W. Although Schubert varieties no longer exist in this setting, all ingredients of the proof of Theorem 2 still make sense (see [5; 14, (4.35)]; the nonvanishing of "equivariant multiplicities" follows from [5, Prop. 1]).

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