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# Gromov–Witten Invariants of a Class of Toric Varieties

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*Dedicated to William Fulton*

## 1. Introduction

*1.1. Background*

Toric varieties admit a combinatorial description, which allows many invariants to be expressed in terms of combinatorial data. Batyrev [Ba2] and Morrison and Plesser [MP] describe the quantum cohomology rings of certain toric varieties in terms of generators (divisors and formal $q$-variables) and relations (linear relations and $q$-deformed monomial relations). The relations are easily obtained from the combinatorial data. Unfortunately, the relations alone do not tell us how to multiply cohomology classes in the quantum cohomology ring $QH^*(X)$, or even how to express ordinary cohomology classes in $H^*(X, \mathbb{Q})$ in terms of the given generators. In this paper, we give a formula that expresses any class in $H^*(X, \mathbb{Q})$—as a polynomial in divisor classes and formal $q$-variables—for any $X$ belonging to a certain class of toric varieties. These expressions, along with the presentation of $QH^*(X)$ via generators and relations, permit computation of any product of cohomology classes in $QH^*(X)$.

Let $X$ be a complete toric variety of dimension $n$ over the complex numbers (all varieties in this paper are over the complex numbers). This means that $X$ is a normal variety with an action by the algebraic torus $(\mathbb{C}^*)^n$ and a dense equivariant embedding $(\mathbb{C}^*)^n \to X$. By the theory of toric varieties (cf. [F]), such $X$ are characterized by a fan $\Delta$ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$, where $N$ is the lattice $\mathbb{Z}^n$. The cones are rational, that is, generated by lattice points. In particular, to every ray (1-dimensional cone) $\sigma$ there is a unique generator $\rho \in N$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot \rho$. There is a one-to-one correspondence between such ray generators $\rho$ and toric (i.e., torus-invariant) divisors of $X$. Given toric divisors $D_1, \dots, D_k$ with corresponding ray generators $\rho_1, \dots, \rho_k$, we have $D_1 \cap \dots \cap D_k \neq \emptyset$ if and only if $\rho_1, \dots, \rho_k$ span a cone in $\Delta$. Hypotheses on $X$ translate as follows into conditions on $\Delta$:

(i) $X$ is nonsingular if and only if every cone is generated by a part of a $\mathbb{Z}$-basis of $N$;

(ii) given that $X$ is nonsingular, $X$ is Fano (i.e., $X$ has ample anticanonical class) if and only if the set of ray generators is strictly convex.

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We need the following terminology from [Ba1].

DEFINITION 1.1. Let $X$ be a complete nonsingular toric variety. $\{D_1, \dots, D_k\}$ is then a *primitive set* for $X$ if $D_1 \cap \cdots \cap D_k = \emptyset$ but $D_1 \cap \cdots \cap \widehat{D_j} \cap \cdots \cap D_k \neq \emptyset$ for all $j$. Equivalently, this means that $\langle \rho_1, \dots, \rho_k \rangle \notin \Delta$ but $\langle \rho_1, \dots, \widehat{\rho_j}, \dots, \rho_k \rangle \in \Delta$ for all $j$. If $S := \{D_1, \dots, D_k\}$ is a primitive set then the element $\rho := \rho_1 + \cdots + \rho_k$ lies in the relative interior of a unique cone of $\Delta$, say the cone generated by $\rho'_1, \dots, \rho'_r$. Then

$$\rho_1 + \cdots + \rho_k = a_1 \rho'_1 + \cdots + a_r \rho'_r \quad (a_i > 0,\ i = 1, \dots, r) \tag{1}$$

is the corresponding *primitive relation*. Correspondingly there is a unique curve class $\beta \in H_2(X, \mathbb{Z})$ such that $\int_\beta D_i = 1$ for $i = 1, \dots, k$ and $\int_\beta D'_j = -a_j$ for $j = 1, \dots, r$, with $\int_\beta D = 0$ for all other toric divisors of $X$. This is called the *primitive class* associated to the primitive set $S$.

We provide more details in Section 2, in particular regarding the fact that, on any nonsingular projective toric variety, every primitive class is effective.

THEOREM 1.2. *Let $X$ be a nonsingular Fano toric variety of dimension $n$ with a corresponding fan $\Delta$ of cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$, with $N = \mathbb{Z}^n$. Let $M = \mathrm{Hom}(N, \mathbb{Z})$. Let $C$ be the cone of effective curve classes on $X$, with $\mathbb{Q}[C]$ the semigroup algebra on $C$. Let $D_1, \dots, D_m$ denote the toric divisors on $X$, with corresponding ray generators $\rho_1, \dots, \rho_m$. Then*

$$QH^*(X) = (\mathbb{Q}[C])[D_1, \dots, D_m]/I, \tag{2}$$

*where $I$ is the ideal generated by*

$$\varphi(\rho_1) D_1 + \cdots + \varphi(\rho_m) D_m \tag{3}$$

*for all $\varphi \in M$ and by*

$$D_1 \cdots D_k - q^\beta (D'_1)^{a_1} \cdots (D'_r)^{a_r} \tag{4}$$

*for every primitive set $\{D_1, \dots, D_k\}$, with corresponding primitive relation* (1) *and primitive curve class $\beta$.*

A general primitive set should perhaps be denoted as $\{D_{i_1}, \dots, D_{i_k}\}$, with $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$; this gets cumbersome, so we let there be an implied shuffling of indices in (4). The element of $\mathbb{Q}[C]$ indexed by $\beta \in C$ is denoted $q^\beta$; these, for nonzero $\beta$, are the quantum correction terms of the quantum cohomology ring. Note that when all the variables $q^\beta$ for $0 \neq \beta \in C$ are set to 0, we recover the presentation of the usual cohomology ring of $X$. In fact, the cohomology ring with integer coefficients of any complete nonsingular toric variety has, as generators, the toric divisor classes and, as relations, the linear relations (3) and the monomial terms (4) (with no $q$-terms).

Theorem 1.2 was stated in [Ba2] and also discussed in [MP]. A suggestive argument was given in [Ba2], but the first proof was supplied by Givental in [Gi], where complete intersections in toric varieties were considered, with the toric varieties themselves as a trivial first case. The argument of [Gi] relied upon a collection of axioms of equivariant Gromov–Witten invariants. For these, the later-supplied equivariant localization theorem of Graber and Pandharipande [GP] is needed. A recently announced formula [Sp] reduces computation of *any* Gromov–Witten invariant on a nonsingular projective toric variety to a certain sum over a finite set of graphs, although deducing the relations (4) from this would be a formidable combinatorial task. Also, [CK, pp. 393–395] and [Sp] exhibit nonsingular projective (but non-Fano) toric varieties $X$ for which (4) fails to vanish in $QH^*(X)$.

When $X$ is Fano, one can identify $QH^*(X) \simeq \mathbb{Q}[C] \otimes_{\mathbb{Q}} H^*(X, \mathbb{Q})$ as $\mathbb{Q}[C]$-modules, where $C$ denotes the semigroup of effective curve classes on $X$. A cohomology class $\alpha \in H^*(X, \mathbb{Q})$ is identified with $1 \otimes \alpha \in QH^*(X)$. To "know" $QH^*(X)$ means to know how to compute $\alpha_1 \cdot \alpha_2$ in $QH^*(X)$ for any $\alpha_1, \alpha_2 \in H^*(X, \mathbb{Q})$. The structure constants in the expression for $\alpha_1 \cdot \alpha_2$ as a linear combination of elements $q^\beta \otimes \alpha'$ are the three-point Gromov–Witten invariants. The three-point Gromov–Witten invariants in turn determine all the Gromov–Witten invariants, by the inductive procedure of the first reconstruction theorem of Kontsevich and Manin [KM] (the needed hypothesis of $H^*(X, \mathbb{Q})$ being generated by divisor classes is satisfied for toric varieties). All the Gromov–Witten invariants are thus determined from having (i) a presentation for $QH^*(X)$ in terms of generators and relations and (ii) an expression for $\alpha$ in $QH^*(X)$ for any $\alpha \in H^*(X, \mathbb{Q})$. This second piece of data, in the context of homogeneous spaces, is referred to as a *quantum Giambelli formula* (see e.g. [Ber]). So the ring presentation of Batyrev and of Morrison and Plesser needs to be supplemented by a quantum Giambelli formula before we can say we "know" $QH^*(X)$.

### *1.2. Main Result*

In this paper, we provide a quantum Giambelli formula for a class of toric varieties. We first need some new terminology.

DEFINITION 1.3. An *exceptional set* is a set of toric divisors $\{D_1, \ldots, D_k\}$ such that the corresponding ray generators $\rho_1, \ldots, \rho_k$ are linearly independent and such that $\rho_1 + \cdots + \rho_k$ is equal to some ray generator $\tilde{\rho}$. Then $\rho_1 + \cdots + \rho_k = \tilde{\rho}$ is the associated *exceptional relation*. There is the corresponding *exceptional divisor* $\tilde{D}$ and *exceptional class* $\beta \in H_2(X, \mathbb{Z})$, with $\int_\beta D_i = 1$ for $i = 1, \ldots, k$, $\int_\beta \tilde{D} = -1$, and $\int_\beta D' = 0$ for all other toric divisors $D'$.

DEFINITION 1.4. Let a cone $\sigma \in \Delta$ be fixed. Then an exceptional set $\{D_1, \ldots, D_k\}$ is called *special* (for $\sigma$) if some $(k-1)$ of $\rho_1, \ldots, \rho_k$, as well as $\tilde{\rho}$, lie in $\sigma$.

DEFINITION 1.5. Let $\{S_1, \ldots, S_t\}$ be a collection of exceptional sets. We say this set of exceptional sets has a *cycle* if there exists $\{i_1, \ldots, i_j\} \subset \{1, \ldots, t\}$ such that the exceptional divisor for $S_{i_{\nu+1}}$ is in $S_{i_\nu}$ for $\nu = 1, \ldots, j-1$ and the exceptional divisor for $S_{i_1}$ is in $S_{i_j}$. Otherwise, we say the set of exceptional sets *has no cycles.*

THEOREM 1.6. *Let $X$ be a nonsingular projective toric variety. Assume $X$ is Fano, and assume further that every toric subvariety of $X$ is Fano and that, for every nonsingular toric variety $X'$ dominated by $X$ such that $X \to X'$ is the blow-up of an irreducible toric subvariety, $X'$ is Fano.*

(i) *Every primitive relation of $X$ is either of the form*

$$\rho_1 + \cdots + \rho_k = 0 \quad or \quad \rho_1 + \cdots + \rho_k = \rho_1'.$$

(ii) *If $\{D_1, \ldots, D_j\}$ is a set of toric divisors such that $D_1 \cap \cdots \cap D_j$ is nonempty and if $\alpha$ denotes the cohomology class Poincaré dual to $[D_1 \cap \cdots \cap D_j]$, then we have*

$$\alpha = \sum_{\{S_1, \ldots, S_t\}} q^{\beta_1 + \cdots + \beta_t} \prod_{\substack{1 \le i \le j \\ D_i \notin S_1 \cup \cdots \cup S_t}} D_i \tag{5}$$

*in $QH^*(X)$, where the sum is over sets of exceptional sets $\{S_1, \ldots, S_t\}$ that are special for the cone associated to $D_1 \cap \cdots \cap D_j$, have distinct exceptional divisors, and have no cycles; for the sum in* (5), *$\beta_i$ denotes the exceptional class associated to $S_i$ for each $i$.*

REMARK 1.7. It is not obvious yet, but the hypotheses in Theorem 1.6 guarantee that, for any $\{S_1, \ldots, S_t\}$ in the sum (5), the sets $S_i$ are pairwise disjoint. This means that the degrees work out correctly: it is a general fact that, if $\{D_1, \ldots, D_m\}$ is the set of all toric divisors on $X$, then we have $-K_X = D_1 + \cdots + D_m$ and, in general, $QH^*(X)$ is a graded ring with $\deg q^\beta = \int_\beta (-K_X)$ and $\deg \alpha = i$ for $\alpha \in H^{2i}(X, \mathbb{Q})$.

After setting up notation in Section 2, we study the class of toric varieties indicated by Theorem 1.6 in Section 3. These toric varieties are all iterated blow-ups of products of projective spaces, along irreducible toric subvarieties, such that the exceptional divisors of the blow-up can be blown down in any order; see the characterization in Theorem 3.9. This is a convenient class of toric varieties, since it is closed under blow-downs and under inclusions of toric subvarieties. In fact, it is the largest category of nonsingular Fano toric varieties that is closed under these operations. Also, it has the nice feature of admitting a neatly presentable quantum Giambelli formula in terms of the given combinatorial data only. And, unlike in the case of products of projective spaces, there are some $q$ correction terms in the quantum Giambelli. Still, it is a limited class of toric varieties; the author has no idea what sort of shape a general quantum Giambelli formula might take (say, for arbitrary nonsingular Fano toric varieties).

The class of toric varieties includes products of projective spaces themselves, for which the results are known, as well as blow-ups of points, which were studied in [Ga]. This class also includes some of the projective bundles over projective spaces [Ma; QR] and over products of projective spaces [CM]. Such toric varieties are generally not convex varieties, so in the theory of quantum cohomology (cf. [FP] and references therein) one needs virtual fundamental classes [B; BF; LT].

The proof of Theorem 1.6 uses no computations of intersection numbers on moduli spaces, but only the following facts regarding $QH^*(X)$: it is a ring (commutative and associative), graded (see Remark 1.7), presented by (2), with multiplicative rule governed by the three-point Gromov–Witten invariants. For $\alpha_1, \alpha_2 \in H^*(X, \mathbb{Q})$, the pairing (via the usual cup product) of $\alpha_1 \cdot \alpha_2$ with $\alpha_3 \in H^*(X, \mathbb{Q})$ is

$$\int_X (\alpha_1 \cdot \alpha_2) \cup \alpha_3 = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta q^\beta.$$

The number $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta$ is a Gromov–Witten invariant; it counts the (virtual) number of rational curves in class $\beta$ passing through cycles that represent Poincaré duals to $\alpha_1$, $\alpha_2$, and $\alpha_3$. So, for instance, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta = 0$ if there are no curves in homology class $\beta$ satisfying such incidence conditions. The Gromov–Witten invariant also vanishes if one of the $\alpha_i$ is a divisor class whose intersection number with $\beta$ is 0, assuming $\beta \neq 0$ (divisor axiom). These facts let us deduce Theorem 1.6 from Theorem 1.2, using some combinatorial reasoning (Section 4). The reader needs to grant that Theorem 1.2 is proved in [Gi], or else work through Exercise 4.13, which derives relations (4) from scratch (for a class of varieties that includes those indicated in Theorem 1.6).

As a valuable exercise, the reader may list all five isomorphism classes of 2-dimensional toric varieties satisfying the hypotheses of Theorem 1.6, and write down the quantum Giambelli. Note there are often several pairs of divisors intersecting in a point, giving several different expressions for the point class in $QH^*(X)$. Any two such expressions must be equal, via the linear relations and deformed monomial relations in $QH^*(X)$. Unlike in the case of homogeneous spaces, there is no canonical basis for $H^*(X, \mathbb{Q})$.

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## 2. Preliminaries

### *2.1. Conventions*

We use the following notation:

$N$ = finite-dimensional integer lattice, $N_\mathbb{R} = N \otimes \mathbb{R}$;
$M$ = dual lattice, $M_\mathbb{R} = M \otimes \mathbb{R}$;
$X$ = nonsingular projective toric variety;
$\Delta$ = corresponding fan of cones in $N_\mathbb{R}$;
$n$ = dimension of the lattice (hence also the dimension of $X$);
$m$ = number of 1-dimensional rays in $\Delta$ (equal to the number of toric divisors of $X$);
$D_1, \dots, D_1', \dots$ = toric divisors;
$\rho_1, \dots, \rho_1', \dots$ = corresponding ray generators;
$\Delta(\sigma)$ = star of the cone $\sigma \in \Delta$: a fan in $N/\langle\sigma\rangle$ whose cones are in one-to-one correspondence with the cones of $\Delta$ containing $\sigma$;
$X(\sigma)$ = corresponding toric subvariety;
$QH^*(X)$ = the small quantum cohomology ring of $X$.

### 2.2. Divisors and Curve Classes

We let $X$ be an arbitrary nonsingular projective toric variety, with notation as just listed. Some standard exact sequences are

$$0 \to M \to \mathbb{Z}^m \to \operatorname{Pic}(X) \to 0$$

and the dual sequence

$$0 \to H_2(X, \mathbb{Z}) \to \mathbb{Z}^m \to N \to 0.$$

The dual exact sequence indicates that any linear relation among ray generators, such as (1), determines a class in $H_2(X, \mathbb{Z})$.

It is known (cf. [O]) that the set of effective curve classes on $X$ is equal to the cone generated by the toric curves on $X$ (simply let an arbitrary curve degenerate by means of the torus action). Shortly we shall see that this is also equal to the cone generated by the primitive classes.

We first recall the characterization of ample divisors. Let the toric divisors on $X$ be denoted $D_1, \ldots, D_m$. Then a divisor $\sum_{i=1}^m a_i D_i$ is ample if and only if the piecewise linear function $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$, linear on every cone of $\Delta$ and defined by $\psi(\rho_i) = -a_i$, is strictly convex. Linearly equivalent divisors correspond to piecewise linear functions that differ by a global linear function. To every such $\psi$ there corresponds a convex polytope in $M_{\mathbb{R}}$:

$$P_\psi = \{\, v \in M_{\mathbb{R}} \mid \langle v, x\rangle \geq \psi(x) \text{ for all } x \in N_{\mathbb{R}} \,\}.$$

Translation of $\psi$ by a global linear function corresponds to translation of $P_\psi$ by an element of $M$. There is a unique translation sending a given vertex of $P_\psi$ to the origin. Correspondingly, for a fixed ample divisor $D$, to every maximal cone $\mu$ there is a unique representative for $D$ of the form $\sum_{i=1}^m a_i D_i$, with $a_i \geq 0$ for all $i$ and $a_i = 0$, if and only if $\rho_i \in \mu$. This implies the following proposition.

PROPOSITION 2.1. *If $\beta \in H_2(X, \mathbb{Z})$ is nonzero and if the toric divisors that $\beta$ intersects negatively have nonempty common intersection, then $\beta$ must have positive intersection with every ample divisor.*

COROLLARY 2.2. *Any $\beta \in H_2(X, \mathbb{Z})$ that intersects every ample divisor positively must satisfy: $\left\{ D_i \mid \int_\beta D_i > 0 \right\}$ contains a primitive set.*

*Proof.* Apply Proposition 2.1 to $-\beta$. □

PROPOSITION 2.3. *Suppose $\beta \in H_2(X, \mathbb{Z})$. If the $D_i$ for which $\int_\beta D_i < 0$ have nonempty common intersection, then $\beta$ is equal to a linear combination, with nonnegative integer coefficients, of primitive curve classes.*

*Proof.* By Corollary 2.2, $\left\{ i \mid \int_\beta D_i > 0 \right\}$ contains a primitive set. Let $\beta_0$ be the primitive curve class corresponding to this primitive set, and write $\beta = \beta_0 + \beta'$. Now $\left\{ i \mid \int_{\beta'} D_i < 0 \right\} \subset \left\{ i \mid \int_\beta D_i < 0 \right\}$ and so we are done, by induction on the degree of $\beta$ (with respect to a fixed projective embedding of $X$). □

Consider a toric curve $\mathbb{P}^1 \simeq C \subset X$. Any toric divisor having negative intersection with $[C]$ must contain $C$. So, by Proposition 2.3, the cone of effective curve classes on $X$ is contained in the cone spanned by primitive curve classes on $X$. This constitutes half of the following known result [O; OP; Re].

THEOREM 2.4. *Let $X$ be a nonsingular projective toric variety. The cone of effective curve classes on $X$ is equal to the cone spanned by primitive curve classes on $X$.*

It is not hard to obtain a proof of Theorem 2.4 by constructing explicitly a tree of toric $\mathbb{P}^1$'s representing a given primitive curve class. This is an easy consequence of some combinatorial results that are needed in this paper (see Exercise 4.3).

Batyrev's approach [Ba2] to $QH^*(X)$ is to study the moduli space of rational curves on $X$ in a curve class that has nonnegative intersection with every toric divisor. Moduli of rational curves in such a homology class is much like that of curves on a homogeneous space, although the situation at the boundary is a bit more complicated. Nevertheless, if one can derive relations in $QH^*(X)$ involving such curve classes then one can deduce the ring presentation (2).

DEFINITION 2.5. A class $\beta \in H_2(X, \mathbb{Z})$ is said to be *very effective* if $\beta \neq 0$ and $\int_\beta D \geq 0$ for every toric divisor $D$.

Batyrev predicted that, if $\beta$ is a very effective curve class on $X$ and if we set $a_i = \int_\beta D_i$ for each $i$, then the relation

$$D_1^{a_1} \cdots D_m^{a_m} = q^\beta \tag{6}$$

holds in $QH^*(X)$. The enumerative interpretation is that, given a general point $x_0$ on $X$ and distinct points $z_0, z_{1,1}, \dots, z_{1,a_1}, \dots, z_{m,1}, \dots, z_{m,a_m}$ in general position on $\mathbb{P}^1$, there is precisely one morphism $\varphi\colon \mathbb{P}^1 \to X$, with $\varphi_*([\mathbb{P}^1]) = \beta$, such that $\varphi(z_0) = x_0$ and $\varphi(z_{i,j}) \in D_i$ for all $i$ and $j$ with $1 \leq i \leq m$ and $1 \leq j \leq a_i$ (and that there are no curves in other homology classes that contribute $q$-terms).

PROPOSITION 2.6. *Given a nonsingular projective toric variety $X$, assume relation* (6) *for every very effective curve class $\beta$. Then the deformed monomial relations* (4) *hold. If, moreover, $X$ is Fano, then $QH^*(X)$ has the claimed presentation* (2).

*Proof*. Let $\beta$ be a primitive curve class, and write $\beta = \beta_2 - \beta_1$ with $\beta_1$ and $\beta_2$ very effective. Then

$$\begin{aligned} q^{\beta_1} \prod_{\int_\beta D_i = 1} D_i &= \Bigg[ \prod_{\int_\beta D_i = 1} D_i \Bigg] D_1^{\int_{\beta_1} D_1} \cdots D_m^{\int_{\beta_1} D_m} \\ &= \Bigg[ \prod_{\int_\beta D_j = 0} D_j^{(-\int_\beta D_j)} \Bigg] D_1^{\int_{\beta_2} D_1} \cdots D_m^{\int_{\beta_2} D_m} = q^{\beta_2} \prod_{\int_\beta D_j = 0} D_j^{(-\int_\beta D_j)}, \end{aligned}$$

and (4) follows since $q^{\beta_1}$ is a nonzero divisor in $QH^*(X)$.

If $X$ is Fano, then a presentation for $QH^*(X)$ is obtained by starting with a presentation for $H^*(X,\mathbb{Q})$ in terms of generators and relations and then replacing each relation by a $q$-deformed relation that holds in $QH^*(X)$ ([ST], or cf. [FP]). The presentation (2) is of this form. □

REMARK 2.7. The proof shows that, for any effective $\beta$ with associated relation

$$c_1\rho_1 + \cdots + c_k\rho_k = a_1\rho'_1 + \cdots + a_r\rho'_r$$

in $N$ $\left(\int_\beta D_i = c_i > 0 \text{ and } -\int_\beta D'_j = a_j > 0\right)$, the relation

$$D_1^{c_1}\cdots D_k^{c_k} = q^\beta (D'_1)^{a_1}\cdots(D'_r)^{a_r} \tag{7}$$

holds in $QH^*(X)$ (assuming relations (6) hold for $X$).

## 3. A Class of Fano Toric Varieties

### *3.1. Fano Conditions*

We relate the shape of the relations among ray generators corresponding to primitive sets of a fan, on the one hand, to a series of increasingly restrictive conditions on the associated toric variety, on the other. We arrive at the following dictionary. We recall the primitive relation associated to a primitive set:

$$\rho_1 + \cdots + \rho_k = a_1\rho'_1 + \cdots + a_r\rho'_r \quad (a_i > 0,\ \langle\rho'_1,\dots,\rho'_r\rangle \in \Delta). \tag{8}$$

The dictionary reads:

| | | |
|---|---|---|
| $\sum a_i < k$ for all relations (8) | $\iff$ | $X$ is Fano; |
| $\sum a_i \le 1$ for all relations (8) | $\iff$ | $X$ is Fano, and every toric subvariety of $X$ is Fano; |
| $\sum a_i \le 1$, and every $\rho'$ appears on the right-hand side of at most one relation (8) | $\iff$ | $X$ is Fano; every toric subvariety and blow-down of $X$ is Fano. |

The first of these conditions is known (cf. [O]). The others are Theorems 3.1 and 3.9.

### *3.2. Conditions for Every Toric Subvariety to be Fano*

Part (i) of Theorem 1.6 is a consequence of the following characterization.

THEOREM 3.1. *Let $X$ be a complete nonsingular toric variety, and let $\Delta$ be the associated fan. Then the following are equivalent.*

(i) *$X$ is Fano, and every toric subvariety of $X$ is Fano.*
(ii) *For every primitive set $\{D_1,\dots,D_k\}$ we have either $\rho_1 + \cdots + \rho_k = 0$ or $\rho_1 + \cdots + \rho_k = \rho'$, where $\rho'$ is a ray generator of $\Delta$.*

(iii) *For every maximal cone* $\mu = \langle \rho_1, \ldots, \rho_n \rangle$ *in* $\Delta$ *and for every ray generator* $\rho$*, if we write* $\rho = b_1\rho_1 + \cdots + b_n\rho_n$ *then we have* $-1 \leq b_j \leq 1$ *for* $j = 1, \ldots, n$*, with* $b_j = 1$ *for at most one* $j$*.*

*Proof*. For (i) $\Rightarrow$ (ii), we induct on the dimension $n$. The case $n = 1$ is trivial, and the base case $n = 2$ is easily verified. For the inductive step, let us suppose $X$ satisfies (i) but that (ii) fails to hold. Then there is a primitive set $\{D_1, \ldots, D_k\}$ whose associated primitive relation (8) satisfies $\sum a_i \geq 2$.

Let $\mu$ be a maximal cone containing $\rho'_1, \ldots, \rho'_r$, and let us denote the remaining generators of $\mu$ by $\rho_1, \ldots, \rho_h, \rho'_{r+1}, \ldots, \rho'_s$ (suitably rearranging indices). We insist that the sets $\{\rho_1, \ldots, \rho_k\}$ and $\{\rho'_1, \ldots, \rho'_s\}$ be disjoint. Now $\mu$ is the cone spanned by

$$T := \{\rho_1, \ldots, \rho_h, \rho'_1, \ldots, \rho'_s\}. \tag{9}$$

Let $\varphi \in M$ be the point corresponding to $\mu$ (so $\varphi(\rho) = 1$ for all $\rho \in T$). We have $\varphi(\rho_1 + \cdots + \rho_k) = \sum a_i \geq 2$.

Since $X$ is Fano, we have $\varphi(\rho) \leq 1$ for every ray generator $\rho$, with equality if and only if $\rho \in T$. So, for $h + 1 \leq j \leq k$ we have $\varphi(\rho_j) = -c_j$ for some nonnegative integer $c_j$. Now

$$\varphi(\rho_1 + \cdots + \rho_k) = h - \sum_{j=h+1}^{k} c_j \geq 2.$$

In particular, $h \geq 2$ and so $k \geq 3$. Consider the fan $\Delta(\rho_1)$ in $N/\langle \rho_1 \rangle$. Let us give $N$ coordinates by identifying the elements of $T$ (in the order listed in (9)) with the standard basis elements. Then $\Delta(\rho_1)$ consists of all cones of $\Delta$ containing $\rho_1$, projected by forgetting the first coordinate. The divisors associated to the projections of $\rho_2, \ldots, \rho_k$ form a primitive set for $X(\rho_1)$. Note that $\rho_1 + \cdots + \rho_k$ has first coordinate equal to zero; so, if we define $\bar{\varphi} \in \mathrm{Hom}(N/\langle \rho_1 \rangle, \mathbb{Z})$ by $\bar{\varphi}(\bar{\rho}) = 1$ for all $\rho \in T \setminus \{\rho_1\}$, then we have $\bar{\varphi}(\bar{\rho}_2 + \cdots + \bar{\rho}_k) = \varphi(\rho_1 + \cdots + \rho_k) \geq 2$. We are assuming every toric subvariety of $X$ is Fano. The induction hypothesis applied to the toric subvariety $X(\rho_1)$ implies that $\bar{\varphi}(\bar{\rho}_2 + \cdots + \bar{\rho}_k) \leq 1$, so we have a contradiction.

For (ii) $\Rightarrow$ (iii), we let $\mu = \langle \rho_1, \ldots, \rho_n \rangle$ be a maximal cone and give $N$ the coordinates thus dictated. Suppose some ray generator $\rho$, when written in coordinates as $(b_1, \ldots, b_n)$, satisfies $b_1 \leq -2$. If the $\mathbb{P}^1$ on $X$ corresponding to the $(n-1)$-dimensional cone $\langle \rho_2, \ldots, \rho_n \rangle$, has fixed points $X(\mu)$ and $X(\mu')$, then in the coordinate system of $\mu'$ we find that $\rho$ has first coordinate $-b_1$. Hence, if (iii) fails then, for some $\mu$ and $\rho$, the coordinates $(b_1, \ldots, b_n)$ for $\rho$ satisfy $b_1 \geq 2$ or $b_1 = b_2 = 1$ (after shuffling indices). Among all such pairs $\mu$ and $\rho$ we may assume that $b_1 + \cdots + b_n$ is as large as possible. Now $\rho, \rho_1, \ldots, \rho_n$ fail to generate a cone and so, by (ii), the sum $\rho'$ of $\rho$ and some nonempty subset of $\{\rho_1, \ldots, \rho_n\}$ is also a ray generator. But $\rho'$ must have either some coordinate $\geq 2$ or at least two coordinates $= 1$, and the sum of the coordinates of $\rho'$ is strictly larger than $b_1 + \cdots + b_n$. This is a contradiction.

Statement (iii) implies that $X$ is Fano; for any cone $\sigma$, statement (iii) for $\Delta$ implies statement (iii) for $\Delta(\sigma)$ and hence that the toric subvariety $X(\sigma)$ is Fano. Thus every toric subvariety of $X$ is Fano, and we have (iii) $\Rightarrow$ (i). □

### 3.3. *Blow-Downs of Fano Toric Varieties*

We show that, for toric varieties satisfying the conditions of Theorem 3.1, the blow-downs of toric divisors are in one-to-one correspondence with primitive relations with nonzero right-hand side. The property that every blow-down is Fano then becomes that every ray generator appears on the right-hand side of at most one primitive relation. Such varieties then enjoy the property of possessing a collection of exceptional divisors that can be blown down in any order, at every stage producing a nonsingular Fano toric variety, and yielding finally a product of projective spaces.

DEFINITION 3.2. If $X$ satisfies the conditions of Theorem 3.1, we say a toric divisor $\hat{D}$ is *exceptional* if $\rho_1 + \cdots + \rho_k = \hat{\rho}$ is a primitive relation for $X$ for some $\rho_1, \ldots, \rho_k$.

LEMMA 3.3. *Suppose $X$ satisfies the conditions of Theorem 3.1. If a ray generator $\rho$ is equal to a nonnegative linear combination of ray generators other than $\rho$, then the toric divisor $D$ associated to $\rho$ is exceptional.*

*Proof.* Induct on the sum of the coefficients, and apply Theorem 3.1(ii). □

LEMMA 3.4. *Assume $X$ satisfies the conditions of Theorem 3.1. Let $\langle \rho'_1, \ldots, \rho'_k \rangle$ be a cone of $\Delta$, and let $w = a_1\rho'_1 + \cdots + a_k\rho'_k$ with $a_i \geq 1$ for each $i$ and $a_1 \geq 2$. If $\{\rho_1, \ldots, \rho_j\}$ is any linearly independent set of ray generators, then $\rho_1 + \cdots + \rho_j \neq w$.*

*Proof.* We induct on $j$. Suppose $\rho_1 + \cdots + \rho_j = w$. Then $\{D_1, \ldots, D_j\}$ must contain a primitive set. The set $\{D_1, \ldots, D_j\}$ itself cannot be a primitive set, since $w$ is not a ray generator in $\Delta$. Hence, we may suppose that $\{D_1, \ldots, D_h\}$ is primitive with $h < j$. Then we have $\rho_1 + \cdots + \rho_h = \rho$ for some ray generator $\rho$, and now $\rho + \rho_{h+1} + \cdots + \rho_j = w$ with $\rho, \rho_{h+1}, \ldots, \rho_j$ linearly independent. This contradicts the induction hypothesis. □

PROPOSITION 3.5. *Assume $X$ satisfies the conditions of Theorem 3.1. Let $\hat{D}$ be an exceptional divisor with primitive relation $\rho_1 + \cdots + \rho_k = \hat{\rho}$. Then there exists a morphism of nonsingular toric varieties $X \to X'$ such that $\sigma := \langle \rho_1, \ldots, \rho_k \rangle$ is a cone of the fan $\Delta'$ corresponding to $X'$ and such that $X \to X'$ is the blowing up of $X'$ along $X'(\sigma)$.*

*Proof.* We need to show that, for all $h$ ($1 \leq h \leq k$) and for every cone $\sigma \in \Delta$ with $\hat{\rho} \in \sigma$,

$$\rho_h \notin \sigma \implies \langle \rho_1, \ldots, \widehat{\rho_h}, \ldots, \rho_k, \sigma \rangle \in \Delta. \tag{10}$$

Suppose (10) fails for $\sigma = \langle \hat\rho \rangle$. We may suppose that $\langle \rho_1, \dots, \rho_{k-1}, \hat\rho \rangle \notin \Delta$ and, in fact, that $\{D_1, \dots, D_r, \hat D\}$ is a primitive set with $1 \le r \le k-1$. Hence $\rho_1 + \cdots + \rho_r + \hat\rho = \rho'$ for some $\rho'$. Now $\rho', \rho_{r+1}, \dots, \rho_k$ are linearly independent and $\rho' + \rho_{r+1} + \cdots + \rho_k = 2\hat\rho$, so we have a contradiction to Lemma 3.4. Suppose that (10) fails for $\sigma \supsetneq \langle \hat\rho \rangle$; that is, we have $\langle \hat\rho, \rho'_1, \dots, \rho'_j \rangle \in \Delta$ but $\langle \rho_1, \dots, \rho_{k-1}, \hat\rho, \rho'_1, \dots, \rho'_j \rangle \notin \Delta$. Then (rearranging indices further) there is a primitive set composed of $D_1$, some subset of $\{D_2, \dots, D_{k-1}, \hat D\}$, and (without loss of generality) all of $\{D'_1, \dots, D'_j\}$ with $j$ positive. Therefore,

$$\rho_1 + c_2\rho_2 + \cdots + c_{k-1}\rho_{k-1} + \hat c\hat\rho + \rho'_1 + \cdots + \rho'_j = \tilde\rho$$

for some $\tilde\rho$ and some $c_2, \dots, c_{k-1}, \hat c \in \{0, 1\}$. We now have

$$\tilde\rho + (1 - c_2)\rho_2 + \cdots + (1 - c_{k-1})\rho_{k-1} + \rho_k + (1 - \hat c)\hat\rho = 2\hat\rho + \rho'_1 + \cdots + \rho'_j.$$

This contradicts Lemma 3.4. □

EXERCISE 3.6. Produce a 3-dimensional toric variety $X$, satisfying the conditions of Theorem 3.1, such that there is a blow-down of an exceptional divisor $X \to X'$ with $X'$ nonsingular and projective but not Fano. For a characterization of when the blow-down of a Fano toric variety fails to be Fano, see [Sa].

LEMMA 3.7. *Assume $X$ satisfies the conditions of Theorem 3.1. Let $\{D_1, \dots, D_j\}$ and $\{\hat D_1, \dots, \hat D_k\}$ be distinct primitive sets, and suppose that $\rho_1 + \cdots + \rho_j = \rho'$ and $\hat\rho_1 + \cdots + \hat\rho_k = \hat\rho'$ are the corresponding primitive relations. If $\rho'$ and $\hat\rho'$ are equal or span a cone of $\Delta$, then $\{\rho_1, \dots, \rho_j\} \cap \{\hat\rho_1, \dots, \hat\rho_k\} = \emptyset$.*

*Proof*. Suppose not: $\rho_1 = \hat\rho_1$, say. In the case $\rho' = \hat\rho'$ we find $\rho_2 + \cdots + \rho_j = \hat\rho_2 + \cdots + \hat\rho_k$; a contradiction. If $\rho' \ne \hat\rho'$ then, by Proposition 3.5, the fact that $\langle \rho', \hat\rho' \rangle \in \Delta$ implies that $\{\rho_2, \dots, \rho_j\} \cup \{\rho', \hat\rho'\}$ and $\{\hat\rho_2, \dots, \hat\rho_k\} \cup \{\rho', \hat\rho'\}$ are two sets of cone generators. Now

$$\rho_2 + \cdots + \rho_j + \hat\rho' = \rho' + \hat\rho' - \rho_1 = \hat\rho_2 + \cdots + \hat\rho_k + \rho',$$

and we have a contradiction. □

PROPOSITION 3.8. *Assume that $X$ satisfies the conditions of Theorem 3.1. Then the following statements are equivalent*.

(i) *Every blow-down of $X$ along an exceptional divisor produces a nonsingular Fano toric variety.*
(ii) *Every blow-down of $X$ along an exceptional divisor produces a nonsingular toric variety which* (a) *is Fano,* (b) *satisfies the condition that all of its toric subvarieties are Fano, and* (c) *is such that every blow-down of an exceptional divisor is nonsingular Fano.*
(iii) *Every ray generator of $\Delta$ appears on the right-hand side of at most one primitive relation of $X$.*

*Proof*. Since a Fano toric variety is determined uniquely by the set of ray generators, we have (i) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (i) is clear. We obtain (iii) $\Rightarrow$ (ii) from the characterization of how primitive relations behave under blow-down. By [Sa, Cor. 4.9], if $X \to X'$ is the blow-down corresponding to the primitive relation $\rho_1 + \cdots + \rho_k = \hat{\rho}$, then the primitive sets of $X'$ are precisely the primitive sets of $X$ not containing $\hat{D}$ (other than $\{D_1, \ldots, D_k\}$), plus the sets $S' := (S \setminus \{\hat{D}\}) \cup \{D_1, \ldots, D_k\}$ (disjoint union, by Lemma 3.7) for some (though perhaps not all) primitive sets $S$ containing $\hat{D}$. For such $S$ and $S'$ (primitive sets for $X$ and $X'$, respectively), the respective primitive relations have the same right-hand sides. Given (iii), then, every blow-down of an exceptional divisor is a toric variety that satisfies condition (ii) of Theorem 3.1 and also condition (iii) of this proposition and hence, by induction on the number of toric divisors, is a Fano toric variety all of whose toric subvarieties and toric blow-downs along divisors are Fano. $\square$

Let $X$ be a toric variety satisfying the conditions of Theorem 3.1, and suppose that each exceptional divisor can be blown down in exactly one way. Then, by Proposition 3.8, we can perform a sequence of blow-downs

$$X = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0$$

and so finally obtain the toric variety $X_0$, which satisfies the conditions of Theorem 3.1 and has no exceptional divisors. Now, by Theorem 3.1(ii), the absence of exceptional divisors implies that every linearly independent set of ray generators spans a cone of $\Delta$. It is apparent, then, that $X_0$ is isomorphic to a product of projective spaces.

By Lemma 3.3, for any iterated blow-down $X'$ of $X$ dominating $X_0$, every toric divisor $D'$ on $X'$ with $\langle \rho' \rangle \notin \Delta_0$ must be exceptional. Hence, starting with $X$, the divisors $\{ D \mid \langle \rho \rangle \notin \Delta_0 \}$ can be blown down in any order to yield a succession of birational morphisms of toric varieties, with each variety satisfying the conditions of Proposition 3.8 and terminating with $X_0$. The results of this section are summarized in the following statement.

THEOREM 3.9. *Let $X$ be a complete nonsingular toric variety. Then the following are equivalent.*

(i) *$X$ is Fano, every toric subvariety of $X$ is Fano, and every nonsingular toric variety $X'$ dominated by $X$, such that $X \to X'$ is the blow-down of a toric divisor, is Fano.*

(ii) *The fan associated to $X$ satisfies: for every primitive set $\{D_1, \ldots, D_k\}$ we have either $\rho_1 + \cdots + \rho_k = 0$ or $\rho_1 + \cdots + \rho_k = \rho'$ for some ray generator $\rho'$, with every $\rho'$ equal to $\rho_1 + \cdots + \rho_j$ for at most one primitive set $\{D_1, \ldots, D_j\}$.*

*Moreover, if $X$ satisfies* (i) *and* (ii)*, then $X$ is an iterated blow-up of a product of projective spaces, along irreducible toric subvarieties, such that the exceptional divisors of the blow-up can be blown down in any order, and every intermediate blow-up is a toric variety satisfying* (i) *and* (ii).

## 4. Rational Curves on Toric Varieties

### *4.1. Curves Joining a Point and a Divisor*

We need the following result, which characterizes the lowest possible degree of a stable, torus-invariant genus-0 curve joining a toric point to a toric divisor. Degree of a curve refers to degree under the anticanonical embedding: $\deg \beta = \int_\beta (-K_X)$.

PROPOSITION 4.1. *Let $X$ be a toric variety satisfying the conditions of Theorem 3.1. Let $\mu = \langle \rho_1, \ldots, \rho_n \rangle$ be a maximal cone of $\Delta$ corresponding to the toric point $x = X(\mu)$, and let us give $N$ coordinates so that $\rho_i$ is the $i$*th *standard basis vector for each $i$. Let $D$ be a toric divisor with corresponding ray generator $\rho = (\rho^{(1)}, \ldots, \rho^{(n)})$ in coordinates. Then there is a tree of toric $\mathbb{P}^1$'s joining $x$ to a point of $D$ and having degree $1 - \sum_{i=1}^n \rho^{(i)}$ and homology class $\beta$ given by*

$$\begin{cases} \beta = 0 & \text{if } \rho \in \{\rho_1, \ldots, \rho_n\}, \\ \int_\beta D = 1,\ \int_\beta D_i = -\rho^{(i)}\ \forall i, \quad \int_\beta D' = 0\ \forall D' \notin \{D_1, \ldots, D_n, D\} & \text{otherwise.} \end{cases} \tag{11}$$

*Any tree of toric $\mathbb{P}^1$'s that joins $x$ to a point of $D_i$ having homology class not equal to $\beta$ must have degree larger than $1 - \sum_{i=1}^n \rho^{(i)}$.*

*Proof*. For a maximal cone $\mu'$, let $\Sigma_{\mu'}$ denote the affine span of the generators of $\mu'$ and let $\mathrm{dist}(-, \Sigma_{\mu'})$ denote (signed) integer distance to $\Sigma_{\mu'}$ in $N$. Then the quantity $1 - \sum_{i=1}^n \rho^{(i)}$ appearing in the statement is $\mathrm{dist}(\rho, \Sigma_\mu)$. We prove the statement by induction on the degree $d$ of a tree of $\mathbb{P}^1$'s. The induction hypothesis is: (i) that, given any tree $C$ of $\mathbb{P}^1$'s of total degree $< d$ meeting $D$, the toric point $X(\mu')$ lies in $C$ only if $\mathrm{dist}(\rho, \Sigma_{\mu'}) \le \deg C$ for any maximal cone $\mu'$; (ii) if $\mathrm{dist}(\rho, \Sigma_{\mu'}) = \deg C < d$ and $X(\mu') \in C$ then the homology class of $C$ is that indicated in (11); and (iii) for any maximal cone $\mu'$ with $\mathrm{dist}(\rho, \Sigma_{\mu'}) < d$, there exists a tree of $\mathbb{P}^1$'s that join the corresponding toric point to a point of $D$ and have degree equal to $\mathrm{dist}(\rho, \Sigma_{\mu'})$.

Let $C$ be a tree of $\mathbb{P}^1$'s, of total degree $d$, joining $x$ to a point of $D$. It suffices to assume that $C = C_0 \cup C_1$, where $C_0$ is a toric $\mathbb{P}^1$ joining $x$ to $y$ for some toric point $y$, and that $C_1$ is a tree of $\mathbb{P}^1$'s joining $y$ to a point of $D_i$. Shuffling coordinates, we may suppose $C_0 = X(\sigma)$, where $\sigma = \langle \rho_2, \ldots, \rho_n \rangle$. Denote the additional generator of the maximal cone $\mu'$ corresponding to $y$ by $\rho_{n+1}$ (i.e., $\mu' = \langle \sigma, \rho_{n+1} \rangle$), and let us write $\rho_{n+1} = (-1, a^{(2)}, \ldots, a^{(n)})$ in coordinates. Then $C_0$ has intersection numbers 1 with $D_1$ and with $D_{n+1}$ and $-a^{(i)}$ with $D_i$ for $2 \le i \le n$. Hence, $\deg C_0 = \mathrm{dist}(\rho_{n+1}, \Sigma_\mu) = 2 - \sum_{i=2}^n a^{(i)}$. We claim

$$\mathrm{dist}(\rho, \Sigma_\mu) \le \mathrm{dist}(\rho, \Sigma_{\mu'}) + \mathrm{dist}(\rho_{n+1}, \Sigma_\mu), \tag{12}$$

with equality if and only if $\rho^{(1)} = -1$. This is a computation: $\mathrm{dist}(\rho, \Sigma_{\mu'}) = 1 + \rho^{(1)} - \sum_{i=2}^n (\rho^{(i)} + a^{(i)} \rho^{(1)})$, so the right-hand side minus the left-hand side of (12) equals

$$1+\rho^{(1)}-\sum_{i=2}^{n}(\rho^{(i)}+a^{(i)}\rho^{(1)})+2-\sum_{i=2}^{n}a^{(i)}-\Bigl(1-\sum_{i=1}^{n}\rho^{(i)}\Bigr)$$
$$=(\rho^{(1)}+1)\Bigl(2-\sum_{i=2}^{n}a^{(i)}\Bigr)$$

and by Theorem 3.1(iii) we have $\rho^{(1)}+1\geq 0$. By the induction hypothesis, then, we have $\deg C\geq \operatorname{dist}(\rho,\Sigma_\mu)$ with equality only if $\rho^{(1)}=-1$ and that the homology class $\beta_1=[C_1]$ satisfies $\beta_1=0$ if $\rho=\rho_{n+1}$; otherwise, $\int_{\beta_1}D=1$, $\int_{\beta_1}D_{n+1}=-1$, $\int_{\beta_1}D_i=-\rho^{(i)}+a^{(i)}$ for $2\leq i\leq n$, and $\int_{\beta_1}D'=0$ for all other $D'$. Therefore, $\beta=[C]=[C_0]+[C_1]$ satisfies (11).

For the existence portion of the inductive step, if $\operatorname{dist}(\rho,\Sigma_\mu)>0$ then $\rho$ must have some coordinate equal to $-1$ and so, without loss of generality, we have $\rho^{(1)}=-1$. We can now take $C$ to be the union of $C_0$ (as defined in the previous paragraph) and a tree $C_1$ of $\mathbb{P}^1$'s joining $y$ to a point of $D$ satisfying $\deg C_1=\operatorname{dist}(\rho,\Sigma_{\mu'})$ (the existence of such $C_1$ follows from the induction hypothesis). □

COROLLARY 4.2. *Assume $X$ satisfies the conditions of Theorem 3.1. Suppose $\beta\in H_2(X,\mathbb{Z})$, and suppose the toric divisors that $\beta$ intersects negatively have nonempty common intersection. Then $\beta$ is represented by a tree of toric $\mathbb{P}^1$'s.*

*Proof.* Let $\{\rho\mid\int_\beta D<0\}=\{\rho_1,\dots,\rho_j\}$, and let $\mu$ be a maximal cone containing $\rho_1,\dots,\rho_j$ with $x=X(\mu)$. For each ray generator $\rho$, let $C_\rho$ be a tree of $\mathbb{P}^1$'s that join $x$ to a point of $D$ and with $\deg C_\rho=\operatorname{dist}(\rho,\Sigma_\mu)$. For each $\rho\notin\mu$, let $a_\rho=\int_\beta D$; we have $a_\rho\geq 0$ for all $\rho\notin\mu$. Now the sum over all $\rho\notin\mu$ of $a_\rho$ copies of $C_\rho$ has homology class $\beta$. □

EXERCISE 4.3. Prove Corollary 4.2 for an arbitrary nonsingular projective toric variety $X$. (The trees $C_\rho$ are constructed as in the existence portion of the inductive step in the proof of Proposition 4.1, except that the $\mathbb{P}^1$ joining toric points $x$ and $y$ is given multiplicity $-\rho^{(1)}$, where ordering of coordinates is chosen so that $\rho^{(1)}<0$.) In particular, every primitive homology class is represented by a tree of $\mathbb{P}^1$'s; see Theorem 2.4.

### 4.2. *Quantum Giambelli*

Here we prove Theorem 1.6(ii). Let $D_1,\dots,D_k$ be toric divisors such that $\rho_1,\dots,\rho_k$ span a cone of $\Delta$. Recall the two facts about quantum cohomology we use. First, for $0\neq\beta\in H_2(X,\mathbb{Z})$ and $\omega\in H^*(X,\mathbb{Q})$, if $D$ is a toric divisor satisfying $\int_\beta D=0$ then the coefficient of $q^\beta$ in $D\cdot\omega$ is 0. Second, if—in the fiber of the moduli space of stable maps $\bar{M}_{0,k+1}(X,\beta)$ over a general point of $\bar{M}_{0,k+1}$ (via the morphism that forgets the map of the curve to $X$ and stabilizes; cf. [FP] for notation and definition)—the intersection $ev_1^{-1}(D_1)\cap\cdots\cap ev_k^{-1}(D_k)\cap ev_{k+1}^{-1}(T)$ is empty for every $T$ among a collection of cycles representing a basis of $H_{2(k-\deg\beta)}(X,\mathbb{Q})$, then the coefficient of $q^\beta$ in $D_1\cdots D_k$ is 0. If the cycles $T$ are toric subvarieties then, to deduce that the intersection is empty, it suffices to verify that the intersection contains no fixed points for the torus action on $\bar{M}_{0,k+1}(X,\beta)$.

Definition 4.4. We say that a collection of exceptional sets $\{S_1, \ldots, S_t\}$ has an *overlap* if the exceptional divisor for $S_i$ is an element of $S_j$ for some $i$ and $j$ in $\{1, \ldots, t\}$. Otherwise, we say the set of exceptional sets *has no overlaps.* We also refer to a set of exceptional curves as having an overlap or not having overlaps, depending on whether the associated set of exceptional sets has or does not have overlaps.

Remark 4.5. Fixing a cone $\sigma$, the exceptional classes that are special for $\sigma$ are linearly independent. Indeed, it suffices to consider $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$, a maximal cone. Let us enumerate the toric divisors as $\{D_1, \ldots, D_n, D_{n+1}, \ldots, D_m\}$. Then $D_{n+1}, \ldots, D_m$ are linearly independent in $H^2(X, \mathbb{Q})$. Each special exceptional class has intersection number 1 with exactly one of $D_{n+1}, \ldots, D_m$ and 0 with all the rest.

Remark 4.6. Every exceptional curve class meets the conditions of Proposition 2.3 and hence is effective and is a nonnegative integer combination of primitive classes. Suppose now that $X$ satisfies the conditions of Theorem 3.9. Let $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$ be a maximal cone, and let us enumerate the divisors of $X$ as $\{D_1, \ldots, D_n, D_{n+1}, \ldots, D_m\}$. The following observations are immediate. First, no effective curve class has negative intersection pairing with $D_{n+1} + \cdots + D_m$. Second, any effective curve class having zero intersection with $D_{n+1} + \cdots + D_m$ must have nonnegative intersection with each of $D_1, \ldots, D_n$. Consequently, if $S$ is a special exceptional set for $\sigma$ with exceptional divisor $D_i$ $(1 \le i \le n)$, then (a) the (unique) primitive set $S'$ with exceptional divisor $D_i$ is a special exceptional set for $\sigma$ and (b) $S' \cap \{D_1, \ldots, D_n\} \subset S$. In particular, any two special exceptional sets with the same exceptional divisor must have some elements in common. Also, the reader should verify (by inductive application of Proposition 3.5 and Lemma 3.7), that any two special exceptional sets with different exceptional divisors and no cycle must be disjoint.

We first need a technical lemma.

Lemma 4.7. *Let $\sigma = \langle \rho_1, \ldots, \rho_k \rangle$ be a cone of $\Delta$. Suppose $\{\beta'_1, \ldots, \beta'_s\}$ is a set of special exceptional classes for $\sigma$. Let $\{\beta_1, \ldots, \beta_t\}$ be a set of exceptional classes such that each associated exceptional set $S_i$ satisfies $|S_i \cap \{D_1, \ldots, D_k\}| = |S_i| - 1$, and suppose that $\int_{\beta_1} D_1 = -1$. If*

$$\beta_1 + \cdots + \beta_t = \beta'_1 + \cdots + \beta'_s,$$

*then at least one of the $\beta'_i$ has nonzero intersection pairing with $D_1$.*

*Proof*. Suppose not. Since $|S_1 \cap \{D_1, \ldots, D_k\}| = |S_1| - 1$ and $\int_{\beta_1} D_1 = -1$, it follows that $\beta_1$ is special for $\sigma$. By Remark 4.5, then, if $\tilde{D}_1$ denotes the unique element of $S_1$ not in $\{D_1, \ldots, D_k\}$, then $\int_{\beta'_i} \tilde{D}_1 = 0$ for every $i$. So $\sum_{j=1}^{t} \int_{\beta_j} \tilde{D}_1 = 0$, and hence some $\beta_j$ has intersection number $-1$ with $\tilde{D}_1$. It follows without loss of generality that $\int_{\beta_2} \tilde{D}_1 = -1$. Then $\beta_1 + \beta_2$ is special exceptional or very effective, with (say) $\tilde{D}_2$ the unique element of the associated exceptional set not in

$\{D_1, \ldots, D_k\}$. As before, $\int_{\beta_i'} \tilde{D}_2 = 0$ for every $i$, and we may iterate this process. We eventually reach a contradiction. □

The quantum Giambelli formula follows quickly from the following pair of propositions, whose proofs occupy the bulk of this section.

PROPOSITION 4.8. *Let $X$ be a toric variety satisfying the conditions of Theorem 3.9. Let $D_1, D_2, \ldots, D_k$ be toric divisors such that corresponding ray generators $\rho_1, \ldots, \rho_k$ span a cone $\sigma \in \Delta$. Then a term $q^\beta$ appears with nonzero ($H^*(X, \mathbb{Q})$-valued) coefficient in the quantum product $D_1 \cdot D_2 \cdots D_k$ only if $\beta = \beta_1 + \cdots + \beta_t$, for some $t$, such that the $\beta_i$ are special (for $\sigma$) exceptional classes that have distinct exceptional divisors and no overlaps.*

PROPOSITION 4.9. *Let $X$ be a toric variety satisfying the conditions of Theorem 3.9. Then the quantum Giambelli formula* (5) *of Theorem 1.6(ii) holds in $QH^*(X)$. Moreover, we have the formula in $QH^*(X)$:*

$$D_1 \cdot D_2 \cdots D_k = \sum_{\{\beta_1, \ldots, \beta_t\}} (-1)^t q^{\beta_1 + \cdots + \beta_t} D_{\{1 \le i \le k \,|\, \int_{\beta_1 + \cdots + \beta_t} D_i \ne 1\}}, \tag{13}$$

*where the sum is over sets of special exceptional classes $\{\beta_1, \ldots, \beta_t\}$ that have distinct exceptional divisors and no overlaps and where $D_I$, for an index set $I$, denotes the cohomology class Poincaré dual to $\bigcap_{i \in I} D_i$.*

We prove Propositions 4.8 and 4.9 jointly, by induction on $k$. For each $k \ge 1$, Proposition 4.8 is proved assuming the statements of Propositions 4.8 and 4.9 for smaller $k$. Then, for each $k$, we deduce Proposition 4.9 for the case of products of $k$ divisors.

Let the maximal cones of $\Delta$ be $\mu_1, \ldots, \mu_s$, with corresponding points $y_1, \ldots, y_s \in M$. Let $\rho$ be a nonzero vector of $N$. Let $\rho'$ be a small perturbation of $\rho$, so that $y_1(\rho'), \ldots, y_s(\rho')$ are all distinct, and let the indices be assigned so that

$$y_1(\rho') > y_2(\rho') > \cdots > y_s(\rho'). \tag{14}$$

For each $i$, let $\tau_i = \mu_i \cap \Big( \bigcap_{\substack{j > i \\ \dim(\mu_j \cap \mu_i) = n-1}} \mu_j \Big)$.

LEMMA 4.10 [F, Sec. 5.2]. *If $X$ is a nonsingular Fano toric variety, then the classes $[X(\tau_i)]$ $(1 \le i \le s)$ form a $\mathbb{Z}$-basis for $H_*(X, \mathbb{Z})$. Moreover, for any $i$ and $j$, if $\tau_i \subset \mu_j$ then $i \le j$.*

This is the basis for homology that we use to detect which $q^\beta$ terms occur in a quantum product of divisors. In using this basis, it is convenient to perform computations in coordinates. Given a maximal cone $\mu_i$, we give $N$ coordinates so that the generators of $\mu_i$ are the $n$ standard basis elements. Then, in dual coordinates, $y_i = (1, 1, \ldots, 1)$. Now suppose $\mu_j$ is a neighboring maximal cone; that is, $\sigma := \mu_j \cap \mu_i$ has dimension $n - 1$. Hence $\mu_j$ is generated by $n - 1$ of the generators of $\mu_i$, say all except the $\nu$th standard basis element; there is one additional generator, $(a^{(1)}, \ldots, a^{(\nu)} = -1, \ldots, a^{(n)})$. It follows that $y_j = \big(1, \ldots, 1, \sum_{\ell=1}^{n} a^{(\ell)}, 1, \ldots, 1\big)$ in the dual coordinates we are using, where the entry $\sum_{\ell=1}^{n} a^{(\ell)}$ appears in the $\nu$th position. Thus,

$$y_i - y_j = (0, \ldots, 0, \deg X(\sigma), 0, \ldots, 0)$$

in coordinates. The degree of $X(\sigma)$ is positive. Hence, for any $i$, the cone $\tau_i$ has dimension equal to the number of negative entries in the coordinate expression for $\rho'$ with respect to the coordinates dictated by $\mu_i$.

We are interested in knowing how large $\dim \tau_j - \dim \tau_i$ can be.

LEMMA 4.11. *Suppose $X$ is a toric variety satisfying the conditions of Theorem 3.9. Let the maximal cones $\{\mu_i\}$ be ordered with respect to pairings with $\rho'$ as in* (14). *Suppose cones $\mu_i$ and $\mu_j$ intersect in an $(n-1)$-dimensional cone $\sigma$. Then* $\dim \tau_j - \dim \tau_i \leq \deg X(\sigma)$; *equality implies that $X(\sigma)$ is an exceptional curve, special for $\mu_i$, and the following condition on coordinates of $\rho'$ must be satisfied. Let coordinates for $N$ be assigned such that the generators of $\mu_i$ are the standard basis vectors, the generators of $\mu_j$ are the second through $n$*th *standard basis vectors, and* $(-1, -1, \ldots, -1, 1, 0, \ldots, 0)$; *the number of $-1$'s is equal to* $d := \deg X(\sigma)$. *Then, the first $d$ coordinates of $\rho'$ must be positive, with the first coordinate larger than any of the second through $d$*th *coordinates*; *moreover, the* $(d+1)$th *coordinate must either be positive or else negative and larger in absolute value than the first coordinate. The change of coordinates to the coordinate system of $\mu_j$ has the effect of negating the first coordinate, making the second through $d$*th *coordinates negative, preserving the sign of the* $(d+1)$th *coordinate and leaving the remaining coordinates unchanged.*

*Proof.* We know that, in the coordinate system dictated by $\mu_i$, $\dim \tau_i$ is the number of negative entries in the coordinate expression for $\rho'$. Let us suppose that $\mu_j$ is generated by the second through $n$th standard basis elements plus one additional vector. By Theorem 3.1(iii), there are two possibilities. First, the additional generator can be of the form $(-1, \ldots, -1, 0, \ldots, 0)$; the number of $-1$'s is $d-1$ and in this case $X(\sigma)$ is not exceptional. The change of coordinates to the coordinate system of $\mu_j$ preserves the last $n-d+1$ entries of $\rho'$. Hence $|\dim \tau_j - \dim \tau_i| \leq d-1$.

In the remaining case, the additional generator of $\mu_j$ is

$$(-1, \ldots, -1, 1, 0, \ldots, 0),$$

where the number of $-1$'s is $d$. In this case, $X(\sigma)$ is exceptional. If, in the coordinates of $\mu_i$, $\rho'$ is

$$(a^{(1)}, \ldots, a^{(d+1)}, a^{(d+2)}, \ldots, a^{(n)})$$

then, in the coordinates of $\mu_j$, the coordinate expression is

$$(-a^{(1)}, a^{(2)} - a^{(1)}, \ldots, a^{(d)} - a^{(1)}, a^{(d+1)} + a^{(1)}, a^{(d+2)}, \ldots, a^{(n)}).$$

So $\dim \tau_j - \dim \tau_i \leq d$, with equality only if $a^{(1)} > 0$, with additionally $0 < a^{(\ell)} < a^{(1)}$ for $2 \leq \ell \leq d$ and either $a^{(d+1)} > 0$ or $a^{(d+1)} < -a^{(1)}$. □

We can now prove Proposition 4.8 for the case of $k$ divisors, assuming the statements of Propositions 4.8 and 4.9 for fewer than $k$ divisors. Let $D_1, \ldots, D_k$ be toric divisors such that $\sigma := \langle \rho_1, \ldots, \rho_k \rangle$ is in $\Delta$. Let $\rho = \rho_1 + \cdots + \rho_k$. Let $\rho'$ be a perturbation of $\rho$, and let the maximal cones $\mu_i$ be ordered as in (14).

Suppose $\beta \in H_2(X, \mathbb{Z})$. Define $T_{\beta,j} = T_{\beta,j}(D_1, \ldots, D_k)$ to be the set of stable maps

$$(\varphi\colon C \to X;\, p_1, \ldots, p_{k+1} \in C) \in \bar{M}_{0,k+1}(X, \beta),$$

invariant for the torus action, with the $i$th marked point mapping into $D_i$ for $i = 1, \ldots, k$ and the $(k+1)$th marked point mapping into $X(\tau_j)$ and such that, when we forget the map to $X$ and stabilize $C$, all the marked points collapse to a single distinguished irreducible component $C_0$ of $C$. The important thing is that we know the coefficient of $q^\beta$ in the quantum product $D_1 \cdots D_k$ is zero unless

$$\dim \tau_j = n - k + \deg \beta \quad \text{for some } j \text{ such that } T_{\beta,j} \neq \emptyset.$$

LEMMA 4.12. *Suppose $X$ satisfies the hypotheses of Theorem 3.9. Let $D_1, \ldots, D_k$ be toric divisors with $D_1 \cap \cdots \cap D_k \neq \emptyset$ and, for $\beta \in H_2(X, \mathbb{Z})$ and $j \in \{1, \ldots, s\}$, let $T_{\beta,j}$ be as previously defined. Then we have*

$$\dim \tau_j \leq n - k + \deg \beta$$

*for every $\beta$ and $j$ such that $T_{\beta,j} \neq \emptyset$. Moreover, given $(\varphi\colon C \to X) \in T_{\beta,j}$ such that $\dim \tau_j = n - k + \deg \beta$, there exists a chain of exceptional curves $X(\sigma_i)$ $(i = 1, \ldots, t)$ on $X$, for some $t$, joining a point on $D_1 \cap \cdots \cap D_k$ to the point $\varphi(p_{k+1}) \in X(\tau_j)$ with total homology class $\beta$ (by "chain" we mean a tree with each irreducible component joined to at most two others; a chain joins two points if removing the indicated points preserves the connectedness of the chain) and such that each $X(\sigma_i)$ has positive intersection with exactly $d_i := \deg X(\sigma_i)$ of the divisors $D_1, \ldots, D_k$ and such that each of divisors in $\{D_1, \ldots, D_k\}$ has positive intersection with at most one of the exceptional curves in the chain.*

*Proof.* Let $\varphi\colon C \to X$ be a torus-invariant genus-0 stable $(k+1)$-pointed map, which stabilizes (upon forgetting the map to $X$) to $k+1$ distinct points on a single irreducible component $C_0 \subset C$, such that the $i$th marked point maps into $D_i$ for $1 \leq i \leq k$ and such that the image of the $(k+1)$th point is $X(\mu_{j'}) \in X(\tau_j)$. By Lemma 4.10, $j \leq j'$ and, in fact (exercise), there exist $j = j_0 < j_1 < \cdots < j_\ell = j'$ for some $\ell$ such that $\dim(\mu_{j_\nu} \cap \mu_{j_{\nu+1}}) = n - 1$, $y_{j_\nu}(\rho') > y_{j_{\nu+1}}(\rho')$, and $\dim \tau_{j_\nu} \leq \dim \tau_{j_{\nu+1}}$ for each $\nu$ (for the last assertion, use (iii) of Theorem 3.1). Hence it suffices to prove $\dim \tau_{j'} \leq n - k + \deg \beta$.

We induct on the degree of $\beta$. The base case is the inequality $k \leq \dim X(\tau_j)$ for every $j$ such that $\langle \rho_1, \ldots, \rho_k \rangle \subset \mu_j$. This is immediate from the characterization of $\dim \tau_j$ as the number of negative entries in the corresponding coordinate expression for $\rho'$. Equality holds only when the coordinate expression for $\rho'$ has exactly $k$ positive entries, each close to 1, and $n-k$ negative entries, each small in magnitude.

We divide the inductive step into two cases. Suppose $(\varphi\colon C \to X) \in T_{\beta,j}$. For the first case, assume the $(k+1)$th marked point $p_{k+1}$ does not lie on the distinguished component $C_0$. Let $C'$ denote the connected component of $C \setminus \{p_{k+1}\}$ containing $C_0$, with the $\mathbb{P}^1$ terminating in $p_{k+1}$ deleted. Assume that this $\mathbb{P}^1$ maps to the toric curve $X(\omega)$ with

$$\omega = \mu_i \cap \mu_{j'}; \quad X(\mu_i) \neq ev_{k+1}(C), \ \ X(\mu_{j'}) = ev_{k+1}(C).$$

Let $\beta'$ denote the homology class of $C'$. Then, by induction,

$$\dim \tau_i \leq n-k+\deg \beta'.$$

By Lemma 4.11, $\dim \tau_{j'} \leq n-k+\deg \beta' + \deg X(\omega) \leq n-k+\deg \beta$ and so the inequality is established. If equality holds, then $X(\omega)$ is exceptional and $C$ is equal to the union of $C'$ and a $\mathbb{P}^1$ mapping with degree 1 to $X(\omega)$. By induction, $C'$ is equivalent in homology to a chain $\tilde{C}'$ of toric curves, each exceptional, joining a point on $D_1 \cap \cdots \cap D_k$ to the point $X(\mu_i)$. Also, equality implies that there are precisely $d := \deg X(\omega)$ divisors $D_\nu \in \{D_1, \ldots, D_k\}$ having positive intersection with $X(\omega)$ and, for any of these, the corresponding $\rho_\nu$ is a generator of $\mu_i$ whose corresponding entry in the coordinate expression of $\rho'$ is positive. It follows that each of these $D_\nu$ has nonpositive intersection with every component of $\tilde{C}'$.

The second case is when $p_{k+1} \in C_0$. As before, let $X(\mu_{j'})$ denote the image of the $(k+1)$th marked point. Choose coordinates on $N$ so that the generators of $\mu_{j'}$ are the standard basis elements, and order these so that $\rho$ has negative first coordinate, $\rho^{(1)} = -c$, with $c \geq 1$. Let $\omega$ be the cone generated by the second through $n$th basis elements; we have $\omega = \mu_{j'} \cap \mu_i$ for some (unique) $i$. Let $d = \deg X(\omega)$. Then $y_i(\rho) - y_{j'}(\rho) = cd$, so in particular $y_i(\rho) - y_{j'}(\rho) \geq d$. Let $C' = C'_1 \cup \cdots \cup C'_k$, where $C'_\nu$ is the tree of $\mathbb{P}^1$'s joining $X(\mu_i)$ to $D_\nu$, as given in Proposition 4.1. The degree of $C'$ is $k - y_i(\rho)$. Hence, the union of $C'$ and $X(\omega)$ is (more precisely, determines) a torus-invariant genus-0 $(k+1)$-pointed stable map whose homology class $\beta'$ satisfies $\deg \beta' = k - y_i(\rho) + d \leq k - y_{j'}(\rho) \leq \deg \beta$, by Proposition 4.1. Moreover, the $(k+1)$th marked point now does not lie on the distinguished component. By the previous case, we have $\dim \tau_j \leq n-k+\deg \beta'$, and the desired equality holds. In case of equality we must have $c = 1$ and $\beta'$ equal to the sum of the homology classes of the curves joining $X(\mu_{j'})$ to $D_1, \ldots, D_k$ of Proposition 4.1, and then we find $\beta' = \beta$. Thus, we are reduced to the previous case. □

Suppose now that the coefficient $c_\beta$ of $q^\beta$ in the quantum product $D_1 \cdots D_k$ is nonzero. By Lemma 4.12, then, $\beta$ is a sum of exceptional curve classes, $\beta = \beta_1 + \cdots + \beta_t$, such that each corresponding primitive set $S_i$ satisfies $|S_i \cap \{D_1, \ldots, D_k\}| = |S_i| - 1$. It remains to show that whenever $i \neq j$ we have $\big(\int_{\beta_i} D_\nu\big)\big(\int_{\beta_j} D_\nu\big) = 0$ for all $1 \leq \nu \leq k$. We must also show that $\beta$ is a sum of special exceptional classes. Suppose, first, that for some $\nu$ $(1 \leq \nu \leq k)$ we have $\big(\int_{\beta_i} D_\nu\big)\big(\int_{\beta_j} D_\nu\big) \neq 0$ for some $i \neq j$. We cannot have $\big(\int_{\beta_i} D_\nu\big)\big(\int_{\beta_j} D_\nu\big) > 0$

(the sets $S_i \cap \{D_1, \ldots, D_k\}$ are pairwise disjoint, and Remark 4.6 rules out $D_\nu$ being exceptional for both $\beta_i$ and $\beta_j$). Thus, without loss of generality, $\int_{\beta_i} D_1 = 1$ and $\int_{\beta_j} D_1 = -1$. It follows that $\int_\beta D_1 = 0$. Applying quantum Giambelli to the $k-1$ divisors $D_2, \ldots, D_k$, we find

$$D_2 \cdots D_k = D_{\{2,3,\ldots,k\}} - \sum_{\emptyset \neq \{S'_1, \ldots, S'_{t'}\}} q^{\beta'_1 + \cdots + \beta'_{t'}} \prod_{2 \le i' \le k,\ D_{i'} \notin S'_1 \cup \cdots \cup S'_{t'}} D_{i'}$$

(notation similar to that of (5)). The coefficient of $q^\beta$ in $D_1 \cdot D_{\{2,3,\ldots,k\}}$ is zero because $\int_\beta D_1 = 0$. The coefficient of $q^\beta$ in each additional term is zero because no sum of special exceptional classes, each having intersection number 0 with $D_1$, can be equal to $\beta$ (Lemma 4.7).

We show by induction on $t$ that $\beta = \beta_1 + \cdots + \beta_t$ can be written as a sum of special exceptional classes (then, by the previous paragraph, the set of special exceptional classes in this sum has no overlaps). Write $\beta_1 + \cdots + \beta_{t-1} = \beta'_1 + \cdots + \beta'_s$ with each $\beta'_j$ special. If the exceptional divisor of $\beta_t$ is in $\{D_1, \ldots, D_k\}$, then $\beta_t$ is special. Otherwise, the exceptional divisor intersects some $\beta'$ positively; in this case, $\beta'_j + \beta_t$ is special. By Remark 4.5, the expression of $\beta$ as a sum of special exceptional classes is unique, and by Remark 4.6, the $\beta'_j$ have distinct exceptional divisors and pairwise disjoint exceptional sets.

We complete the proof of Proposition 4.9 for the case of $k$ divisors by demonstrating (13) and then deducing quantum Giambelli from (13). Let $\beta = \beta_1 + \cdots + \beta_t$ be a sum of special exceptional classes with distinct exceptional divisors and no overlaps. We need to show that the coefficient of $q^{\beta_1 + \cdots + \beta_t}$ in $D_1 \cdots D_k$ is $(-1)^t D_{\{1 \le i \le k \mid \int_\beta D_i \neq 1\}}$. (We assume the result is known for products of smaller numbers of divisors.) If $\beta$ has zero intersection with some $D_i$, say with $D_1$, then we write

$$D_1 \cdot D_2 \cdots D_k = D_1 \cdot \left[ \sum_{\{\beta'_1, \ldots, \beta'_s\}} (-1)^s q^{\beta'_1 + \cdots + \beta'_s} D_{\{2 \le i \le k \mid \int_{\beta'_1 + \cdots + \beta'_s} D_i \neq 1\}} \right].$$

Note that, on the right-hand side, the curve class $\beta - (\beta'_1 + \cdots + \beta'_s)$ has zero intersection with $D_1$ for every term. Therefore, the coefficient of $q^\beta$ in $D_1 \cdots D_k$ is the classical product of $D_1$ with the coefficient of $q^\beta$ inside the brackets, and this is $(-1)^t D_{\{1 \le i \le k \mid \int_{\beta_1 + \cdots + \beta_t} D_i \neq 1\}}$.

If $\int_\beta D_\nu \neq 0$ for all $1 \le \nu \le k$ and if $t \ge 2$, then we separate off the divisors meeting $\beta_1$, apply (13), and use linear relations (3) to conclude that no term from (13) (save that with maximal $q$-term) contributes anything to the coefficient of $q^\beta$ in $D_1 \cdots D_k$.

For the remaining case, where (with suitable indices) $\{D_1, D_2, \ldots, D_{k-1}, \tilde{D}\}$ is an exceptional set with $\rho_1 + \cdots + \rho_{k-1} + \tilde{\rho} = \rho_k$, we apply a linear relation (3) followed by a $q$-deformed monomial relation (7): $D_1 \cdots D_{k-1} \cdot D_k = D_1 \cdots D_{k-1} \cdot (-\tilde{D} + \cdots) = -q^\beta D_k + \cdots$.

Finally, quantum Giambelli (5) follows from the formula (13) as follows. Applying known cases of quantum Giambelli to (13), we obtain

$$D_{\{1,2,\ldots,k\}} = D_1 \cdots D_k - \sum_{\emptyset \neq \{\beta'_1,\ldots,\beta'_s\}} (-1)^s q^{\beta'} \sum_{\{S_1,\ldots,S_t\}} q^{\beta} \prod_{\substack{\int_{\beta'} D_i \neq 1 \\ D_i \notin S_1 \cup \cdots \cup S_t}} D_i$$

$$= D_1 \cdots D_k - \left[ \sum_{\{\beta'_1,\ldots,\beta'_s\}} (-1)^s \sum_{\{S_1,\ldots,S_t\}} q^{\beta'+\beta} \prod_{\substack{\int_{\beta'} D_i \neq 1 \\ D_i \notin S_1 \cup \cdots \cup S_t}} D_i \right] + (*),$$

where $\beta'$ (resp. $\beta$) denote $\beta'_1 + \cdots + \beta'_s$ (resp. $\beta_1 + \cdots + \beta_t$) with $\beta_j$ the exceptional class associated to $S_j$; where the sums are over sets of exceptional classes, special for $\langle \rho_1, \ldots, \rho_k \rangle$, with distinct exceptional divisors and no overlaps (resp. sets of exceptional sets, special for $\left\langle \rho_i \mid \int_{\beta'} D_i \neq 1 \right\rangle$, with distinct exceptional divisors and no cycles); and where $(*)$ denotes the expression on the right-hand side of (5) from Theorem 1.6(ii). We thus need to show that the quantity in brackets in the right-hand side has no $q$-terms. Fix some curve class $\beta^* \neq 0$, and consider decompositions $\beta^* = \beta' + \beta$ that occur in this term. We may choose a special exceptional class $\gamma$, which is a summand of $\beta^*$, such that if $\int_\gamma D_\nu = 1$ ($1 \le \nu \le k$) then $D_\nu$ is not exceptional for any of special exceptional classes that are summands of $\beta^*$. But now the terms that contribute to the coefficient $q^{\beta^*}$ can be paired off according to whether $\gamma$ is among the $\beta'_i$ or is the exceptional curve class of some $S_j$. Corresponding pairs of terms add with opposite sign, so the total coefficient of $q^{\beta^*}$ is zero in this term, and we have established the quantum Giambelli formula.

### 4.3. *Elementary Derivation of Quantum Cohomology Ring Presentation*

By Proposition 2.6, to prove that relations (4) hold for a given nonsingular projective toric variety $X$ it suffices to establish (6) for every very effective curve class $\beta$; Theorem 1.2 then follows. As promised, we outline here an elementary derivation (not relying upon equivariant localization techniques) of Theorem 1.2 for toric varieties $X$ satisfying the hypotheses of Theorem 3.1. This is essentially the approach outlined in [Ba2].

EXERCISE 4.13. Suppose $X$ satisfies the hypotheses of Theorem 3.1. Let $\beta \in H_2(X, \mathbb{Z})$ be a very effective curve class. Let $D_1, \ldots, D_m$ denote the toric divisors of $X$, and set $a_i = \int_\beta D_i$ for $i = 1, \ldots, m$. Obtain the relation

$$D_1^{a_1} \cdots D_m^{a_m} = q^\beta$$

in $QH^*(X)$ by the following four steps.

(i) If we write $D_1^{a_1} \cdots D_m^{a_m} = \sum_{\beta'} c_{\beta'} q^{\beta'}$ with $c_{\beta'} \in H^*(X, \mathbb{Q})$, then $c_{\beta'} = 0$ unless $\beta' = \beta$. (Use Proposition 4.1 to see that there are no torus-invariant genus-0 stable maps $\varphi\colon C \to X$ whose marked points collapse to distinct points on a distinguished component of $C$—and that satisfy the required incidence conditions—unless $\beta' = \beta$).

(ii) $c_\beta$ can be computed by counting maps $\mathbb{P}^1 \to X$; precisely, if

$$\pi : \bar{M}_{0,r}(X, \beta) \to \bar{M}_{0,r}$$

denotes the forgetful map with $r = \left(\sum a_i\right) + 1$, and if $z \in M_{0,r} \subset \bar{M}_{0,r}$ is a general point and $x \in X$ a general point, then with

$$\bar{M}_z := \{z\} \times_{\bar{M}_{0,r}} \bar{M}_{0,r}(X, \beta),$$

$$M_z := \bar{M}_z \cap M_{0,r}(X, \beta),$$

$$M_z^\circ := \left\{ (\varphi\colon \mathbb{P}^1 \to X) \in M_z \;\middle|\; \varphi(\mathbb{P}^1) \cap \left( \bigcup_{\substack{\sigma\in\Delta \\ \dim\sigma\geq 2}} X(\sigma) \right) = \emptyset \right\},$$

we have

$$\left( \bigcap_{1\leq i\leq a_1} ev_i^{-1}(D_1) \right) \cap \cdots \cap \left( \bigcap_{r-a_m\leq i\leq r-1} ev_i^{-1}(D_m) \right) \cap ev_r^{-1}(x) \subset M_z^\circ$$

in $\bar{M}_z$. (Hint: Let $\varphi\colon C \to X$ be in $\bar{M}_z$ and consider separately the cases where the distinguished component of $C$ maps into a boundary divisor, or into the open torus orbit.)

(iii) Identify $M_z^\circ$ with the space of $m$-tuples of homogeneous polynomials

$$(p_1(s,t), \ldots, p_m(s,t))$$

such that $\deg p_i = a_i$ for each $i$ and, for $i \neq j$, $p_i$ and $p_j$ have no common roots among $[s:t] \in \mathbb{P}^1$ modulo $(p_1, \ldots, p_m) \sim (p_1', \ldots, p_m')$ if there exists $g \in H_2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*$ such that $p_i' = \left(\int_g D_i\right) p_i$ for each $i$ (see [C, Thm. 3.1]).

(iv) Compute

$$c_\beta = \int_{\bar{M}_z} ev_1^*(D_1) \cdots ev_{r-1}^*(D_m) \cdot ev_r^*(\{x\}) = 1.$$

(Note that $M_z$ is smooth of the expected dimension for $z$ general, and by (ii) there are no contributions from virtual moduli cycle classes supported on boundary components.)

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