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Globally F-Regular Varieties: Applications to Vanishing Theorems for Quotients of Fano Varieties

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Dedicated to Professor William Fulton on the occasion of his sixtieth birthday

1. Introduction

A smooth projective variety is said to be *Fano* if its anti-canonical bundle is ample. The Kodaira vanishing theorem easily implies vanishing of all higher cohomology modules of numerically effective line bundles on any Fano variety, at least in characteristic 0. Indeed, for any positive i, it implies that $H^i(X, \mathcal{L}) = H^i(X, (\mathcal{L} \otimes \omega^{-1}) \otimes \omega)$ vanishes when $\mathcal{L} \otimes \omega^{-1}$ is ample, and hence vanishing holds in particular whenever $\mathcal{L}$ is numerically effective and ω^{-1} is ample.

In this paper, a class of algebraic varieties is introduced, the class of *globally F-regular varieties.* Globally F-regular varieties have strong vanishing properties, including the vanishing of the higher cohomology groups for any numerically effective line bundle (as discussed above for Fano varieties). Indeed, the class of globally F-regular varieties of characteristic 0 is shown to include Fano varieties, so the vanishing just described is recovered. A nice feature of the class of globally F-regular varieties is that it is preserved under the operation of forming certain (and conjecturally: any) GIT quotients by linearly reductive groups.

Globally F-regular varieties are closely related to Frobenius split varieties [MRn]. Both Frobenius splitting and global F-regularity are notions defined using the Frobenius morphism in characteristic p; by reduction to characteristic p, both Frobenius splitting and global F-regularity make sense in characteristic 0 as well. As explained within, global F-regularity turns out to be a stable version of the notion of *Frobenius split along a divisor* that has arisen in the Indian school of algebraic groups [MRn; RR; R1; R2]. However, the definition of global F-regularity is based on the theory of tight closure introduced by Hochster and Huneke in [HH1]: roughly speaking, a projective algebraic variety is globally F-regular if it has a coordinate ring in which all ideals are tightly closed.

The original motivation for this work was a question of Allen Knutson in his study [Kn] of torus actions in symplectic geometry: Let G be a semi-simple complex algebraic group with fixed Borel subgroup B and maximal torus $T \subset B$. Consider the geometric invariant theory (GIT) quotient X of the homogeneous space G/B with respect to some choice of linearization of the natural left action

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of T. If $\mathcal{L}$ is an ample line bundle on the quotient X, then does $H^i(X, \mathcal{L})$ vanish for $i > 0$? Vanishing does not follow immediately from the Kodaira vanishing theorem because X need not be Fano (even though G/B is). In his thesis, Knutson posed this question for a particular choice of linearization. Knutson's specific question turned out to have an answer in [Sj], but it led to the general problem of vanishing theorems for positive line bundles on GIT quotients of Fano varieties. In this paper, Knutson's question is answered affirmatively as a special case of vanishing theorems for torus quotients of any Fano variety. A special case of one of our main theorems is the following.

1.1. THEOREM. *Let X be a quotient variety obtained by the action of a finite group or a torus on a complex Fano variety with rational Gorenstein singularities. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-modules that is numerically effective* (*nef*). *Then $H^i(X, \mathcal{L}) = 0$ for all $i > 0$.*

1.2. REMARK. Here, the group action is assumed to be algebraic, and by "quotient" we mean a geometric invariant theory quotient in the sense of Mumford [MFK] with respect to any choice of ample linearization of the action. In particular, although such quotients are not unique (see [DH; Th]), the vanishing theorem holds for any of them.

1.3. REMARK. Versions of Theorem 1.1 for an arbitrary reductive group G are proved in Theorems 7.6 and 7.7, but at present a tricky technical point in the theory of tight closure prevents me from stating Theorem 7.1 for arbitrary reductive quotients of Fano varieties. This difficulty is explained at the end of Section 7.

The usefulness of Frobenius splitting and related techniques in establishing vanishing theorems is well known; see [HR1; HR2; MRn; RR; R1; R2]. On the other hand, the idea of F-regularity first arose in the theory of *tight closure* in commutative algebra and had nothing to do with projective geometry. One theme of this paper is the relationship between local (commutative algebra) and global (projective geometry) issues. The properties of F-purity and F-regularity in commutative algebra [HH1; HH3; HR1; HR2; Hu; S3] and the notion of Frobenius splitting and related techniques [MRn; RR; R1; R2] are equivalent from a certain point of view, as explained within. I hope this paper shows the fruitfulness of combining these points of view and encourages more experts in one of these points of view to embrace the other.

In order to make the presentation accessible to a larger audience, this paper is partially expository and with various arguments using the Frobenius (which are standard for experts) repeated here in detail. I hope the experts will forgive me this lack of novelty. A basic reference for the commutative algebra language (Cohen–Macaulayness, Gorensteinness, injective hull, etc.) used here is [Ma].

I am grateful to Allen Knutson and Michael Thaddeus for making me aware of these interesting questions about vanishing and for conversations that piqued my interest in them.

2. F-Purity and F-Regularity

This section contains definitions and a quick review of some commutative algebra, which can be revisited as necessary for reference.

Let R be an arbitrary Noetherian commutative ring. A map of R-modules $M_1 \xrightarrow{f} M_2$ is *pure* if the induced map $M_1 \otimes M \xrightarrow{f\otimes \mathrm{id}} M_2 \otimes M$ is injective for every R-module M. If M_2/M_1 is finitely presented, then the purity of f is equivalent to the splitting of f; see [Ma, 7.14].

For finitely generated modules over a local ring, there is a convenient criterion for purity that (according to Hochster) goes back to Auslander, at least in a primitive form. Suppose that R is local and that E is an injective hull of the residue field of R. The map $M_1 \xrightarrow{f} M_2$ is pure if and only if $M_1 \otimes E \xrightarrow{f\otimes \mathrm{id}} M_2 \otimes E$ is injective [HH3, 2.1e].

Now assume that R has prime characteristic p. By definition, R is *F-pure* if the Frobenius map

$$R \xrightarrow{F} {}^1R,$$

$$r \mapsto r^p$$

is pure [HR1]. The notation ${}^eR$ denotes R itself, but viewed as an R-module via Frobenius: $r \in R$ acts on ${}^ex \in {}^eR$ by $r \cdot {}^ex = {}^e(r^{p^e}x) \in {}^eR$. In particular, the iterated Frobenius $R \xrightarrow{F^e} {}^eR$ is an R-module map for each $e \geq 0$.

Some basic facts about F-purity are proved in [HR1]. Among them: F-pure rings are reduced; every regular ring is F-pure; if an iterate F^e is pure, then F is pure; a ring is F-pure if and only if all its local rings are F-pure. This latter property enables us to say that any scheme over $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$ is F-pure if all its local rings are.

We assume throughout that all rings are F-finite. A ring of characteristic p is said to be F-finite it is finitely generated over its subring of pth powers. This mild hypothesis is satisfied, for instance, by any finitely generated algebra over a perfect field k, and it is preserved under localization. With this assumption, each ${}^eR$ is a Noetherian R-module; in particular, F-purity is equivalent to the splitting of the Frobenius map. The assumption that R is finite over R^p implies that R is excellent [Ku].

The point of this paper is to apply the notion of strong F-regularity to projective varieties. We first recall the local notion from [HH1].

2.1. Definition. An F-finite ring R of prime characteristic p is *strongly F-regular* if, for every $c \in R$ not in any minimal prime of R, there exists an integer $e \geq 0$ such that the R-module map

$$R \xrightarrow{{}^ecF^e} {}^eR,$$

$$1 \mapsto {}^ec$$

splits. This map is the composition of the Frobenius map $R \to {}^eR$ followed by multiplication by the element ${}^ec$, where the notation ${}^ec$ denotes c regarded as an element in ${}^eR$. This notation is used for extra clarity, to distinguish between elements in ${}^eR$ and the action of elements in R on the unit element in ${}^eR$. For instance, note that $r \cdot {}^e1 = {}^er^{p^e}$.

2.2. Some basic facts about strong F-regularity are recorded in [HH1]. Among them: strongly F-regular rings are F-pure, reduced, normal, and Cohen–Macaulay; regular rings are strongly F-regular; if $c \in R$ is an element (not in any minimal prime of R) such that R_c is strongly F-regular and if ${}^ecF^e$ splits for some $e > 0$, then R is strongly F-regular; the ring R is strongly F-regular if and only if all of its local rings are strongly F-regular. This final property motivates the definition of a scheme X over $\mathbb{Z}/p\mathbb{Z}$ as strongly F-regular if all its local rings are.

How much stronger than F-purity is the concept of strong F-regularity? An answer is provided in [S1]: an F-pure ring R is strongly F-regular if and only if R is a simple module over the ring of $\mathbb{Z}$-linear differential operators on R.

Strong F-regularity first arose in connection with tight closure. Tight closure is a closure operation performed on ideals in a Noetherian rings containing a field; its precise definition is not important here (see [HH1; HH3]). Strong F-regularity is conjectured to be equivalent to weak F-regularity, which is the property that all ideals are tightly closed. Strong and weak F-regularity are known to be equivalent for $\mathbb{Q}$-Gorenstein schemes [Mac] and for $\mathbb{N}$-graded rings [LS]. In this paper, the qualifier "strongly" is frequently dropped in discussing strongly F-regular rings. Because we will not need the concept of weak F-regularity and since we will primarily be discussing the graded case, this should not cause any confusion.

3. Relationship between Local and Global Properties

In this section, we discuss how the local properties of F-purity and F-regularity give rise to global properties for projective varieties. This gives rise to the concept of a *globally F-regular* projective variety.

Throughout this section, X will be a connected projective variety over a perfect (or, more generally, F-finite) field k of prime characteristic p.

The absolute Frobenius map $X \xrightarrow{F} X$ is the identity on the underlying topological space of X, but its corresponding sheaf map $\mathcal{O}_X \to F_*\mathcal{O}_X$ is the pth power map locally on sections. (Locally, this is the map $R \xrightarrow{F} {}^1R$ described previously.) The variety X is said to be *Frobenius split* if the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules [MRn]. Note that, because X is of finite type over a F-finite field, X itself is F-finite and hence the Frobenius map is finite.

If X is regular then the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits on every stalk, but the existence of a global splitting of Frobenius is quite rare and forces strong restrictions on X. Thus the local condition of F-purity defined in the previous section is not the same as Frobenius splitting. Rather, we can think of Frobenius splitting of a variety X as a sort of *global F-purity*.

Nonetheless, global properties of X can be studied, using the techniques of local algebra, by passing to an *affine cone over X* determined by some polarization. A *polarization* for X is a choice of an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. The cone determined by $\mathcal{L}$ is the spectrum of the *section ring*

$$S = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n).$$

The section ring S is a finitely generated algebra over the F-finite field $H^0(X, \mathcal{O}_X)$, and $\operatorname{Proj} S \cong X$. If X is normal, then so is S. Let m denote the unique homogeneous maximal ideal $m = S_{>0}$ of S (the *irrelevant ideal*).

The following proposition relates the local property of F-purity to the global property of F-splitting (see also [S2, 4.10; W2, 3.3]).

3.1. Proposition. *Let X be a connected projective variety over an F-finite* (*e.g. perfect*) *field of prime characteristic. Then the following are equivalent*:

(1) *X is Frobenius split*;
(2) *the section ring of X with respect to every polarization is F-pure*;
(3) *the section ring of X with respect to some polarization is F-pure.*

The proof follows readily by tensoring the Frobenius map $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ with the sheaf of algebras $\bigoplus \mathcal{L}^i$ and considering global sections. It is not included here, since it can be found in [S2, Prop. 4.10] and is, in any case, a simplified version of the proof of Theorem 3.10 offered here.

By analogy, this motivates the following definition.

3.2. Definition. A projective variety over an F-finite field is *globally F-regular* if it admits some section ring that is F-regular.

By the conventions agreed upon in Section 2, the unqualified term "F-regular" here means strongly F-regular. However, because strong and weak F-regularity are equivalent for $\mathbb{N}$-graded F-finite rings [LS], there is no possibility of confusion.

Of course, any globally F-regular variety is locally F-regular, but the converse is far from true. Indeed, any smooth variety is locally F-regular whereas a globally F-regular variety, for example, has the property that $H^i(X, \mathcal{O}_X)$ for any $i > 0$, as we will soon prove.

3.3. Note that globally F-regular varieties are Frobenius split. Indeed, passing to section rings, this is immediate from the fact that strongly F-regular rings are F-pure.

As with F-splitting, if X admits a section ring that is F-regular then *every* section ring for X is F-regular, as we will prove in Theorem 3.10. We first recall the notion of *Frobenius splitting along a divisor*, as defined by Ramanan and Ramanathan [RR], and then introduce a stable version of it.

3.4. Stable Frobenius Splitting along a Divisor. Let D be an effective Cartier divisor, let s be a section of $\mathcal{O}_X(D)$ vanishing precisely along D, and let

e be a positive integer. Consider the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ sending $1 \mapsto s$. This induces a map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to F_*\mathcal{O}_X \to F_*\mathcal{O}_X(D),$$

where the first arrow is the Frobenius map and the second arrow is (the push-forward of) the map $1 \mapsto s$. Consistent with the notion introduced in Section 2, we will denote by ${}^1s$ the element s considered as an element of $F_*\mathcal{O}_X(D)$. The variety X is said to be *Frobenius D-split* if this composition map splits, that is, if there is a map $F_*\mathcal{O}_X(D) \to \mathcal{O}_X$ sending ${}^1s \mapsto 1$. This notion appears to have been first exploited in [RR] but was not named as such until [R2].

It is convenient to consider not just the Frobenius map but also its iterates. Accordingly, we introduce the notion of stable Frobenius D-splitting.

3.5. DEFINITION. Let X be an algebraic variety over an F-finite field, and let D be an effective Cartier divisor on X defined by a section s. The variety X is said to be *e-Frobenius D-split* (or *e-Frobenius split along D*) if there is a map $F_*^e\mathcal{O}_X(D) \to \mathcal{O}_X$ sending ${}^es \mapsto 1$. (Again, the notation ${}^es$ denotes the element s considered as an element of $F_*^e\mathcal{O}_X(D)$.) We will say that X is *stably* Frobenius D-split if it is e-Frobenius D-split for some e.

It is easy to check that if X is e-Frobenius D-split then it is e'-Frobenius split for all $e' \geq e$, thus motivating the adjective "stable". Stable Frobenius D-splitting is weaker than Frobenius D-splitting, yet virtually all the good properties of Frobenius D-splitting are applicable with only the assumption that eventually, for large enough e, we have splitting.

Basic Properties

3.6. Stable Frobenius splitting along any divisor implies Frobenius splitting. Indeed, because $\mathcal{O}_X \to F_*^e\mathcal{O}_X(D)$ factors through the Frobenius map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$, any splitting will also split $\mathcal{O}_X \to F_*^e\mathcal{O}_X$ (sending 1 to 1). On the other hand, any splitting of the natural map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$ must split the Frobenius $\mathcal{O}_X \to F_*\mathcal{O}_X$ because we have a factorization $\mathcal{O}_X \to F_*\mathcal{O}_X \to F_*F_*^{(e-1)}\mathcal{O}_X \cong F^e\mathcal{O}_X$ (where all maps are induced by sending 1 to 1). Thus the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ also splits. In particular, *stable Frobenius splitting along the zero divisor is equivalent to Frobenius splitting.* However, stable splitting along the zero divisor is, in general, a strictly weaker condition than stable splitting along an effective divisor.

3.7. If X is stably Frobenius split along some effective divisor D, then X is stably Frobenius split along any effective divisor D' with $D' \leq D$. (Here, $D' \leq D$ means that the multiplicity of each irreducible component of D' is at most the multiplicity of the corresponding component in D.) Indeed, when $D' \leq D$ and s' is a defining equation of D', then a defining equation for D will have the form $s't$ for some t. We have maps of sheaves $\mathcal{O}_X \to \mathcal{O}_X(D') \to \mathcal{O}_X(D)$, given by multiplication by s' and by t, that induce maps $F_*^e\mathcal{O}_X \to F_*^e\mathcal{O}_X(D') \to F_*^e\mathcal{O}_X(D)$. Putting this together with the Frobenius map, we have

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X \to F^e_*\mathcal{O}_X(D') \to F^e_*\mathcal{O}_X(D),$$
$$1 \mapsto {}^e1 \mapsto {}^es' \mapsto {}^e(s't),$$

which shows that any any splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ will induce a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D')$. So if X is e-Frobenius D-split, it is also e-Frobenius D'-split for any effective $D' \leq D$.

3.8. We have remarked that if X is e-Frobenius D-split then it is e'-Frobenius D-split for all $e' \geq e$. However, it can happen that X is e-Frobenius split along D yet not $(e-1)$-Frobenius split along D. On the other hand, the following lemma ensures that one can get $(e-1)$-Frobenius splitting from e-Frobenius splitting along a divisor that is p-divisible.

3.9. LEMMA. *Let X be a variety and D an effective Cartier divisor. Then X is e-Frobenius D-split if and only if X is Frobenius $(e+1)$-split along pD. In particular, X is stably Frobenius split along D if and only if X is stably Frobenius split along pD.*

Proof. Say that X is e-Frobenius D-split. Fix a section s defining D. We know that the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ sending 1 to s splits. By Remark 3.5, the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ sending 1 to 1 also splits. But the splitting of $\mathcal{O}_X \to F_*\mathcal{O}_X$ implies the splitting of $\mathcal{O}_X \otimes \mathcal{O}_X(D) \to F_*\mathcal{O}_X \otimes \mathcal{O}(D) \cong F_*\mathcal{O}(pD)$, with this isomorphism coming from the projection formula. (The map $\mathcal{O}_X(D) \to F_*\mathcal{O}(pD)$ sends s to ${}^es^p$.) Pushing down via F^e, we have the splitting of $F^e_*\mathcal{O}_X(D) \to F^e_*F_*\mathcal{O}(pD) = F^{e+1}_*\mathcal{O}_X(pD)$. Putting this together with the original splitting $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$, we see that the map $\mathcal{O}_X \to F^{e+1}_*\mathcal{O}_X(pD)$ sending 1 to ${}^{e+1}s^p$ splits over $\mathcal{O}_X$.

Conversely, say that X is $(e+1)$-Frobenius pD-split. Then the map $\mathcal{O}_X \to F^{e+1}_*\mathcal{O}_X(pD)$ sending 1 to ${}^{e+1}s^p$ splits. But this map factors through the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ sending 1 to ${}^es$. To see this factorization, first note that the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$ induces a map $\mathcal{O}_X(D) \to F_*\mathcal{O}_X \otimes \mathcal{O}_X(D) \cong F_*(pD)$, with the isomorphism following from the projection formula. Pushing down via F^e yields a map $F^e_*\mathcal{O}_X(D) \to F^{e+1}_*\mathcal{O}_X(pD)$, which sends ${}^es$ to ${}^{e+1}s^p$. Thus we have the factorization

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(D) \to F^{e+1}_*\mathcal{O}_X(pD),$$
$$1 \mapsto {}^es \mapsto {}^{e+1}s^p.$$

So any splitting of $\mathcal{O}_X \to F^{e+1}_*\mathcal{O}_X(pD)$ induces a splitting for $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$, and X is e-Frobenius D-split. □

3.10. THEOREM. *Let X be a projective variety over a perfect field. Then the following statements are equivalent.*

(1) *X is globally F-regular; that is, the section ring of X with respect to some ample divisor is strongly F-regular.*

(2) *X is stably Frobenius split along some ample effective divisor D such that the open set $X - D$ is* (*locally*) *strongly F-regular.*
(3) *X is stably Frobenius split along every effective Cartier divisor.*
(4) *The section ring of X with respect to every ample divisor is strongly F-regular.*

Proof. We show that (1) is equivalent to (2) and then that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

To see that (1) and (2) are equivalent, fix any ample effective D and let s be a defining section for D. Consider the $\mathcal{O}_X$-module map

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D)$$

sending 1 to ${}^e s$.

We tensor this map of $\mathcal{O}_X$-modules with the sheaf of algebras $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(nD)$ and take global sections to construct the corresponding map of graded S-modules. Of course, this process applied to $\mathcal{O}_X$ produces a section ring S. What is the corresponding graded S-module this process produces for $F^e_*\mathcal{O}_X(D)$? Note that, by the projection formula, $F^e_*\mathcal{O}_X(D) \otimes \mathcal{O}_X(nD) \cong F^e_*(\mathcal{O}_X(D + p^e nD))$ and so the graded S-module associated with $F^e_*\mathcal{O}_X(D)$ is the module

$$\bigoplus_{n\in\mathbb{Z}} H^0(X, F^e_*\mathcal{O}_X(D + p^e nD)) = {}^e[S]_{1 \bmod p^e}(1),$$

where (a) the notation $[S]_{1 \bmod p^e}$ indicates the subset of elements of S consisting of (sums of) homogeneous elements whose degrees are congruent to 1 modulo p^e and (b) the shift $[S]_{1 \bmod p^e}(1)$ indicates that a homogeneous element c of degree 1 in S should be considered to have degree 0 in $[S]_{1 \bmod p^e}(1)$. The notation ${}^e[S]_{1 \bmod p^e}(1)$ indicates the natural S-module structure on this set given by Frobenius—that is, $r \in S$ acts on ${}^e c$ in ${}^e[S]_{1 \bmod p^e}(1)$ to produce ${}^e\big(r^{p^e} c\big)$. Note that $[S]_{1 \bmod p^e}(1)$ is a finitely generated graded S-module whose degree-n piece is $S_{1+p^e n} = H^0(X, \mathcal{O}_X(D + p^e nD))$. Thus, the degree-preserving map of graded S-modules corresponding to the map of sheaves $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(D)$ sending 1 to ${}^e s$ is the map

$$S \longrightarrow {}^e[S]_{1 \bmod p^e}(1)$$

sending 1 to ${}^e s$.

Splitting the $\mathcal{O}_X$-module map $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(D)$ is equivalent to splitting the map $S \longrightarrow {}^e[S]_{1 \bmod p^e}(1)$ as a map of graded S-modules. But because the module ${}^e[S]_{1 \bmod p^e}(1)$ is a graded direct summand of ${}^e S$ as an S-module, this is equivalent to splitting the map $S \longrightarrow {}^e S(1)$ sending 1 to ${}^e s$ in the category of graded S-modules. But because ${}^e S$ is a finitely generated S-module and s is homogeneous, this is equivalent to splitting the map $S \longrightarrow {}^e S$ sending 1 to ${}^e s$ as a map of S-modules (without worrying about whether there is a homogeneous splitting). Thus, for an ample effective divisor D defined by s, we have shown that stable Frobenius D-splitting is equivalent to splitting the S-module map $S \longrightarrow {}^e S$ sending 1 to ${}^e s$ for some e.

Now consider the assumption that $X - D$ is strongly F-regular. If again S denotes the section ring of X with respect to D, then this open affine set is defined as $\operatorname{Spec}[S_s]_0$, where $[S_s]_0$ denotes the 0th graded piece of the localization ring S_s. Because S_s is a Laurent ring over its degree-0 subring (see [S4, proof of 3.4]), $[S_s]_0$ is strongly F-regular if and only if S_s is strongly F-regular (see e.g. [LS, 4.1]). Hence condition (2) is equivalent to the condition that there is a splitting of S-modules $S \longrightarrow {}^eS$ sending 1 to ${}^es$ for some e and some s with S_s strongly F-regular. According to the basic "strongly F-regular" properties described in Section 2.2, we see that this is equivalent to strong F-regularity of S. This completes the proof of the equivalence of conditions (1) and (2).

Now assume (1). Let S be a section ring for X with respect to the ample line bundle $\mathcal{L}$, and assume that S is strongly F-regular. Let D be an effective divisor defined by the section s, and let $\mathcal{O}_X(D)$ denote the corresponding invertible sheaf. Consider the $\mathcal{O}_X$-module map

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X(D),$$
$$1 \mapsto s.$$

By tensoring with the "polarization algebra" $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n$ and taking global sections, we have a degree-preserving map of graded S-modules

$$S \longrightarrow M = \bigoplus H^0(X, \mathcal{O}_X(D) \otimes \mathcal{L}^n),$$
$$1 \mapsto s.$$

Note that (a) the S-module M is a torsion-free rank-1 S-module with a natural grading and (b) the map $S \longrightarrow M$ sending 1 to s defines a degree-preserving map of S-modules. Thus, M is isomorphic to a finitely generated graded submodule of the fraction field of S, and there is a "clearing denominators" map of graded S-modules $M \overset{r}{\hookrightarrow} S(d)$, given by multiplication by r, for some homogeneous element r (of degree, say, d) in S.

Let $c = rs \in S$. Because S is strongly F-regular, there exists an e and an S-module map ${}^eS \overset{\phi}{\longrightarrow} S$ that sends ${}^ec$ to 1. Consider the compositions of S-module maps

$$S \longrightarrow {}^eS \longrightarrow {}^eM \overset{{}^er}{\longrightarrow} {}^e(S(d)) \overset{\phi}{\longrightarrow} S,$$
$$1 \mapsto 1 \mapsto {}^es \mapsto {}^ec \mapsto 1.$$

Because the degree of s in ${}^eM$ is 0, the composition $\psi = \phi \circ {}^er$ gives a degree-preserving map from ${}^eM$ to S. That is, there are degree-preserving maps of graded S-modules

$$S \overset{F^e}{\longrightarrow} {}^eS \longrightarrow {}^eM \overset{\psi}{\longrightarrow} S,$$
$$1 \mapsto 1 \mapsto {}^es \mapsto 1.$$

The corresponding map of sheaves (after passing to the appropriate direct summand, as in the proof of the equivalence of (1) and (2)) is

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X,$$

showing that X is stably Frobenius split along D. This shows that (1) implies (3).

To see that (3) implies (4), consider a section ring S for X with respect to some ample invertible sheaf $\mathcal{L}$. Because X is Frobenius split along the zero divisor, it is reduced and hence there is an open affine subset that is regular. Thus we can find an ample effective divisor D defined by a section s of some power of $\mathcal{L}$ such that $X - D$ is regular and hence strongly F-regular. Equivalently, on the algebraic side this means that $s \in S$ is such that the ring S_s is regular and hence strongly F-regular. We know that X is stably Frobenius D-split by assumption and so, as in the proof of the equivalence of (1) and (2), we see equivalently that the map $S \longrightarrow {}^eS$ sending 1 to ${}^es$ splits. So again, this implies that S is strongly F-regular, and (4) is proved.

Finally, (4) trivially implies (1). □

3.11. EXAMPLE. It is worth remarking that Frobenius splitting along a divisor depends, in general, on the divisor itself and not on the divisor class. For example, let S be the graded ring $\mathbb{F}_7[x, y, z, w]/(x^3 + y^3 + z^3)$, which is the section ring of a projectivized cone X over an ordinary elliptic curve with respect to the natural $\mathcal{O}(1)$. One can check that we have splitting of the map $S \longrightarrow {}^eS$ sending 1 to ${}^ew$ for large e although the map $S \longrightarrow {}^eS$ sending 1 to ${}^ex$ never splits for any e. Thus, X is stably Frobenius split along $D := \{w = 0\}$ but not along the linearly equivalent divisor $D' := \{x = 0\}$. Accordingly, Theorem 3.10 ensures that $X - D'$ ought not be locally strongly F-regular. This is indeed the case, as one can check directly, although $X - D'$ (and in fact X itself) is F-split.

4. Globally F-Regular Varieties: Properties and Vanishing Theorems

Globally F-regular varieties have remarkably strong properties. In this section, we summarize some of these.

4.1. PROPERTIES OF GLOBALLY F-REGULAR VARIETIES. *Let X be a connected projective variety over an F-finite field of prime characteristic. If X is globally F-regular, then X enjoys the following properties.*

(1) *X is normal.*
(2) *X is Cohen–Macaulay.*
(3) *Every ample invertible sheaf on X is arithmetically Cohen–Macaulay; that is, every section ring for X is Cohen–Macaulay.*
(4) *X is Frobenius split; that is, every section ring is F-pure.*
(5) *Every section ring for X is pseudorational (a desingularization-free analog of rational singularities that makes sense for arbitrary schemes; see* [LT]).

Proof. Let $X = \operatorname{Proj} S$, where S is an arbitrary section ring. If X is globally F-regular then S is strongly F-regular, by Theorem 3.10. Hence S is normal and Cohen–Macaulay (by Section 2.2), so properties (1), (2), and (3) follow. Strongly F-regular rings are also F-pure, so property (4) follows from Proposition 3.1. We establish (5) as follows. Strongly F-regular rings have the property that all ideals—in particular, all parameter ideals—are tightly closed. But any excellent domain in which all parameter ideals are tightly closed is pseudorational [S3, 3.1]. □

We now prove a vanishing theorem for globally F-regular varieties. The idea of using Frobenius splitting and its variants to prove vanishing theorems is not new: it goes back at least to Hochster and Roberts in their famous proof [HR1] that invariant rings are Cohen–Macaulay; the language here is closer to [MRn]. The following argument is standard for experts.

4.2. THEOREM. *Let X be a projective variety that is globally F-regular, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

(1) *If $H^i(X, \mathcal{L}^n) = 0$ for $n \gg 0$, then $H^i(X, \mathcal{L}) = 0$.*
(2) *If there exists an effective Cartier divisor D such that $H^i(X, \mathcal{L}^n(D)) = 0$ for $n \gg 0$, then $H^i(X, \mathcal{L}) = 0$.*

Proof. Let $\mathcal{L}$ be an arbitrary invertible $\mathcal{O}_X$-module. By the projection formula applied to the Frobenius map on X, we have $F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} \cong F_*F^*\mathcal{L}$ [H, p. 124]. Also, note that $F^*\mathcal{L} \cong \mathcal{L}^p$. Thus, the split inclusion

$$\mathcal{L} \to F_*\mathcal{O}_X \otimes \mathcal{L} \cong F_*\mathcal{L}^p$$

induces an injective map of cohomology

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, F_*\mathcal{L}^p) \cong H^i(X, \mathcal{L}^p),$$

with the latter isomorphism arising because the Frobenius map is affine [H, p. 252]. We may iterate this process to obtain a series of inclusions

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^p) \hookrightarrow H^i\big(X, \mathcal{L}^{p^2}\big) \hookrightarrow H^i\big(X, \mathcal{L}^{p^3}\big) \hookrightarrow \cdots .$$

Thus, any line bundle $\mathcal{L}$ having the property the $H^i\big(X, \mathcal{L}^{p^e}\big) = 0$ for some natural number $e \in \mathbb{N}$ must have $H^i(X, \mathcal{L}) = 0$ as well.

The proof of the second statement is similar. Because X is stably Frobenius D-split for every effective D, the map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$$

sending 1 to a defining equation for D splits for all large e. Now, tensoring with any invertible sheaf $\mathcal{L}$, the map

$$\mathcal{L} \to F^e_*\mathcal{O}_X(D) \otimes \mathcal{L} \cong F^e_*(\mathcal{O}_X(D) \otimes F^{e*}\mathcal{L}) \cong F^e_*\big(\mathcal{O}_X(D) \otimes \mathcal{L}^{p^e}\big)$$

splits, where the first isomorphism is obtained from the projection formula. This induces a split inclusion of cohomology groups

$$H^i(X, \mathcal{L}) \hookrightarrow H^i\big(X, F^e_*\big(\mathcal{L}^{p^e}(D)\big)\big) \cong H^i\big(X, \mathcal{L}^{p^e}(D)\big),$$

with the isomorphism holding because the Frobenius map F^e, like any finite map, is affine. Now it is immediate that, if $H^i\big(X, \mathcal{L}^{p^e}(D)\big)$ vanishes for any $e \gg 0$, then $H^i(X, \mathcal{L})$ vanishes as well. □

4.3. COROLLARY. *If X is a projective variety that is globally F-regular, then $H^i(X, \mathcal{L}) = 0$ for any invertible $\mathcal{L}$ that is numerically effective (nef). In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. First, say that $\mathcal{L}$ is ample. By Serre vanishing, $H^i(X, \mathcal{L}^n) = 0$ for $n \gg 0$. Thus $H^i(X, \mathcal{L}) = 0$ follows from (1). Now say $\mathcal{L}$ is nef. Fix an ample effective divisor H. Then $\mathcal{L}^n(H)$ is ample for all $n \geq 0$, and we just checked that cohomology vanishes for an ample invertible sheaf, so $H^i(X, \mathcal{L}^n(H)) = 0$ for all $n \geq 0$ and all $i \geq 1$. Now it follows from Theorem 4.2(2) that $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$. □

We also have the following form of the Kawamata–Viehweg vanishing theorem for globally F-regular varieties.

4.4. COROLLARY. *Let X be a globally F-regular projective variety, and let $\mathcal{L}$ be a big and nef invertible sheaf on X. Then $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$.*

Proof. Because $\mathcal{L}$ is big and nef, we can find an effective Cartier divisor D such that $\mathcal{L}^m(-D)$ is ample for all $m \gg 0$ [KM, 2.6]. But then $H^i(X, (L^{-m}(D))^n) = 0$ for all $i < \dim X$ and all $n \gg 0$ because X is Cohen–Macaulay, so by Theorem 4.2 we conclude that $H^i(X, \mathcal{L}^{-m}(D)) = 0$ for all $i < \dim X$. By the second part of Theorem 4.2, we conclude that $H^i(X, \mathcal{L}^{-1}) = 0$. □

5. The Characteristic-0 Theory

The notions of Frobenius splitting and strong F-regularity can be adapted to varieties of characteristic 0, where we will call them Frobenius split type and strongly F-regular type. We briefly recall the idea of reduction modulo p.

Let X be a scheme of finite type over a field k of characteristic 0. Choose a finitely generated $\mathbb{Z}$-algebra A contained in k over which X is defined. Let X_A be the "thickened" scheme of finite type over A, so that $X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k$ is naturally identified with X. Each closed fiber of the *family of models* $X_A \longrightarrow \operatorname{Spec} A$ is a scheme of finite type over a finite field (of different characteristics p). In this way, notions defined in characteristic p make sense also over k by requiring them to hold on a dense set of closed fibers of the family $X_A \longrightarrow \operatorname{Spec} A$.

5.1. DEFINITION. A scheme of finite type over a field of characteristic 0 is *Frobenius split type* (resp., *strongly F-regular type*) if it admits a family of models in which a dense set of closed fibers are Frobenius split (resp., strongly F-regular). Consequently, a variety projective over a field of characteristic 0 is *globally F-regular type* if it admits a family of models in which a dense set of closed fibers is globally F-regular. None of these properties depends on the choice of the family of models.

Furthermore, given any finite collection $\{\mathcal{M}_i\}$ of coherent $\mathcal{O}_X$-modules, we can choose A so that these sheaves are defined over A. The resulting coherent $\mathcal{O}_{X_A}$-modules $\mathcal{M}_{Ai}$ pull back on $X = X_A \times \operatorname{Spec} k$ to $\mathcal{M}_i$. By pulling back to a closed fiber, we get a collection of coherent sheaves $\bar{\mathcal{M}}_i$ on the prime characteristic scheme $X_A \times \operatorname{Spec}(A/\mu)$ for each μ in $\max \operatorname{Spec}(A)$. In this way, we can study coherent modules (e.g. line bundles) on a variety of characteristic 0 by "reduction to prime characteristic".

5.2. CAUTION. One can define the operation of tight closure on ideals in an algebra of finite type over a field of characteristic 0 using reduction to characteristic p (see [HH4]). Because the property of strong F-regularity is equivalent to the property that all ideals are tightly closed for $\mathbb{N}$-graded rings over a perfect field of prime characteristic [LS], we know that global F-regularity in prime characteristic is equivalent to the property that all ideals are tightly closed in some section ring for X. It is tempting to believe that the property of globally F-regular type is equivalent to the property that all ideals are tightly closed in a section ring of characteristic 0 as well. This may be true, but it is currently an open problem whether the property of F-regular type is equivalent to all ideals being tightly closed, even for a finitely generated graded ring over a field. See Conjecture 7.5.

The following are immediate corollaries of the theorems in the preceding section.

5.3. COROLLARY. *Let X be a connected projective variety over a field of characteristic* 0. *If X is of globally F-regular type, then the conclusion of Theorem 4.1 holds, with "Frobenius split" and "F-pure" in* (4) *replaced by "Frobenius split type" and "F-pure type", respectively. Furthermore, X and all its section rings have rational singularities; and if X is $\mathbb{Q}$-Gorenstein then it will have log terminal singularities. Finally, any $\mathbb{Q}$-Gorenstein section ring of X will have log terminal singularities.*

Proof. This is essentially already proved. Items (1)–(3) and (5) in Theorem 4.1 follow from a general fact about properties in families: if a dense set of closed fibers has property "P" then so does the generic fiber, where "P" can be (for instance) normality or Cohen–Macaulayness. The claim about rational singularities follows from [S3, 3.1]. That $\mathbb{Q}$-Gorenstein rings of strongly F-regular type have log terminal singularities was first proved by Watanabe [W2]; alternatively, it follows easily from the result on rational singularities using a canonical cover trick (see [Ha] or [S2, 4.16]). □

5.4. COROLLARY. *Let X be a projective variety of characteristic* 0 *that is globally F-regular type, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

(1) *If $H^i(X, \mathcal{L}^n) = 0$ for $n \gg 0$, then $H^i(X, \mathcal{L}) = 0$.*
(2) *If there exists an effective divisor D such that $H^i(X, \mathcal{L}^n(D)) = 0$ for $n \gg 0$, then $H^i(X, \mathcal{L}) = 0$.*

Proof. This follows easily from Theorem 4.2 by semi-continuity. □

As a corollary, we have a very strong version of vanishing as follows.

5.5. COROLLARY. *Let X be a projective algebraic variety of characteristic* 0 *which has globally F-regular type, and let $\mathcal{L}$ be a nef line bundle on X. Then $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$.*

Also, we have Kawamata–Viehweg vanishing: $H^i(X, \mathcal{L}^{-1}) = 0$ *for* $i < \dim X$ *if* $\mathcal{L}$ *is big and nef.*

Note that this is the same vanishing discussed in the introduction for Fano varieties. The reason for this is that Fano varieties are of globally F-regular type, as explained in next section.

6. Examples of Globally F-Regular Varieties

In this section, we identify basic classes of varieties that are globally F-regular, including Fano varieties and toric varieties. As we shall see in the final section, these lead to further examples of globally F-regular (type) varieties obtained from these as quotients. First, a few propositions are recorded; these are easy and possibly standard, but I do not know a reference.

The first proposition indicates one way in which rational singularities can be regarded as a natural extension of Cohen–Macaulayness.

6.1. PROPOSITION.

(1) *Let* X *be a Cohen–Macaulay projective variety. Then* X *has a section ring that is Cohen–Macaulay if and only if* $H^i(X, \mathcal{O}_X) = 0$ *for* $0 < i < \dim X$.

(2) *Let* X *be a projective variety* (*of characteristic* 0) *that has rational singularities. Then* X *has a section ring that has rational singularities if and only if* $H^i(X, \mathcal{O}_X) = 0$ *for all* $i > 0$.

Proof. Choose any ample invertible $\mathcal{L}$. Because X is Cohen–Macaulay, we know from Serre vanishing that $H^i(X, \mathcal{L}^n) = 0$ for all $0 < i < \dim X$ and all $|n| \gg 0$. Replacing $\mathcal{L}$ by a large power, we may assume that $H^i(X, \mathcal{L}^n) = 0$ for all $n \neq 0$ and all $0 < i < \dim X$. The corresponding section ring S is Cohen–Macaulay if and only if the local cohomology modules $H^i_m(S)$ are zero for $i < \dim S$. Since $[H^{i+1}_m(S)]_n \cong H^i(X, \mathcal{L}^n)$ for $0 < i < \dim X$ and all n, this is equivalent to $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. (Note that $H^1_m(S)$ and $H^0_m(S)$ vanish in any case because S satisfies Serre's S_2 condition.)

In general, a normal Cohen–Macaulay graded ring S with rational singularities away from $\{m\}$ has rational singularities itself if and only if $H^d_m(S)$ vanishes in nonnegative degree. (To see this, note that blowing up the vertex of the cone $\operatorname{Spec} S$ yields a variety X with rational singularities whose higher direct images $R^i\pi_*\mathcal{O}_X$ on $\operatorname{Spec} S$ are isomorphic to $[H^{i+1}_m(S)]_{\geq 0}$; see [Fl; W1].) Thus the second claim follows similarly to the first. □

Let X be a connected normal projective variety over a field k. The canonical sheaf of X is the unique reflexive sheaf that agrees with the sheaf $\bigwedge^d \Omega_{X/k}$ of top differential forms on the smooth locus of X.

By definition, a *Fano variety* is a normal projective variety whose anti-canonical sheaf $\omega_X^{-1} = \operatorname{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$ is an ample invertible sheaf. In particular, Cohen–Macaulay Fano varieties are Gorenstein. Unlike many authors, I do not require that Fano varieties be smooth.

Fano varieties are very special. Accordingly, so are the affine cones over them, as indicated by the following basic proposition characterizing Fano varieties in terms of the cones over them.

6.2. PROPOSITION.

(i) *A normal variety X is Fano if and only if it admits a section ring S such that $\omega_S \cong S(r)$ with $r < 0$, where ω_S is the canonical module of the normal graded ring S and where $S(r)$ indicates a degree shift by r.*

(ii) *The anti-canonical ring of a Fano variety (of characteristic 0) with rational singularities is Gorenstein and has rational singularities. Conversely, if X admits a Gorenstein section ring with rational singularities, then X is Fano and has rational singularities.*

Proof. Suppose that ω_X^{-1} is ample invertible on X, and let S be the section ring with respect to ω_X^{-1}. In general, if $\mathcal{L}$ is an ample invertible sheaf on a normal variety X and if S is the corresponding section ring, then the graded canonical module for S (in the sense of [BH]) can be identified with the graded S-module $\bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{L}^n \otimes \omega_X)$; see [S1]. In particular, when $\mathcal{L} = \omega_X^{-1}$ we have $\omega_S \cong S(-1)$. Conversely, suppose S is the section ring for $\mathcal{L}$. Because $\omega_S \cong S(r)$, we see that $\omega_X \cong \mathcal{L}^r$; since $r < 0$, we see that ω_X^{-1} is ample.

For the second statement, let S be the section ring for ω_X^{-1}. By Kodaira vanishing, which is valid for all rings with rational singularites, $H^i(X, \omega_X^n) = 0$ for $n > 0$ and all $i < d$, because ω_X^{-1} is ample. Likewise, $H^i(X, \omega_X^{-n}) = H^i(X, \omega_X^{-n-1} \otimes \omega_X) = 0$ for $i > 0$ and $n \geq 0$. Thus, for $1 \leq i \leq \dim X$, $H^i(X, \omega_X^n) = [H_m^{i+1}(S)]_n = 0$ for all n and, since S is normal, $H_m^0(S) = H_m^1(S) = 0$ in any case. This implies that S is Cohen–Macaulay, and hence S is Gorenstein by (1). But since $H_m^{\dim S}(S)$ vanishes in nonnegative degrees, the criterion for rational singularities mentioned in the proof of Proposition 6.1 shows that S has rational singularities. Indeed, we have already seen that S is Cohen–Macaulay and that $H_m^{\dim S}(S) = \bigoplus_{n\in\mathbb{Z}} H^{\dim X}(X, \omega_X^{-n})$ vanishes nonnegative degrees. □

This proof shows that X has a section ring S whose canonical module is locally free if and only if either ω_X or its dual is an ample invertible sheaf.

6.3. PROPOSITION. *A Fano variety (of characteristic 0) with rational singularities is globally F-regular type.*

Proof. The anti-canonical cone of X has Gorenstein rational singularities. But Gorenstein rational singularities are log terminal singularities. But finitely generated algebras of characteristic 0 with log terminal singularities are strongly F-regular type, by [Ha; MS]. □

An alternate proof that log terminal singularities are strongly F-regular can be found in [S2], still based on Hara's work [Ha]. In [S2] the strong regularity is shown to follow from the "canonical cover trick" (in a form due to Watanabe) and the “strong Kodaira vanishing theorem”. The strong Kodaira vanishing theorem (conjectured in [HS] and proved in [Ha]) states: If $X_A \to \operatorname{Spec} A$ is a generically smooth projective map to an affine scheme Spec Z (where A is a finitely generated $\mathbb{Z}$-algebra containing $\mathbb{Z}$) and if $\mathcal{L}_A$ is a relatively ample invertible sheaf on X_A, then for a generic closed fiber X_μ the map $H^i(X_\mu, \mathcal{L}_\mu^{-1}) \to H^i(X_\mu, F_*F^*\mathcal{L}_\mu^{-1})$ induced by Frobenius is injective for all indices i. Hara deduces this from the Kodaira–Akizuki–Nakano vanishing theorem (see also [MS]). The weaker statement that smooth Fano varieties are Frobenius split follows directly from strong Kodaira vanishing, as pointed out in [Ha, 3.7] and [S2, 4.11].

6.4. PROPOSITION. *Projective toric varieties are globally F-regular* (*type*).

Proof. Projective toric varieties always admit a (normal) section ring of the form $k[m_1, \dots, m_r]$, where the m_i are monomials in a polynomial ring $k[t_1, \dots, t_d]$ (see e.g. [St, p. 131]). Such monomial rings are F-regular, and hence every toric variety is globally F-regular. □

7. Geometric Invariant Theory Quotients of Globally F-Regular Varieties

7.1. THEOREM. *Let X be any projective Frobenius-split* (*resp., globally F-regular*) *algebraic variety of prime characteristic on which a linearly reductive group T acts algebraically. Then the GIT quotient $X//T$ with respect to any linearization of the action of T is also a Frobenius-split* (*resp., globally F-regular*) *variety.*

Recall that “linearly reductive” means that every representation is completely reducible; in characteristic 0, linearly reductive is equivalent to reductive. Of course, linearly reductive groups are somewhat limited in prime characteristic: they include tori and finite groups whose order is not divisible by p as well as extensions of these. By semi-continuity, we have the following theorem in characteristic 0.

7.2. COROLLARY. *Let X be any projective variety over a field of characteristic* 0 *that is Frobenius-split type* (*resp., globally F-regular type*). *Suppose that a group T that is either a torus or finite group* (*or an extension of these*) *acts algebraically on X. Then the GIT quotient $X//T$ with respect to any linearization of the action of T is also a Frobenius-split type* (*resp., globally F-regular type*) *variety.*

It is natural to expect that GIT quotients of globally F-regular varieties by arbitrary reductive groups are again globally F-regular. This appears to be true, but it depends on a tricky technical point in the theory of tight closure that is still open. See Theorem 7.6 and subsequent remarks.

Before proving Theorem 7.1, let us point out that the question of Knutson mentioned in Section 1 is now answered. The complex homogeneous spaces G/P (where G is semi-simple and P is parabolic) are all smooth Fano varieties; hence they are of globally F-regular type (by Proposition 6.3) and have vanishing higher cohomology for all ample (or even nef) line bundles. Alternatively, we can also deduce this directly in arbitrary characteristic using a result of Ramanathan: in any characteristic, the G/P are stably Frobenius split along an ample divisor [R2, 3.1] and hence globally F-regular by Theorem 3.10. Explicitly, we have the following corollary.

7.3. COROLLARY. *Let $P \subset G$ be any parabolic subgroup of a semi-simple algebraic group of arbitrary characteristic. Let T be a torus or a finite group acting on G/P. For any numerically effective line bundle $\mathcal{L}$ on any GIT quotient $X = (G/P)//T$, the cohomology modules $H^i(X, \mathcal{L})$ vanish for $i > 0$.*

Of course, similar corollaries can be drawn for geometric invariant theory quotients of any globally F-regular variety.

Proof of Theorem 7.1. Let X be a connected projective variety over an F-finite ground field of prime characteristic p.

Let T act on X. Choose any lift of this action to an ample line bundle $\mathcal{L}$ on X. Let S be the corresponding section ring, with the given T action. As we have seen in Proposition 3.1 (resp., Theorem 3.10), X is Frobenius split (resp., globally F-regular) if and only if S is F-pure (resp., F-regular).

Let R be the ring of invariants for this action. By definition, the projective scheme defined by R is the geometric invariant quotient of X with respect to our choice of the linearization of T. Because T is linearly reductive, the Reynolds operator provides a splitting of $R \subset S$ as an R-module map. We claim that R is F-pure (resp., F-regular) whenever S is.

In general, if $R \subset S$ splits as a map of R modules and if S is F-pure (resp., F-regular), then R is F-pure (resp., F-regular). This can be seen by considering the commutative diagram

$$\begin{array}{ccc} R^{1/p} & \xrightarrow{R^{1/p}\text{-split}} & S^{1/p} \\ \uparrow & & \uparrow \\ R & \xrightarrow{R\text{-split}} & S, \end{array}$$

together with the fact that the vertical arrow on the left splits as a map of S-modules. Composing the inclusion of $R^{1/p}$ in $S^{1/p}$ with this splitting and then with the splitting of the bottom horizontal arrow gives an R-linear splitting of $R \hookrightarrow R^{1/p}$ [HR2]. A similar argument takes care of the (strongly) F-regular case [HH1].

If the ring of invariants $R = S^T$ is F-pure (resp., F-regular), then the same is true for any Veronese subring $R^{(n)} = \bigoplus_{i \in \mathbb{N}} R_{in}$ generated by elements of R that have degrees a multiple of n, since this Veronese subring splits off from R. Because R is a finitely generated algebra, some such Veronese subring $R^{(n)}$ is generated by its elements $R_n = R_1^{(n)}$ of degree 1.

If a graded ring R is normal and generated by its elements of degree 1, then R is the section ring for the natural $\mathcal{O}(1)$ it defines on Proj R. (In general, a graded ring need not be a section ring for the projective scheme it defines, as the resulting $\mathcal{O}(1)$ need not even be an invertible sheaf. This difficulty does not arise if the ring is generated in degree 1.)

Now, the GIT quotient $X//T$ is Proj $R \cong$ Proj $R^{(n)}$. Because $R^{(n)}$ may be assumed normal and generated in degree 1, it is a section ring for $X//T$. Because $R^{(n)}$ is F-pure (resp., F-regular), we conclude that $X//T$ is Frobenius split (resp., globally F-regular). $\square$

7.4. CAUTION. The corollary does not apply to reductive groups G such as $\mathrm{SL}(n)$ that are not linearly reductive in prime characteristic, because the splitting of $S^G \subset S$ that exists in characteristic 0 is not preserved after reduction to prime characteristic. (See, however, Theorem 7.6 and subsequent remarks.)

If S is an $\mathbb{N}$-graded ring of characteristic 0 and if the linearly reductive group G acts on S, then the ring of invariants $R = S^G$ is a direct summand of S as an R-module. One can show directly in this case that R then has the property that all ideals are tightly closed (in characteristic 0); see the appendix by Hochster in [Hu]. The subtlety is that this is not known to imply that, after reducing modulo p, all ideals are tightly closed for infinitely many closed fibers in a family of models. That is, we do not know whether F-regularity in characteristic 0 is equivalent to F-regular type, although this is conjectured to be true.

7.5. CONJECTURE. *If R is a finitely generated $\mathbb{N}$-graded algebra over a field of characteristic* 0 *in which all ideals are tightly closed, then R has F-regular type.*

The converse of Conjecture 7.5 is known to be true. Furthermore, it is not hard to check that Conjecture 7.5 holds true for Gorenstein rings R, because in this case it is enough to check that a single ideal in each closed fiber is tightly closed—namely, any ideal coming from a homogeneous system of parameters for R. See [HH4] for the theory of tight closure in characteristic 0, including these basic facts.

7.6. THEOREM. *Assume that Conjecture 7.5 is true. Let X be any projective variety over a field of characteristic* 0 *that is Frobenius-split type* (*resp., globally F-regular type*). *Suppose that a reductive group G acts algebraically on X. Then the GIT quotient $X//G$ with respect to any linearization of the action of G is also a Frobenius split type* (*resp., globally F-regular type*) *variety. In particular, every GIT quotient of a Fano variety by a linearly reductive group acting algebraically would be globally F-regular type.*

There are a few cases where we can conclude that rings of invariants of reductive groups have strongly F-regular type. If the group acts on a strongly F-regular ring G that also happens to be a unique factorization domain, then the ring of invariants has strongly F-regular type (see e.g. [SV]). The point is that any reductive group G is an extension of a semi-simple group H by a group T that is linearly reductive for generic prime characteristic reduction. Then $S^G = (S^H)^{G/H}$. But one can show that G^H is Gorenstein and hence has strongly F-regular type, so the invariant ring under G/H has strongly F-regular type by Theorem 7.1. Thus, for certain kinds of linearized actions of reductive groups on certain special varieties, the quotients are globally F-regular. Two examples follow.

7.7. COROLLARY. *Suppose that a reductive group G acts algebraically on projective space $\mathbb{P}^n$ over a field of characteristic* 0 *and that its action is linearized by* $\mathcal{O}(1)$. *Then the GIT quotient $\mathbb{P}^n//G$ is of globally F-regular type.*

Similarly, if a reductive group G acts algebraically on a Grassmannian Y of characteristic 0, *with its action linearized by the Plücker embedding. Then the GIT quotient $Y//G$ is of globally F-regular type.*

Proof. In both cases, the rings on which G acts are strongly F-regular unique factorization domains, so it follows from the proof of [SV, 5.23]. □

The story of F-splitting and global F-regularity for quotients by reductive groups in characteristic p that are not linearly reductive is much more subtle and complicated. See [MR1; MR2] for a treatment of some specific quotients.

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