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The Bergman Metric in the Normal Direction: A Counterexample

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1. Introduction

Let $D \subset\subset \mathbb{C}^n$ be a bounded domain. By $B_D(z; X)$ we denote its Bergman metric and by $d_B(z, w)$ the distance function associated to B_D. The question of the completeness of D with respect to d_B has found much interest. Kobayashi [13; 14] proved criteria for the Bergman completeness of a bounded domain D that are based on a representation of d_B by means of the Fubini–Study metric in the projective space $\mathbb{P}(H^2(D))$ over the Hilbert space $H^2(D)$ of all holomorphic functions on D that are square integrable with respect to Lebesgue measure. Almost all qualitative completeness results for the Bergman metric were obtained by means of this criterion (see e.g. [1; 11; 12; 16]).

Another interesting (yet more refined) method for studying the Bergman completeness of D consists of looking for quantitative estimates for d_B and B_D implying it. In this direction, a very general result was obtained by Diederich and Ohsawa [8], who proved that—for those hyperconvex domains admitting a plurisubharmonic exhaustion function ρ satisfying

$$c\,\mathrm{dist}(\cdot, \partial D)^m \le |\rho| \le C\,\mathrm{dist}(\cdot, \partial D)^{1/m}$$

with suitable constants $c, C, m > 0$—the Bergman distance grows at least like a constant times $\log\log(1/\mathrm{dist}(z, \partial D))$ for z sufficiently close to ∂D. This result applies in particular to all finite intersections of C^2-smooth pseudoconvex domains.

Let now $D = \{r < 0\}$ be a bounded pseudoconvex domain with smooth boundary and let $z^0 \in \partial D$. In this paper we study the following related question on the boundary behavior of the Bergman metric B_D near z^0:

Does there exist a constant $C > 0$ and an open neighborhood $U \ni z^0$ such that, for all directions $X \in \mathbb{C}^n$, one has the lower bound

$$B_D(z; X) \ge C\frac{|(\partial r(z), X)|}{|r(z)|} \tag{1}$$

on $D \cap U$?

Here $(\partial r(z), X) = \sum_{j=1}^n X_j \partial r/\partial z_j(z)$.

The inequality (1) has long been known to be true under certain additional hypotheses on the domain D. It holds for example when z^0 is strongly pseudoconvex

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(shown for the first time in [5]) or if there exists an open neighborhood $V \ni z^0$ and a continuous function $F\colon V \times (V \cap \partial D) \to \mathbb{C}$ such that $F(\cdot, \zeta)$ is holomorphic, $F(\zeta, \zeta) = 0$, and $\operatorname{Re} F(\cdot, \zeta) \leq 0$ on $\bar{D} \cap V$ for any $\zeta \in V \cap \partial D$. (This condition is, for instance, trivially satisfied on convex domains.)

If the boundary point z^0 is supposed to be of finite type in the sense of D'Angelo, a situation where local holomorphic supporting surfaces need not exist in general, then the question of whether (1) is satisfied is settled only in some cases: for example, if $n = 2$ (see [3]); or for $n \geq 3$ if the Levi form of ∂D has at most one degenerate eigenvalue at z^0 [4; 10]. If, however, the approach of the point $z \in D$ to z^0 is restricted to be nontangential, then positive results are obtained if z^0 is of finite semiregular type (see [2; 6; 9]).

Many people have asked whether (1) holds on any bounded smooth pseudoconvex domain, and it is often conjectured that an affirmative answer should be possible. However, by the following theorem, the estimate (1) does not hold, in general, on a smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$.

1.1. THEOREM. *Let $1 > a > 0$ be arbitrary. Then there exists a pseudoconvex domain $D \subset\subset \mathbb{C}^2$, having smooth boundary with $0 \in \partial D$, that is described by a defining function r of the form*

$$r(z, w) = \operatorname{Re} w + b|w|^2 + \rho(z), \tag{2}$$

where ρ denotes a subharmonic function with $\rho(0) = 0$ and where $b > 0$ is suitably chosen, such that:

(i) *∂D is regular, as the weakly pseudoconvex points are of the form $(0, w)$, where w lies on a circle; and*

(ii) *there is no constant $C > 0$ such that*

$$B_D(z; X) \geq C \frac{|(\partial r(z), X)|}{|r(z)|(\log 1/|r(z)|)^{1/1+2a}} \tag{3}$$

holds for $z \in D \cap U$ with any open neighborhood U of 0.

The construction of the counterexample given in Section 2 is inspired by ideas of Krantz, who indicated in [15] how to construct a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ for which the corresponding lower bound

$$F_D^K(z; X) \geq C \frac{|(\partial r(z), X)|}{|r(z)|} \tag{4}$$

does not hold for the infinitesimal Kobayashi metric F_D^K of D.

2. Construction of the Example

We start with the construction of the function ρ appearing in Theorem 1.1. For this we will modify the ingredients going into the construction of the corresponding function ρ from [15, pp. 8, 9] in order to be able to control the Bergman kernel function. For the reader's convenience, we give all main details.

2.1. PROPOSITION. *There exists on $\mathbb{C}$, for any number $0 < a < 1$, a subharmonic smooth function ρ and sequences $(\delta_n)_n$ and $(r_n)_n$ tending to zero such that*:

(i) *for any $z \in \mathbb{C}$ with $|z| \le r_n$,*

$$\rho(z) \le (1 + c_0 a^{-1}(n+1)^3 e^{-n^a})\delta_n + \frac{n\delta_n}{r_n} \operatorname{Re} z \tag{5}$$

with an unimportant constant $c_0 > 0$; and

(ii) *the Laplacian of ρ is positive outside the origin, and all derivatives of ρ vanish at the origin.*

Proof. For $\nu \ge 100$ and $\alpha < 1/48$, let

$$w_\nu(z) := \frac{1}{2} - 4\alpha + \operatorname{Re} z + \alpha\frac{|z|^2}{\nu^2} + \frac{1}{2C_\nu}\log(|z|^2 + e^{-2C_\nu})$$

with

$$C_\nu = 2\log(\nu+1). \tag{6}$$

We proceed in five steps as follows.

Step 1: Definition of certain auxiliary functions. We define the functions

$$R_\nu(z) := \begin{cases} \max\{w_\nu(z), 0\} & \text{if } \operatorname{Re} z > -\frac{1}{4}, \\ w_\nu(z) & \text{if } \operatorname{Re} z \le -\frac{1}{4}, \end{cases}$$

and we claim that R_ν has the following properties:

(a) R_ν is subharmonic;
(b) $R_\nu(z) = 0$ for $|z| < \frac{3}{2}e^{-C_\nu}$;
(c) $R_\nu(z) \le 1 + \operatorname{Re} z$ for $z \in \mathbb{C}$ with $|z| \le \nu + \frac{1}{2}$.

For the proof of (a), we observe at first that w_ν is subharmonic. Hence we need only show that the definition of R_ν is consistent. This follows from the fact that $w_\nu(z) \ge 0$ for any $z \in \mathbb{C}$ with $\operatorname{Re} z = -\frac{1}{4}$. Namely, for $\operatorname{Re} z = -\frac{1}{4}$ we have

$$w_\nu(z) \ge \frac{1}{4} - 4\alpha - \frac{\log 4}{C_\nu} = \frac{1}{4} - 4\alpha - \frac{\log 2}{\log(\nu+1)} \ge 0,$$

since

$$\log(\nu+1) \ge \log 64 = 6\log 2 \ge \frac{\log 2}{\frac{1}{4} - 4\alpha}.$$

For the proof of (b), let $z \in \mathbb{C}$ and $|z| < \frac{3}{2}e^{-C_\nu}$. Then, in particular, $\operatorname{Re} z > -\frac{1}{4}$; hence $R_\nu(z) = \max\{w_\nu(z), 0\}$. We check that $w_\nu(z) \le 0$. Namely:

$$\begin{aligned} w_\nu(z) &= \frac{1}{2} - 4\alpha + \operatorname{Re} z + \alpha\frac{|z|^2}{\nu^2} + \frac{1}{2C_\nu}\log(|z|^2 + e^{-2C_\nu}) \\ &\le \frac{1}{2} - 4\alpha + \frac{3}{2}e^{-C_\nu} + \frac{9\alpha e^{-2C_\nu}}{4\nu^2} + \frac{1}{2C_\nu}\log\left(\left(\frac{9}{4}+1\right)e^{-2C_\nu}\right) \\ &\le \frac{1}{2} - 4\alpha + \frac{3}{2(\nu+1)^2} + \frac{9}{192\nu^2(\nu+1)^4} - 1 + \frac{\log 13/4}{4\log(\nu+1))} \\ &< 0, \end{aligned}$$

since ν was chosen sufficiently large.

In order to prove (c), we choose $z \in \mathbb{C}$ such that $|z| < \nu + \frac{1}{2}$. Then we have

$$\begin{aligned}
w_\nu(z) - 1 - \operatorname{Re} z &= -\frac{1}{2} - 4\alpha + \alpha\frac{|z|^2}{\nu^2} + \frac{1}{2C_\nu}\log(|z|^2 + e^{-2C_\nu}) \\
&\le -\frac{1}{2} + \frac{1}{2C_\nu}\log(|z|^2 + e^{-2C_\nu}) \\
&\le -\frac{1}{2} + \frac{1}{2C_\nu}\log\left(\left(\nu + \frac{1}{2}\right)^2 + e^{-2C_\nu}\right) \\
&= -\frac{1}{2} + \frac{1}{4\log(\nu+1)}\log\left((\nu+1)^2\left(\frac{\left(\nu+\frac{1}{2}\right)^2}{(\nu+1)^2} + \frac{1}{(\nu+1)^6}\right)\right) \\
&= \frac{1}{4\log(\nu+1)}\log\left(\frac{\left(\nu+\frac{1}{2}\right)^2(\nu+1)^4 + 1}{(\nu+1)^6}\right) \\
&< 0,
\end{aligned}$$

because

$$\begin{aligned}
\left(\nu + \tfrac{1}{2}\right)^2(\nu+1)^4 + 1 &= \left(\nu + 1 - \tfrac{1}{2}\right)^2(\nu+1)^4 + 1 \\
&= (\nu+1)^6 - (\nu+1)^5 + \tfrac{1}{4}(\nu+1)^4 + 1 \\
&= (\nu+1)^6 + \tfrac{1}{4}(\nu+1)^4(1 - 4(\nu+1)) + 1 \\
&< (\nu+1)^6
\end{aligned}$$

for $\nu \ge 2$.

This proves (c) for all z with $\operatorname{Re} z \le -\frac{1}{4}$ and for all z with $\operatorname{Re} z > -\frac{1}{4}$ and $w_\nu(z) > 0$. If $\operatorname{Re} z \ge -\frac{1}{4}$ and $w_\nu(z) \le 0$, then one has trivially that $1 + \operatorname{Re} z > 0 = \max\{w_\nu(z), 0\} = R_\nu(z)$. Hence, (c) holds in each case.

Step 2: Smoothing of the functions R_ν. We fix a radially symmetric nonnegative smooth function ϕ_1, with support in the unit disc $\mathbb{D}$ in $\mathbb{C}$, such that $\|\phi_1\|_{L^1} = 1$. For $\varepsilon > 0$ we put $\phi_\varepsilon(z) = \varepsilon^{-2}\phi_1(z/\varepsilon)$. Then we define smooth subharmonic functions u_ν by

$$u_\nu = R_\nu \star \phi_{\varepsilon_\nu}, \quad \text{where } \varepsilon_\nu = \frac{1}{4(\nu+1)^2} = \frac{1}{4}e^{-C_\nu}. \tag{7}$$

Our claim is now that

$$u_\nu(\tilde z) = 0 \quad \text{if } |\tilde z| \le e^{-C_\nu}. \tag{8}$$

Indeed, for such points $\tilde z$ we have

$$u_\nu(\tilde z) = \int_{|\zeta| < \varepsilon_\nu} R_\nu(\tilde z - \zeta)\phi_{\varepsilon_\nu}(\zeta)\, d^2\zeta = 0,$$

since for $\zeta \in \operatorname{supp}(\phi_{\varepsilon_\nu}) = \Delta\left(0, \frac{1}{4}e^{-C_\nu}\right)$ and $|\tilde z| < e^{-C_\nu}$ we have that $\tilde z - \zeta \in \Delta\left(0, \frac{5}{4}e^{-C_\nu}\right)$, where R_ν is identically zero.

Furthermore, we have

$$u_\nu(\tilde z) \le 1 + \operatorname{Re}\tilde z \quad \text{if } |\tilde z| \le \nu. \tag{9}$$

Namely, one can estimate

$$\begin{aligned} u_\nu(\tilde z) &= \int_{|\zeta|<\varepsilon_\nu} R_\nu(\tilde z - \zeta)\phi_{\varepsilon_\nu}(\zeta)\, d^2\zeta \\ &\le \int_{|\zeta|<\varepsilon_\nu} (1 + \operatorname{Re}(\tilde z - \zeta))\phi_{\varepsilon_\nu}(\zeta)\, d^2\zeta \\ &\qquad\qquad \bigl(\text{since } |\tilde z - \zeta| \le \nu + \tfrac12 \text{ for } \zeta \in \operatorname{supp}(\phi_{\varepsilon_\nu})\bigr) \\ &= 1 + \operatorname{Re}\tilde z, \end{aligned}$$

owing to the harmonicity of $\tilde z \mapsto \operatorname{Re}\tilde z$.

Next we estimate the derivatives of the functions u_ν. For an integer $k \ge 0$, let $D^{(k)}$ denote some kth-order derivative. Then one has

$$D^{(k)}u_\nu(\zeta) = R_\nu \star D^{(k)}\phi_{\varepsilon_\nu}(\zeta);$$

this vanishes for $|\zeta| \le e^{-C_\nu}$.

For any point w, we have

$$\begin{aligned} |R_\nu(w)| &\le |w_\nu(w)| \\ &\le \frac12 + |w| + \frac{|w|^2}{\nu^2} + \frac{2C_\nu + \log(1+|w|^2)}{2C_\nu}; \end{aligned}$$

this follows because, for $|w|^2 < 1 - e^{-2C_\nu}$, the log term can be estimated by

$$-2C_\nu \le \log(|w|^2 + e^{-2C_\nu}) \le 0.$$

Hence $|\log(|w|^2 + e^{-2C_\nu})| \le 2C_\nu$ and, for $|w|^2 \ge 1 - e^{-2C_\nu}$,

$$0 \le \log(|w|^2 + e^{-2C_\nu}) \le \log(1 + |w|^2).$$

If now $\zeta \in \mathbb{C}$ is arbitrarily chosen and $w \in \operatorname{supp}(\phi_{\varepsilon_\nu}(\zeta - \cdot))$, then

$$\begin{aligned} |R_\nu(w)| &\le \frac12 + |w| + \frac{|w|^2}{\nu^2} + \frac{2C_\nu + \log(1+|w|^2)}{2C_\nu} \\ &\le \frac34 + |\zeta| + 2\frac{|\zeta|^2 + e^{-2C_\nu}}{\nu^2} + 1 + \frac{\log(2+|\zeta|^2)}{2C_\nu} \end{aligned}$$

and therefore

$$\begin{aligned} |R_\nu \star D^{(k)}\phi_{\varepsilon_\nu}(\zeta)| &= \left| \int_{|\zeta - w|<\varepsilon_\nu} R_\nu(w) D^{(k)}\phi_{\varepsilon_\nu}(\zeta - w)\, d^2w \right| \\ &\le \int_{|\zeta - w|<\varepsilon_\nu} |R_\nu(w)||D^{(k)}\phi_{\varepsilon_\nu}(\zeta - w)|\, d^2w \\ &\le \left(2 + |\zeta| + 2\frac{|\zeta|^2}{\nu^2} + \frac{\log(2+|\zeta|^2)}{2C_\nu}\right)\|D^{(k)}\phi_{\varepsilon_\nu}\|_{L^1} \\ &\le \left(2 + |\zeta| + 2\frac{|\zeta|^2}{\nu^2} + \frac{\log(2+|\zeta|^2)}{2C_\nu}\right)\varepsilon_\nu^{-k}\|D^{(k)}\phi_1\|_{L^1}. \end{aligned}$$

Step 3: Scaling of the u_ν. For numbers $0 < s_\nu < 1$ to be chosen later, we put

$$v_\nu(z) = u_\nu(z/s_\nu).$$

From the considerations in Step 2, we obtain immediately that

$$v_\nu(z) = 0 \quad \text{for } |z| \leq e^{-C_\nu} s_\nu \tag{10}$$

and

$$|D^{(k)} v_\nu(z)| \leq \left(2 + \frac{|z|}{s_\nu} + 2\frac{|z|^2}{\nu^2 s_\nu^2} + \frac{\log(2 + |z/s_\nu|^2)}{2C_\nu}\right)(s_\nu \varepsilon_\nu)^{-k} \|D^{(k)}\phi_1\|_{L^1} \tag{11}$$

for any kth-order derivative $D^{(k)}$.

If, furthermore, $|z| \leq r_\nu := \nu s_\nu$, then $|z/s_\nu| \leq \nu$ and hence

$$v_\nu(z) \leq 1 + \frac{1}{s_\nu} \operatorname{Re} z = 1 + \frac{\nu}{r_\nu} \operatorname{Re} z. \tag{12}$$

With the functions v_ν just constructed, we can now come to the decisive step.

Step 4: Definition of the function ρ. With positive numbers δ_ν (to be chosen shortly and depending on s_ν), we put

$$\rho(z) = \sum_{\nu=100}^{\infty} \delta_\nu v_\nu(z).$$

Assume now that the s_ν and δ_ν have been chosen such that the following requirements (13)–(15) are satisfied (we will show that this is indeed possible):

$$s_n \leq \frac{1}{n} \min_{100 \leq \nu \leq n-1} s_\nu e^{-C_\nu} \quad \text{for } n \geq 101. \tag{13}$$

For any integer k, the following series converges:

$$\sum_{\nu=100}^{\infty} \frac{\delta_\nu}{s_\nu^{k+2} \varepsilon_\nu^k} < +\infty; \tag{14}$$

furthermore, there is a constant $c^* > 0$ such that, for any $n \geq 100$,

$$\sum_{\nu=n+1}^{\infty} \frac{\delta_\nu}{s_\nu^l} \leq c^* \frac{1}{a}(n+1) e^{-n^a} \frac{\delta_n}{s_n^l} \tag{15}$$

for $l = 0, 1, 2$.

Claim: The series with the terms $\delta_\nu v_\nu$ converges together with all its derivatives uniformly on compact subsets of $\mathbb{C}$.

In order to show this, we choose an arbitrary radius $R > 0$. Then, from (11) we obtain

$$\sup_{|z| \leq R} |D^{(k)} v_\nu(z)| \leq (\varepsilon_\nu s_\nu)^{-k} \|D^{(k)}\phi_1\|_{L^1} \left(2 + \frac{R}{s_\nu} + \frac{2R^2}{\nu^2 s_\nu^2} + \frac{1}{2C_\nu} \log\left(2 + \frac{R^2}{s_\nu^2}\right)\right). \tag{11a}$$

From (14), we conclude that the series

$$\sum_{\nu=100}^{\infty} \delta_\nu \sup_{|z|\le R} |D^{(k)} v_\nu(z)|$$

converges. In particular, it now follows that ρ is smooth and subharmonic throughout $\mathbb{C}$.

Next we prove property (5). Suppose that $|z| \le r_n$ for some $n \ge 100$. We split the series ρ into

$$\rho(z) = \sum_{\nu=100}^{n-1} \delta_\nu v_\nu(z) + \delta_n v_n(z) + \sum_{\nu=n+1}^{\infty} \delta_\nu v_\nu(z).$$

Because of (13), for the terms of the first sum we have

$$\frac{|z|}{s_\nu} \le \frac{r_n}{s_\nu} = \frac{n s_n}{s_\nu} \le e^{-C_\nu};$$

hence, by (10), $v_\nu(z) = 0$ for $\nu < n$. From (12) for $\nu = n$ we obtain

$$\delta_n v_n(z) \le \delta_n + \frac{n\delta_n}{r_n} \operatorname{Re} z.$$

Assume now that $\nu > n$. Then, for $|z| \le r_n$, using first (11a) with $R = r_n$ and then (15) for $l = 0, 1, 2$, we can estimate

$$\begin{aligned}
\sum_{\nu=n+1}^{\infty} \delta_\nu v_\nu(z) &\le \sum_{\nu=n+1}^{\infty} \delta_\nu \left(2 + \frac{r_n}{s_\nu} + \frac{2r_n^2}{\nu^2 s_\nu^2} + \frac{1}{2C_\nu} \log\left(2 + \frac{r_n^2}{s_\nu^2}\right)\right) \\
&\le 2 \sum_{\nu=n+1}^{\infty} \delta_\nu + \sum_{\nu=n+1}^{\infty} \delta_\nu \frac{r_n}{s_\nu} + \sum_{\nu=n+1}^{\infty} \delta_\nu \frac{2r_n^2}{\nu^2 s_\nu^2} + \sum_{\nu=n+1}^{\infty} \delta_\nu \frac{r_n^2}{2C_\nu s_\nu^2} \\
&\le c_0' \left(\sum_{\nu=n+1}^{\infty} \delta_\nu + r_n \sum_{\nu=n+1}^{\infty} \frac{\delta_\nu}{s_\nu} + r_n^2 \sum_{\nu=n+1}^{\infty} \frac{\delta_\nu}{s_\nu^2} \right) \\
&\le c_0 \frac{1}{a} (n+1)^3 e^{-n^a} \delta_n,
\end{aligned}$$

because $r_n = n s_n$ by definition. This implies (5).

Also, Proposition 2.1(ii) holds for ρ. By construction, all the derivatives of ρ vanish at the origin. If now $z \in \mathbb{C} \setminus \{0\}$, then there exists an index ν for which $R_\nu = w_\nu$ in a disc with center z/s_ν and radius ε_ν. Because of the appearance of the term $\alpha(1/\nu^2)|z|^2$ in the definition of w_ν, this implies that also the Laplacian of v_ν is positive near z.

We still have to show that the parameters going into the construction of the function ρ can indeed be chosen such that the inequalities (13)–(15) are satisfied.

Step 5: Choice of the parameters. We put $s_\nu = 1/(\nu!)^3$ and $r_\nu = \nu s_\nu$. Furthermore, for a positive number $0 < a < 1$, let

$$\delta_\nu = s_\nu^2 \exp(-\nu^{1+a}) \tag{16}$$

We will now check that (13)–(15) are satisfied.

The proof of (13) is easy:

$$\begin{aligned}\frac{1}{n}\min_{100\le\nu\le n-1} s_\nu e^{-C_\nu} &= \frac{1}{n}\min_{100\le\nu\le n-1}\frac{s_\nu}{(\nu+1)^2}\\ &= \frac{1}{n}\min_{100\le\nu\le n-1}\frac{1}{(\nu!)^3(\nu+1)^2}\\ &\ge \frac{1}{n}\frac{1}{((n-1)!)^3 n^2}\\ &= s_n.\end{aligned}$$

The proof of (14) is also not difficult:

$$\begin{aligned}\frac{\delta_\nu}{\varepsilon_\nu^k s_\nu^{k+2}} &= 4^k(\nu+1)^{2k}(\nu!)^{3k}e^{-\nu^{a+1}}\\ &\le 4^k\exp(-\nu^{a+1}+2k\log(\nu+1)+3k\nu\log\nu).\end{aligned}$$

The latter terms belong to a convergent majorant of the series $\sum_\nu \delta_\nu/\varepsilon_\nu^k s_\nu^{k+2}$.

Inequality (15) also holds. In fact, let $l\in\{0,1,2\}$. Then, for $\nu\ge n+1$ we have

$$\frac{\delta_\nu}{s_\nu^l}\cdot\frac{s_n^l}{\delta_n} = \left(\frac{s_\nu}{s_n}\right)^{2-l} e^{-\nu^{1+a}+n^{1+a}} \le e^{-\nu^{1+a}+n^{1+a}}.$$

We now estimate the difference $-\nu^{1+a}+n^{1+a}$:

$$\begin{aligned}-\nu^{1+a}+n^{1+a} &= -\nu(\nu^a-n^a)-n^a(\nu-n)\\ &\le -\nu(\nu^a-n^a)-n^a\\ &\le -\nu((n+1)^a-n^a)-n^a\\ &\le -a\nu\int_n^{n+1}x^{a-1}\,dx-n^a\\ &\le -a(n+1)^{a-1}\nu-n^a.\end{aligned}$$

Altogether, these estimates give

$$\begin{aligned}\frac{s_n^l}{\delta_n}\sum_{\nu=n+1}^{\infty}\frac{\delta_\nu}{s_\nu^l} &\le e^{-n^a}\sum_{\nu=n+1}^{\infty}\exp(-a(n+1)^{a-1}\nu)\\ &= e^{-n^a}\frac{\exp(-a(n+1)^{a-1}(n+1))}{1-\exp(-a(n+1)^{a-1})}\\ &\le e^{-n^a}a^{-1}(n+1)^{1-a}\\ &\le \frac{1}{a}(n+1)e^{-n^a},\end{aligned}$$

showing (15).

Together with the arguments from Step 4, Proposition 2.1 is now proved. □

We now can define the pseudoconvex domain D, which will serve as a counter-example to (3).

2.2. LEMMA. *Let*

$$r(z, w) = \operatorname{Re} w + b|w|^2 + \rho(z) = \left|\sqrt{b}w + \frac{1}{2\sqrt{b}}\right|^2 - \frac{1}{4b} + \rho(z),$$

where ρ is as in Proposition 2.1 and $b > 0$ is such that $1/4b$ is not a critical value of ρ. Then the pseudoconvex domain

$$D = \{(z, w) \in \mathbb{C}^2 : r(z, w) < 0\}$$

is smoothly bounded and is regular near 0. The Levi degeneracy set is given by

$$\left\{(0, w) \;\middle|\; \left|\sqrt{b}w + \frac{1}{2\sqrt{b}}\right| = \frac{1}{2\sqrt{b}}\right\}.$$

Proof. This can be seen immediately from the properties of ρ. The boundedness of D follows from the observation that $\rho(z) \gtrsim |z|^2$ for large $|z|$. Furthermore, if $r_w(z, w) = 0$ and $r(z, w) = 0$, then $\rho(z) = 1/4b$; hence $\rho_z(z, w) \neq 0$. This proves the smoothness of ∂D. The Levi function of ∂D is

$$\lambda_D = |r_w|^2 \rho_{z\bar{z}} + b|r_z|^2 = \left|\tfrac{1}{2} + b\bar{w}\right|^2 \rho_{z\bar{z}} + b|\rho_z|^2.$$

If $(z, w) \in \partial D$ and $z \neq 0$, then $\rho_{z\bar{z}}(z) > 0$. If now $\frac{1}{2} + b\bar{w} = 0$, then $\rho(z) = 1/4b$ and hence $|\rho_z(z)|^2 > 0$. The only weakly pseudoconvex points in ∂D must therefore be of the form $(0, w)$. This proves the claim. □

2.3. LEMMA. *Let $(r_n)_n$, $(\delta_n)_n$ be as defined in* (16) *and let $p_n = (0, -5\delta_n/4)$. Then the Bergman kernel K_D of D* (*on the diagonal*) *at p_n can be estimated by*

$$\frac{1}{C}(\delta_n r_n)^{-2} \leq K_D(p_n) \leq C(\delta_n r_n)^{-2}$$

with some unimportant constant C.

Proof. We denote by f_n the following change of coordinates:

$$f_n(z, w) = \left(z, w + \frac{n\delta_n}{r_n} z\right). \tag{17}$$

Its inverse is given by

$$f_n^{-1}(z', w') = \left(z', w' - \frac{n\delta_n}{r_n} z'\right).$$

The mapping f_n leaves p_n fixed and transforms D into the domain $D_n = f_n(D) = \{\psi_n < 0\}$, with

$$\begin{aligned}\psi_n(z', w') &= r(f_n^{-1}(z', w')) \\ &= \operatorname{Re} w' + \rho(z') - \frac{n\delta_n}{r_n} \operatorname{Re} z' + b\left|w' - \frac{n\delta_n}{r_n} z'\right|^2.\end{aligned} \tag{18}$$

The upper bound for K_D then can be deduced as follows. The bidisc

$$\Delta_n := \Delta(0, r_n) \times \Delta\left(-\frac{5\delta_n}{4}, \frac{\delta_n}{8}\right) \tag{19}$$

is, for sufficiently large n, contained in D_n. This follows from property (5) of the function ρ. Namely, for $(z', w') \in \Delta_n$ we have

$$\operatorname{Re} w' + b\left|w' - \frac{n\delta_n}{r_n} z'\right|^2 < -\frac{9\delta_n}{8} + b\left(\frac{11}{8} + n\right)^2 \delta_n^2 < -\frac{17}{16}\delta_n \quad \text{if } n \gg 1.$$

Moreover,

$$\rho(z') - \frac{n\delta_n}{r_n} \operatorname{Re} z' \leq (1 + c_0 a^{-1}(n+1)^3 e^{-n^a})\delta_n.$$

Inserting this, we obtain $\psi_n(z', w') < 0$; hence $\Delta_n \subset D_n$ if $n \gg 1$.

The Bergman kernel increases if the domain is shrunk. This gives us the upper bound:

$$K_D(p_n) = K_{D_n}(p_n) \leq K_{\Delta_n}(p_n) = \frac{1}{\operatorname{Vol}(\Delta_n)} = \frac{64}{\pi}(\delta_n r_n)^{-2}.$$

We now come to the proof of the lower bound for K_D.

Let us first recall a result of Ohsawa from [17]: If $\Omega \subset \mathbb{C}^d$ is a pseudoconvex domain, if $E \subset \mathbb{C}^d$ is a hyperplane with $\Omega' = \Omega \cap E \neq \emptyset$, and if $\varphi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is a plurisubharmonic function such that

$$C_\varphi := \sup_\Omega(\varphi(z) + 2\log \operatorname{dist}(z, E)) < +\infty$$

(these weights are called *negligible*), then any holomorphic function f satisfying

$$I_\varphi(f) := \int_{\Omega'} |f|^2 e^{-\varphi}\, d\lambda_{d-1} < +\infty$$

admits a holomorphic extension $\tilde{f}\colon \Omega \to \mathbb{C}$ with an L^2-norm controlled by $I_\varphi(f)$, namely,

$$\int_\Omega |\tilde{f}|^2\, d\lambda_d \leq C_d e^{C_\varphi} I_\varphi(f)$$

with an unimportant constant C_d. (Here, by $d\lambda_k$ we denote the Lebesgue measure in complex dimension k.)

If we apply this to the Bergman kernel (see [7]), we obtain

$$K_\Omega(p) \geq C_d^{-1} e^{-C_\varphi} K_{\Omega',\varphi}(p) \tag{20}$$

for $p \in \Omega'$. Here $K_{\Omega',\varphi}(p)$ denotes the weighted Bergman kernel of Ω' at p for the space of all holomorphic functions f on Ω' with $I_\varphi(f) < +\infty$.

We want to apply this to $\Omega = D_n$ and $E = \{z' = 0\}$, in which case we have

$$D_n' = D_n \cap E = \left\{ w' \;\middle|\; \left|\sqrt{b}w' + \frac{1}{2\sqrt{b}}\right| < \frac{1}{2\sqrt{b}} \right\}.$$

If φ is plurisubharmonic on D_n, then we obtain (using [7, Thm. 3.5]) that

$$K_{D'_n,\varphi}(p) \geq (\operatorname{dist}(p, \partial D'_n))^{-2} e^{\varphi(p)} \tag{21}$$

for any $p \in D'_n$.

What we need to find is, for each sufficiently large n, a negligible weight $\varphi = \varphi_n$ on D_n (with $C_{\varphi_n} \leq 0$) for which

$$\varphi_n(p_n) \geq C + 2\log\frac{1}{r_n} \tag{22}$$

with some unimportant constant C. If we have found this then, combining and applying (20) and (21) to $p = p_n$, we are done (note that $\operatorname{dist}(p_n, \partial D'_n) \approx \delta_n$).

Choice of a suitable negligible weight. Let $n \geq 100$ be arbitrary. From the definition of the function u_n we have $u_n \geq R_n$, so

$$v_n(z') = u_n\left(\frac{z'}{s_n}\right) \geq R_n\left(\frac{z'}{s_n}\right) \geq w_n\left(\frac{z'}{s_n}\right)$$

and (noting that $\log(|z'|^2 + e^{-2C_n}) \geq -2C_n$)

$$w_n\left(\frac{z'}{s_n}\right) \geq -\frac{1}{2} - 4\alpha + \operatorname{Re}\frac{z'}{s_n} + \alpha\left|\frac{z}{r_n}\right|^2.$$

We choose a smooth function $\chi : \mathbb{R} \to (-\infty, 1]$ with $\chi(t) = t$ for $t \leq 1/2$ and $\chi(t) = 1$ for $t \geq 3/4$. Then, for a small enough constant $c_1 > 0$, the function $z' \mapsto |z'|^2 + c_1 \log\chi(|z'|^2)$ becomes subharmonic. This implies the subharmonicity of the functions

$$\hat{v}_n(z') = -\frac{1}{2} - 4\alpha + \operatorname{Re}\frac{z'}{s_n} + \alpha\left(\left|\frac{z'}{r_n}\right|^2 + c_1\log\chi\left(\frac{|z'|^2}{r_n^2}\right)\right)$$

and

$$\tilde{v}_n(z') := \hat{v}_n(z') - \alpha c_1 \log|z'|^2.$$

From the choice of χ, it also follows that

$$v_n \geq \hat{v}_n.$$

We now claim that

$$\varphi_n(z', w') := \frac{1}{\alpha c_1 \delta_n}\left(\operatorname{Re} w' + \sum_{\nu=100}^{n-1} \delta_\nu v_\nu(z') + \delta_n \tilde{v}_n(z') - \frac{n\delta_n}{r_n}\operatorname{Re} z' + \sum_{\nu=n+1}^{\infty} \delta_\nu v_\nu(z')\right)$$

is the desired negligible weight. In fact, for any $(z', w') \in D_n$ one has

$$\begin{aligned}
\alpha c_1 \delta_n(\varphi_n(z, w') + 2\log|z'|) &= \operatorname{Re} w' + \sum_{\nu=100}^{n-1} \delta_\nu v_\nu(z') \\
&\quad + \delta_n \underbrace{\hat{v}_n(z')}_{\leq v_n(z')} - \frac{n\delta_n}{r_n}\operatorname{Re} z' + \sum_{\nu=n+1}^{\infty} \delta_\nu v_\nu(z') \\
&\leq \operatorname{Re} w' + \rho(z') - \frac{n\delta_n}{r_n}\operatorname{Re} z' \\
&\leq \psi_n(z', w') < 0.
\end{aligned}$$

We therefore obtain $C_{\varphi_n} \le 0$. The function φ_n is obviously plurisubharmonic and has the right behavior at p_n, namely,

$$\begin{aligned}\varphi_n(p_n) &= \frac{1}{\alpha c_1}\left(-\frac{5}{4} + \tilde{v}_n(0)\right)\\ &= \frac{1}{\alpha c_1}\left(-\frac{5}{4} - \frac{1}{2} - 4\alpha + \alpha c_1 \log r_n^{-2}\right) \ge C + \log r_n^{-2}\end{aligned}$$

with an unimportant constant C.

Lemma 2.3 is thus proved. □

3. Final Proof of Theorem 1.1

For a vector $X \in \mathbb{C}^2$, we compute

$$f_n'(p_n)X = \begin{pmatrix} 1 & 0 \\ \frac{n\delta_n}{r_n} & 1 \end{pmatrix} X = \begin{pmatrix} X_1 \\ \frac{n\delta_n}{r_n} X_1 + X_2 \end{pmatrix}.$$

From the Bergman theory we recall that, for $(z, w) \in D$ and $X \in \mathbb{C}^2$, the functional

$$b_D((z, w); X) := \sqrt{K_D((z, w))}\, B_D((z, w); X)$$

increases if D is replaced by a subdomain of D. Let us now assume that a lower bound of the form (3) would exist with a suitable constant $C > 0$. This would yield

$$\begin{aligned} C\frac{|(\partial r(p_n), X)|}{|r(p_n)||\log|r(p_n)||^{1/1+2a}} &\le B_D(p_n; X)\\ &= B_{D_n}(p_n; f_n'(p_n)X)\\ &= \frac{b_{D_n}(p_n; f_n'(p_n)X)}{\sqrt{K_{D_n}(p_n)}}\\ &\le \frac{b_{\Delta_n}(p_n; f_n'(p_n)X)}{\sqrt{K_D(p_n)}}\\ &= \frac{\sqrt{K_{\Delta_n}(p_n)}}{\sqrt{K_D(p_n)}} B_{\Delta_n}(p_n; f_n'(p_n)X)\\ &\le C' B_{\Delta_n}(p_n; f_n'(p_n)X) \quad \text{(by Lemma 2.3)}\\ &= C'\left(2\frac{|X_1|^2}{r_n^2} + 128\frac{\left|\frac{n\delta_n}{r_n}X_1 + X_2\right|^2}{\delta_n^2}\right)^{1/2}. \end{aligned}$$

Now we choose

$$X = X_{(n)} := \begin{pmatrix} 1 \\ -\frac{n\delta_n}{r_n} \end{pmatrix}$$

and insert this into the previous estimate. Then

$$(\partial r(p_n), X_{(n)}) = -\frac{n\delta_n}{2r_n}.$$

With some new constant C^*, we obtain

$$\frac{n}{r_n\left(\log \frac{1}{\delta_n}\right)^{1/1+2a}} \le C^* \frac{1}{r_n};$$

in particular,

$$n \le C^* \left(\log \frac{1}{\delta_n}\right)^{1/1+2a}. \tag{23}$$

On the other hand, by the definition of δ_n (see (16)) we have

$$\delta_n = s_n^2 e^{-n^{1+a}} = e^{-n^{1+a} - 6 \log n!} \ge e^{-n^{1+a} - 6n \log n} \ge e^{-2n^{1+a}}$$

for $n \gg 1$; hence $\log \delta_n \ge -2n^{1+a}$ and

$$n \ge \left(\frac{1}{2} \log \frac{1}{\delta_n}\right)^{1/1+a},$$

which contradicts (23).

This proves the theorem. □

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