**Michigan Math. J. 47 (2000)**

# On Elliptic $K3$ Surfaces

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## 1. Introduction

By virtue of Torelli's theorem for the period map on the moduli of complex $K3$ surfaces [4; 13; 18], we can study many aspects of $K3$ surfaces from the lattice-theoretic point of view. In this paper, we determine all possible $ADE$-types of singular fibers of elliptic $K3$ surfaces using Nikulin's theory of discriminant forms of even integral lattices. We also determine, for each $ADE$-type of singular fibers, all possible torsion parts of the Mordell–Weil groups. Throughout this paper, we use the term "an elliptic $K3$ surface" for "a complex elliptic $K3$ surface with a distinguished zero section" and the term "an elliptic fibration" for "a complex Jacobian elliptic fibration".

A finite formal sum of the symbols $A_l$ ($l \ge 1$), $D_m$ ($m \ge 4$), and $E_n$ ($n = 6, 7, 8$) with nonnegative integer coefficients is called an *$ADE$-type.* For an $ADE$-type

$$\Sigma := \sum a_l A_l + \sum d_m D_m + \sum e_n E_n,$$

we denote by $L(\Sigma)^-$ the negative definite root lattice generated by a root system of type $\Sigma$, and by $\operatorname{rank}(\Sigma)$ the rank of $L(\Sigma)^-$. By definition, we have $\operatorname{rank}(\Sigma) = \sum a_l l + \sum d_m m + \sum e_n n$.

Let $f : X \to \mathbb{P}^1$ be an elliptic $K3$ surface, and let $O : \mathbb{P}^1 \to X$ be the zero section of $f$. Let $\mathrm{MW}_f$ be the Mordell–Weil group of $f$. The torsion part of $\mathrm{MW}_f$ is a finite abelian group, which we shall denote by $G_f$. We put

$$R_f := \{\, p \in \mathbb{P}^1 \mid f^{-1}(p) \text{ is reducible} \,\}$$

and, for each $p \in R_f$, we denote by $f^{-1}(p)^\sharp$ the union of irreducible components of $f^{-1}(p)$ that are disjoint from the zero section. It is known that the cohomology classes of irreducible components of $f^{-1}(p)^\sharp$ span a negative definite root lattice generated by an indecomposable root system of type $A_l$, $D_m$, or $E_n$. Let $\tau_{f,p}$ be the type. The type of singular fiber $f^{-1}(p)$ in the list of Kodaira's classification [7] is related to $\tau_{f,p}$ in an almost one-to-one way (cf. Table 2.8). We define the $ADE$-type $\Sigma_f$ of $f : X \to \mathbb{P}^1$ by

$$\Sigma_f := \sum_{p \in R_f} \tau_{f,p}.$$

Received July 9, 1999. Revision received July 28, 2000.

The Néron–Severi lattice $\mathrm{NS}_X$ of $X$ contains the sublattice $S_f$ generated by the cohomology classes of the irreducible components of $\bigcup_{p\in R_f} f^{-1}(p)^\sharp$, which is isomorphic to $L(\Sigma_f)^-$.

Through computer-aided calculation, we have made the complete list of pairs $(\Sigma, G)$ of an $ADE$-type $\Sigma$ and a finite abelian group $G$ that can be realized as the data $(\Sigma_f, G_f)$ of an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$. This list $\mathcal{P}$ consists of 3693 pairs. In this paper, we present the list $\mathcal{P}$, deduce some geometric facts from it, and explain the algorithm for obtaining it.

The list $\mathcal{P}$ is too large to be included here in a naive way. We therefore describe $\mathcal{P}$ by giving a subset $\mathcal{S}$ of $\mathcal{P}$ and a set of transformation rules of $ADE$-types that generate $\mathcal{P}$ from $\mathcal{S}$ (see Section 2). The reader can obtain $\mathcal{P}$ easily using this description (the list can also be retrieved from the author's homepage: http://www.math.sci.hokudai.ac.jp/~shimada/K3.html).

An elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$ is said to be *extremal* if the sublattice $S_f$ attains the maximal rank 18. After the work of Miranda and Persson [10]—supplemented by Artal-Bartolo, Tokunaga, and Zhang [1] and Ye [23]—the $ADE$-types of singular fibers of extremal elliptic $K3$ surfaces and their Mordell–Weil groups were completely determined in [16]. The list consists of 336 pairs.

One of the remarkable facts that can be read off from the list $\mathcal{P}$ is that an $ADE$-type $\Sigma$ is an $ADE$-type of an elliptic $K3$ surface with trivial Mordell–Weil torsion if and only if $\Sigma$ is obtained from an $ADE$-type of an extremal elliptic $K3$ surface with trivial Mordell–Weil torsion by *elementary transformation*, that is, by deleting vertices from the corresponding Dynkin graph (see Theorem 2.3). However, in order to describe the list of $ADE$-types of elliptic $K3$ surfaces with nontrivial Mordell–Weil torsion, we must forbid the use of some types of elementary transformation (Theorems 2.4–2.7).

By Nishiyama [12] and Besser [2], the technique of discriminant forms was used to find out all possible elliptic fibrations on special $K3$ surfaces. In [19; 20], Urabe investigated possible configurations of singular points on $K3$ surfaces and suggested the existence of a set of simple rules that generates all possible configurations. In [21; 22], Yang made the complete list of all possible configurations of singularities of $ADE$-type on plane sextic curves and quartic surfaces using the technique of discriminant forms and a computer.

This paper is organized as follows. In Section 2 we describe $\mathcal{P}$ and state some facts about elliptic $K3$ surfaces that can be derived from the list $\mathcal{P}$. In Section 3 we recall the definition and properties of local invariants of lattices over $\mathbb{Z}$ according to Conway and Sloane [6, Chap. 15]. In Sections 4 and 5, we review Nikulin's theory [11] of discriminant forms of even lattices over $\mathbb{Z}$. A criterion for establishing whether there exists an even integral lattice of a given signature and a discriminant form is described in detail in Section 5; this criterion is slightly different from [11, Thm. 1.10.1] and is more suited to machine calculation. In Section 6 we recall the properties of root lattices. In Section 7, we show that it is possible to determine by a purely lattice-theoretic calculation whether a given pair $(\Sigma, G)$ can be realized as $(\Sigma_f, G_f)$ of an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$. Here we use

**Table 2.1** Cardinalities of $\mathcal{P}^G$

| $G$ | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [2, 2] | [4, 2] | [6, 2] | [3, 3] | [4, 4] | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{P}^G\rvert$ | 2746 | 732 | 85 | 41 | 6 | 10 | 1 | 1 | 61 | 5 | 1 | 3 | 1 | 3693 |

Kondo–Nishiyama's lemma on the Néron–Severi lattice of an elliptic $K3$ surface. In Section 8, we explain our algorithm.

The program for making $\mathcal{P}$ was written by Maple V. The author would like to thank Waterloo Maple, Inc. for developing the nice software. The author also would like to thank the referee for suggesting some improvements on the first version of this paper.

## 2. Main Results

All results in this section are obtained simply by looking at the list $\mathcal{P}$.

### 2.1. Torsion Parts of Mordell–Weil Groups

THEOREM 2.1. *The torsion part of the Mordell–Weil group of an elliptic $K3$ surface is isomorphic to one of the following*:

$$\begin{gathered}(0),\ \mathbb{Z}/(2),\ \mathbb{Z}/(3),\ \mathbb{Z}/(4),\ \mathbb{Z}/(5),\ \mathbb{Z}/(6),\ \mathbb{Z}/(7),\ \mathbb{Z}/(8),\\ \mathbb{Z}/(2)\times\mathbb{Z}/(2),\ \mathbb{Z}/(4)\times\mathbb{Z}/(2),\ \mathbb{Z}/(6)\times\mathbb{Z}/(2),\\ \mathbb{Z}/(3)\times\mathbb{Z}/(3),\ \mathbb{Z}/(4)\times\mathbb{Z}/(4).\end{gathered}\tag{2.1}$$

For a group $G$ in (2.1), we denote by $\mathcal{P}^G$ the set of all $ADE$-types $\Sigma$ such that there exists an elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with $\Sigma_f = \Sigma$ and $G_f \cong G$. The cardinalities of $\mathcal{P}^G$ are given in Table 2.1. Here, $[a]$ denotes the cyclic group $\mathbb{Z}/(a)$, and $[a, b]$ denotes $\mathbb{Z}/(a) \times \mathbb{Z}/(b)$. In particular, [1] denotes the trivial group.

For a positive integer $r$, let $\mathcal{P}^G_r$ be the subset of $\mathcal{P}^G$ that consists of $\Sigma \in \mathcal{P}^G$ with $\operatorname{rank}(\Sigma) = r$. Let $f : X \to \mathbb{P}^1$ be an elliptic $K3$ surface. Since the Néron–Severi lattice $\mathrm{NS}_X$ of $X$ contains the orthogonal direct sum of $S_f \cong L(\Sigma_f)^-$ and the lattice of rank 2 generated by the cohomology classes of the zero section and a general fiber, and since the Néron–Severi rank of $X$ is at most 20, we always have

$$\operatorname{rank}(\Sigma_f) \le 18.$$

Hence $\mathcal{P}^G_r$ is empty for $r > 18$.

### 2.2. $ADE$-Types of Singular Fibers

Next we describe the list $\mathcal{P}^G$ for each abelian group $G$ in (2.1). We carry out this task by three different methods according to the size of $\mathcal{P}^G$.

**Table 2.2** Substitutions

| | |
|---|---|
| $A_l \;\mapsto\; A_{l'} + A_{l-1-l'}\;\; (0 \le l' \le l/2);$ | |
| $D_m \;\mapsto\; \begin{cases} A_{m-1},\;\; 2A_1 + A_{m-3},\;\; A_3 + A_{m-4}, \\ D_{m'} + A_{m-1-m'}\;\; (4 \le m' \le m-1); \end{cases}$ | |
| $E_n \;\mapsto\; \begin{cases} A_{n-1},\;\; D_{n-1},\;\; A_1 + A_{n-2},\;\; A_1 + A_2 + A_{n-4},\;\; A_4 + A_{n-5}, \\ D_5 + A_{n-6},\;\; E_{n'} + A_{n-1-n'}\;\; (6 \le n' \le n-1). \end{cases}$ | |

*Case 1:* $G \in \{[1], [2], [3], [4], [2,2]\}$. We describe $\mathcal{P}^G$ by giving a subset $\mathcal{S}^G \subset \mathcal{P}^G$ and a set of transformation rules on $ADE$-types that generate the whole $\mathcal{P}^G$ from the subset $\mathcal{S}^G$.

Let $\Gamma(\Sigma)$ be the Dynkin graph of the $ADE$-type $\Sigma$. If we remove a vertex $P$ of $\Gamma(\Sigma)$ and the edges emitting from $P$, we obtain the Dynkin graph $\Gamma(\Sigma')$ of another $ADE$-type $\Sigma'$ with $\operatorname{rank}(\Sigma') = \operatorname{rank}(\Sigma) - 1$. In this case, we say that $\Sigma'$ is obtained from $\Sigma$ by deleting a vertex. In other words, an $ADE$-type $\Sigma'$ is obtained from $\Sigma$ by deleting a vertex if and only if $\Sigma'$ is obtained by applying to $\Sigma$ one of the substitutions listed in Table 2.2. In this table, we understand that $A_0 := 0$.

DEFINITION 2.2. When we can obtain an $ADE$-type $\Sigma'$ from an $ADE$-type $\Sigma$ by applying substitutions in Table 2.2 several times, we say that $\Sigma'$ is obtained from $\Sigma$ by *elementary transformation*.

THEOREM 2.3. (1) *The list* $\mathcal{P}_{18}^{[1]}$ *consists of* 199 *elements as follows.*

$2E_8 + A_2,\; 2E_8 + 2A_1,\; E_8 + E_7 + A_3,\; E_8 + E_7 + A_2 + A_1,\; E_8 + E_6 + D_4,$
$E_8 + E_6 + A_4,\; E_8 + E_6 + A_3 + A_1,\; E_8 + D_{10},\; E_8 + D_9 + A_1,\; E_8 + D_7 + A_2 + A_1,$
$E_8 + D_6 + A_4,\; E_8 + D_6 + 2A_2,\; E_8 + 2D_5,\; E_8 + D_5 + A_5,\; E_8 + D_5 + A_4 + A_1,$
$E_8 + A_{10},\; E_8 + A_9 + A_1,\; E_8 + A_8 + A_2,\; E_8 + A_8 + 2A_1,\; E_8 + A_7 + A_2 + A_1,$
$E_8 + A_6 + A_4,\; E_8 + A_6 + A_3 + A_1,\; E_8 + A_6 + 2A_2,\; E_8 + A_6 + A_2 + 2A_1,\; E_8 + 2A_5,$
$E_8 + A_5 + A_4 + A_1,\; E_8 + A_5 + A_3 + A_2,\; E_8 + 2A_4 + 2A_1,\; E_8 + A_4 + A_3 + A_2 + A_1,$
$E_8 + 2A_3 + 2A_2,\; 2E_7 + A_4,\; 2E_7 + 2A_2,\; E_7 + E_6 + D_5,\; E_7 + E_6 + A_5,$
$E_7 + E_6 + A_4 + A_1,\; E_7 + E_6 + A_3 + A_2,\; E_7 + D_{11},\; E_7 + D_9 + A_2,\; E_7 + D_7 + A_4,$
$E_7 + D_5 + A_6,\; E_7 + D_5 + A_4 + A_2,\; E_7 + A_{11},\; E_7 + A_{10} + A_1,\; E_7 + A_9 + A_2,$
$E_7 + A_8 + A_3,\; E_7 + A_8 + A_2 + A_1,\; E_7 + A_7 + A_4,\; E_7 + A_7 + 2A_2,\; E_7 + A_6 + A_5,$
$E_7 + A_6 + A_4 + A_1,\; E_7 + A_6 + A_3 + A_2,\; E_7 + A_6 + 2A_2 + A_1,\; E_7 + A_5 + A_4 + A_2,$
$E_7 + A_4 + A_3 + 2A_2,\; 2E_6 + D_6,\; 2E_6 + A_6,\; 2E_6 + 2A_3,\; E_6 + D_{12},\; E_6 + D_{11} + A_1,$
$E_6 + D_9 + A_3,\; E_6 + D_9 + A_2 + A_1,\; E_6 + D_8 + A_4,\; E_6 + D_7 + D_5,\; E_6 + D_7 + A_4 + A_1,$
$E_6 + D_6 + A_6,\; E_6 + D_6 + A_4 + A_2,\; E_6 + D_5 + A_7,\; E_6 + D_5 + A_6 + A_1,$
$E_6 + D_5 + A_4 + A_3,\; E_6 + A_{12},\; E_6 + A_{11} + A_1,\; E_6 + A_{10} + A_2,\; E_6 + A_{10} + 2A_1,$
$E_6 + A_9 + A_3,\; E_6 + A_9 + A_2 + A_1,\; E_6 + A_8 + A_4,\; E_6 + A_8 + A_3 + A_1,\; E_6 + A_7 + A_5,$
$E_6 + A_7 + A_4 + A_1,\; E_6 + A_6 + A_5 + A_1,\; E_6 + A_6 + A_4 + A_2,\; E_6 + A_6 + A_4 + 2A_1,$
$E_6 + A_6 + A_3 + A_2 + A_1,\; E_6 + A_5 + A_4 + A_3,\; E_6 + 2A_4 + A_3 + A_1,\; D_{18},\; D_{17} + A_1,$
$D_{15} + A_2 + A_1,\; D_{14} + A_4,\; D_{14} + 2A_2,\; D_{13} + D_5,\; D_{13} + A_5,\; D_{13} + A_4 + A_1,$

$D_{11}+A_6+A_1,\ D_{11}+A_5+A_2,\ D_{11}+A_4+A_2+A_1,\ D_{11}+A_3+2A_2,\ D_{10}+A_8,$
$D_{10}+A_6+A_2,\ D_{10}+2A_4,\ 2D_9,\ D_9+D_5+A_4,\ D_9+A_9,\ D_9+A_8+A_1,$
$D_9+A_6+A_2+A_1,\ D_9+A_5+A_4,\ D_9+A_4+2A_2+A_1,\ D_8+A_6+2A_2,$
$2D_7+2A_2,\ D_7+A_{10}+A_1,\ D_7+A_9+A_2,\ D_7+A_6+A_5,\ D_7+A_6+A_4+A_1,$
$D_7+A_6+A_3+A_2,\ D_7+2A_4+A_2+A_1,\ D_6+A_{12},\ D_6+A_{10}+A_2,\ D_6+A_8+A_4,$
$D_6+2A_6,\ D_6+A_6+A_4+A_2,\ D_6+2A_4+2A_2,\ 2D_5+A_8,\ 2D_5+2A_4,$
$D_5+A_{13},\ D_5+A_{12}+A_1,\ D_5+A_{10}+A_2+A_1,\ D_5+A_9+A_4,\ D_5+A_9+2A_2,$
$D_5+A_8+A_5,\ D_5+A_8+A_4+A_1,\ D_5+2A_6+A_1,\ D_5+A_6+A_5+A_2,$
$D_5+A_6+A_4+A_2+A_1,\ D_5+A_6+A_3+2A_2,\ D_5+A_5+2A_4,\ A_{18},\ A_{17}+A_1,$
$A_{16}+A_2,\ A_{16}+2A_1,\ A_{15}+A_2+A_1,\ A_{14}+A_4,\ A_{14}+A_3+A_1,\ A_{14}+A_2+2A_1,$
$A_{13}+A_5,\ A_{13}+A_4+A_1,\ A_{13}+A_3+A_2,\ A_{13}+2A_2+A_1,\ A_{12}+A_6,\ A_{12}+A_5+A_1,$
$A_{12}+A_4+A_2,\ A_{12}+A_4+2A_1,\ A_{12}+A_3+A_2+A_1,\ A_{12}+2A_2+2A_1,\ A_{11}+A_6+A_1,$
$A_{11}+A_4+A_2+A_1,\ A_{10}+A_8,\ A_{10}+A_7+A_1,\ A_{10}+A_6+A_2,\ A_{10}+A_6+2A_1,$
$A_{10}+A_5+A_3,\ A_{10}+A_5+A_2+A_1,\ A_{10}+2A_4,\ A_{10}+A_4+A_3+A_1,\ A_{10}+A_4+2A_2,$
$A_{10}+A_4+A_2+2A_1,\ A_{10}+2A_3+A_2,\ A_{10}+A_3+2A_2+A_1,\ 2A_9,\ A_9+A_8+A_1,$
$A_9+A_7+A_2,\ A_9+A_6+A_3,\ A_9+A_6+A_2+A_1,\ A_9+A_5+A_4,\ 2A_8+2A_1,$
$A_8+A_7+A_2+A_1,\ A_8+A_6+A_4,\ A_8+A_6+A_3+A_1,\ A_8+A_6+A_2+2A_1,$
$A_8+A_5+A_4+A_1,\ A_8+2A_4+2A_1,\ A_8+A_4+A_3+A_2+A_1,\ 2A_7+2A_2,$
$A_7+A_6+A_5,\ A_7+A_6+A_4+A_1,\ A_7+A_6+A_3+A_2,\ A_7+A_6+2A_2+A_1,$
$A_7+A_5+A_4+A_2,\ A_7+A_4+A_3+2A_2,\ 2A_6+A_4+A_2,\ 2A_6+2A_3,$
$2A_6+2A_2+2A_1,\ A_6+A_5+A_4+A_3,\ A_6+A_5+A_4+A_2+A_1,\ A_6+2A_4+A_3+A_1,$
$A_6+2A_4+A_2+2A_1,\ A_6+A_4+2A_3+A_2,\ A_6+A_4+A_3+2A_2+A_1,\ 2A_5+2A_4,$
$2A_4+2A_3+2A_2.$

(2) *An $ADE$-type $\Sigma$ with $r:=\operatorname{rank}(\Sigma)<18$ is a member of $\mathcal{P}_r^{[1]}$ if and only if $\Sigma$ is obtained from a member of $\mathcal{P}_{18}^{[1]}$ by elementary transformation.*

THEOREM 2.4. (1) *The list $\mathcal{P}_{18}^{[2]}$ consists of* 84 *elements as follows.*

$2E_7+D_4,\ 2E_7+A_3+A_1,\ E_7+D_{10}+A_1,\ E_7+D_8+A_2+A_1,\ E_7+D_7+A_3+A_1,$
$E_7+D_6+D_5,\ E_7+D_6+A_5,\ E_7+D_6+A_3+A_2,\ E_7+D_5+A_5+A_1,\ E_7+A_9+A_2,$
$E_7+A_9+2A_1,\ E_7+A_7+A_3+A_1,\ E_7+A_7+A_2+2A_1,\ E_7+A_5+A_4+2A_1,$
$E_7+A_5+2A_3,\ E_7+A_5+A_3+A_2+A_1,\ E_7+A_4+2A_3+A_1,\ D_{16}+A_2,\ D_{16}+2A_1,$
$D_{14}+A_3+A_1,\ D_{14}+A_2+2A_1,\ D_{12}+D_6,\ D_{12}+D_5+A_1,\ D_{12}+A_4+2A_1,$
$D_{12}+A_3+A_2+A_1,\ D_{12}+2A_2+2A_1,\ D_{10}+D_7+A_1,\ D_{10}+D_6+A_2,$
$D_{10}+D_5+A_2+A_1,\ D_{10}+A_5+A_3,\ D_{10}+A_4+A_3+A_1,\ D_9+A_7+2A_1,$
$D_9+A_5+A_3+A_1,\ D_8+2D_5,\ D_8+A_9+A_1,\ D_8+A_7+A_2+A_1,\ D_8+2A_5,$
$D_8+A_5+A_4+A_1,\ D_8+2A_3+2A_2,\ D_7+D_6+A_5,\ D_7+D_5+A_5+A_1,$
$D_7+A_9+2A_1,\ D_7+A_7+A_2+2A_1,\ D_6+D_5+A_7,\ D_6+D_5+A_5+A_2,$
$D_6+A_{11}+A_1,\ D_6+A_9+A_3,\ D_6+A_9+A_2+A_1,\ D_6+A_7+A_4+A_1,$
$D_6+A_7+A_3+A_2,\ D_6+A_7+2A_2+A_1,\ D_6+A_5+A_4+A_3,\ D_5+A_{11}+A_2,$
$D_5+A_9+A_3+A_1,\ D_5+A_9+A_2+2A_1,\ D_5+A_7+A_4+2A_1,\ D_5+2A_5+A_3,$
$D_5+A_5+A_4+A_3+A_1,\ A_{15}+A_2+A_1,\ A_{13}+A_4+A_1,\ A_{13}+A_3+2A_1,$
$A_{13}+2A_2+A_1,\ A_{13}+A_2+3A_1,\ A_{11}+A_5+2A_1,\ A_{11}+A_4+3A_1,\ A_{11}+A_3+A_2+2A_1,$
$A_9+A_6+3A_1,\ A_9+A_5+A_4,\ A_9+A_5+A_3+A_1,\ A_9+A_5+A_2+2A_1,$

**Table 2.3** Forbidden Substitutions for [2]

| | | |
|---|---|---|
| $A_l$ | $\mapsto$ | $A_{l'}+A_{l-1-l'}$ with $l$ odd and $l'$ even $(0\le l'<l/2)$, |
| $D_m$ | $\mapsto$ | $A_{m-1}$, |
| $D_m$ | $\mapsto$ | $A_3+A_{m-4}$ with $m$ even, |
| $D_m$ | $\mapsto$ | $D_{m'}+A_{m-1-m'}$ with $m$ even and $m'$ odd $(5\le m'\le m-1)$, |
| $E_7$ | $\mapsto$ | $A_6,\ A_4+A_2,\ E_6$. |

**Table 2.4** Forbidden Substitutions for [3]

| | | |
|---|---|---|
| $A_l$ | $\mapsto$ | $A_{l_1}+A_{l_2}\ (l_1+l_2=l-1,\ 0\le l_1\le l_2)$ with $l \bmod 3=2,\ l_1 \bmod 3\ne 2,\ l_2 \bmod 3\ne 2$; |
| $E_6$ | $\mapsto$ | $A_4+A_1,\ D_5$. |

$A_9+A_4+A_3+2A_1,\ A_9+A_4+A_2+3A_1,\ A_9+2A_3+A_2+A_1,\ A_9+A_3+2A_2+2A_1,$
$2A_7+2A_2,\ A_7+A_6+A_3+2A_1,\ A_7+2A_5+A_1,\ A_7+A_5+A_4+2A_1,$
$A_7+A_5+A_3+A_2+A_1,\ A_7+A_4+A_3+A_2+2A_1,\ A_6+2A_5+2A_1,$
$A_6+A_5+2A_3+A_1,\ 2A_5+A_4+A_3+A_1,\ A_5+A_4+2A_3+A_2+A_1.$

(2) *Let* $\mathcal{S}^{[2]}$ *be the union of* $\mathcal{P}^{[2]}_{18}$ *and the following list*.

$2E_7+A_3,\ E_7+D_{10},\ E_7+D_5+A_5,\ D_{12}+D_5,\ 2D_8+A_1,\ D_7+A_9+A_1,\ D_7+2A_5,$
$D_6+A_{11},\ 2D_5+A_7,\ A_{15}+A_2,\ E_7+A_9,\ D_{16},\ 2D_8,\ D_5+A_{11},\ A_{15}.$

*Then an* $ADE$*-type* $\Sigma$ *is a member of* $\mathcal{P}^{[2]}$ *if and only if* $\Sigma$ *is a member of* $\mathcal{S}^{[2]}$ *or obtained from a member of* $\mathcal{S}^{[2]}$ *by applying substitutions listed in Table 2.2 but not in Table 2.3.*

THEOREM 2.5. (1) *The list* $\mathcal{P}^{[3]}_{18}$ *consists of* 19 *elements as follows.*

$3E_6,\ 2E_6+A_5+A_1,\ E_6+A_{11}+A_1,\ E_6+A_8+2A_2,\ E_6+A_8+A_2+2A_1,$
$E_6+2A_5+A_2,\ E_6+A_5+A_3+2A_2,\ A_{17}+A_1,\ A_{14}+2A_2,\ A_{14}+A_2+2A_1,$
$A_{11}+A_5+A_2,\ A_{11}+A_3+2A_2,\ A_{11}+3A_2+A_1,\ 2A_8+2A_1,\ A_8+A_5+A_3+A_2,$
$A_8+A_5+2A_2+A_1,\ A_8+A_4+3A_2,\ A_8+A_3+3A_2+A_1,\ 2A_5+A_4+2A_2.$

(2) *Let* $\mathcal{S}^{[3]}$ *be* $\mathcal{P}^{[3]}_{18}$. *Then an* $ADE$*-type* $\Sigma$ *is a member of* $\mathcal{P}^{[3]}$ *if and only if* $\Sigma$ *is a member of* $\mathcal{S}^{[3]}$ *or obtained from a member of* $\mathcal{S}^{[3]}$ *by applying substitutions listed in Table 2.2 but not in Table 2.4.*

THEOREM 2.6. (1) *The list* $\mathcal{P}^{[4]}_{18}$ *consists of* 11 *elements as follows.*

$D_7+A_{11},\ D_7+A_7+A_3+A_1,\ D_7+3A_3+A_2,\ 2D_5+A_7+A_1,\ D_5+A_{11}+2A_1,$
$D_5+A_7+A_3+A_2+A_1,\ A_{15}+A_3,\ A_{15}+3A_1,\ A_{11}+2A_3+A_1,\ A_{11}+A_3+A_2+2A_1,$
$A_7+3A_3+A_2.$

**Table 2.5** Forbidden Substitutions for [4]

| | | |
|---|---|---|
| $A_1$ | $\mapsto$ | $0$; |
| $A_l$ | $\mapsto$ | $A_{l_1} + A_{l_2}$ $(l_1 + l_2 = l - 1,\ 0 \le l_1 \le l_2)$ with $l \bmod 4 = 3,\ l_1 \bmod 4 \ne 3,\ l_2 \bmod 4 \ne 3$; |
| $D_m$ | $\mapsto$ | $A_{m-1},\ D_{m-1},\ 2A_1 + A_{m-3}$ with $m$ odd; |
| $D_m$ | $\mapsto$ | $D_{m-3} + A_2$ with $m$ odd and $m > 6$. |

**Table 2.6** Forbidden Substitutions for [2, 2]

| | | |
|---|---|---|
| $A_l$ | $\mapsto$ | $A_{l'} + A_{l-1-l'}$ with $l$ odd and $l'$ even $(0 \le l' < l/2)$, |
| $D_m$ | $\mapsto$ | $A_{m-1},\ A_3 + A_{m-4}$ with $m$ even, |
| $D_m$ | $\mapsto$ | $D_{m'} + A_{m-1-m'}$ with $m$ even and $m'$ odd $(5 \le m' \le m-1)$. |

(2) *Let $\mathcal{S}^{[4]}$ be the union of $\mathcal{P}^{[4]}_{18}$ and the following list.*

$2D_5 + A_7,\ A_{15} + 2A_1.$

*Then an $ADE$-type $\Sigma$ is a member of $\mathcal{P}^{[4]}$ if and only if $\Sigma$ is a member of $\mathcal{S}^{[4]}$ or obtained from a member of $\mathcal{S}^{[4]}$ by applying substitutions listed in Table 2.2 but not in Table 2.5.*

THEOREM 2.7. (1) *The list $\mathcal{P}^{[2,2]}_{18}$ consists of* 11 *elements as follows.*

$D_{10} + A_5 + 3A_1,\ D_{10} + 2A_3 + 2A_1,\ 2D_8 + 2A_1,\ D_8 + D_6 + A_3 + A_1,\ D_8 + A_5 + A_3 + 2A_1,$
$3D_6,\ 2D_6 + 2A_3,\ D_6 + 2A_5 + 2A_1,\ D_6 + A_5 + 2A_3 + A_1,\ A_7 + A_5 + A_3 + 3A_1,$
$2A_5 + 2A_3 + 2A_1.$

(2) *Let $\mathcal{S}^{[2,2]}$ be the union of $\mathcal{P}^{[2,2]}_{18}$ and the list*

$4D_4.$

*Then an $ADE$-type $\Sigma$ is a member of $\mathcal{P}^{[2,2]}$ if and only if $\Sigma$ is a member of $\mathcal{S}^{[2,2]}$ or obtained from a member of $\mathcal{S}^{[2,2]}$ by applying substitutions listed in Table 2.2 but not in Table 2.6.*

By these theorems, we can easily generate the complete list $\mathcal{P}^G$ for $G = [1], [2], [3], [4], [2, 2]$. Table 2.7 shows the cardinalities of $\mathcal{P}^G_r$.

*Case 2: $G \in \{[5], [6], [4, 2]\}$.* We simply give the table of $\mathcal{P}^G$. For each $G$, the $ADE$-types are listed according to the rank and the lexicographical order.

$G = [5]$:

$2A_9,\ A_9 + 2A_4 + A_1,\ 4A_4 + 2A_1,\ A_9 + 2A_4,\ 4A_4 + A_1,\ 4A_4.$

**Table 2.7** Cardinalities of $\mathcal{P}_r^G$

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{P}_r^{[1]}\rvert$ | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 39 | 57 | 88 | 127 | 189 | 262 | 360 | 448 | 500 | 416 | 199 | 2746 |
| $\lvert\mathcal{P}_r^{[2]}\rvert$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 6 | 13 | 29 | 53 | 92 | 133 | 164 | 155 | 84 | 732 |
| $\lvert\mathcal{P}_r^{[3]}\rvert$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 6 | 12 | 21 | 24 | 19 | 85 |
| $\lvert\mathcal{P}_r^{[4]}\rvert$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 10 | 15 | 11 | 41 |
| $\lvert\mathcal{P}_r^{[2,2]}\rvert$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 5 | 10 | 16 | 16 | 11 | 61 |

$G = [6]$:

$A_{11}+A_5+2A_1,\ A_{11}+A_3+2A_2,\ A_{11}+2A_2+3A_1,\ 3A_5+A_3,\ 2A_5+A_3+2A_2+A_1,$
$A_{11}+2A_2+2A_1,\ 3A_5+2A_1,\ 2A_5+A_3+2A_2,\ 2A_5+2A_2+3A_1,\ 2A_5+2A_2+2A_1.$

$G = [4, 2]$:

$2A_7+4A_1,\ A_7+3A_3+2A_1,\ A_7+2A_3+4A_1,\ 5A_3+2A_1,\ 4A_3+4A_1.$

*Case 3: $G \in \{[7], [8], [6, 2], [3, 3], [4, 4]\}$.* In this case, the $ADE$-type determines the torsion of the Mordell–Weil group uniquely.

THEOREM 2.8. *Let $f\colon X \to \mathbb{P}^1$ be an elliptic $K3$ surface. Then the following hold.*

(a) $G_f \cong \mathbb{Z}/(7) \iff \Sigma_f = 3A_6$.
(b) $G_f \cong \mathbb{Z}/(8) \iff \Sigma_f = 2A_7 + A_3 + A_1$.
(c) $G_f \cong \mathbb{Z}/(6) \times \mathbb{Z}/(2) \iff \Sigma_f = 3A_5 + 3A_1$.
(d) $G_f \cong \mathbb{Z}/(4) \times \mathbb{Z}/(4) \iff \Sigma_f = 6A_3$.
(e) $G_f \cong \mathbb{Z}/(3) \times \mathbb{Z}/(3) \iff \Sigma_f \in \{2A_5 + 4A_2, A_5 + 6A_2, 8A_2\}$.

REMARK 2.9. Elliptic $K3$ surfaces with $G_f = [7], [8], [6, 2], [4, 4]$ are constructed as elliptic modular surfaces (cf. [14; 17]). The corresponding congruence groups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ are as follows.

| $G_f$ | $[7]$ | $[8]$ | $[6, 2]$ | $[4, 4]$ |
|---|---|---|---|---|
| $\Gamma$ | $\Gamma_1(7)$ | $\Gamma_1(8)$ | $\Gamma_0(3) \cap \Gamma(2)$ | $\Gamma(4)$ |

### 2.3. *From $ADE$-Types to Configurations of Singular Fibers*

The correspondence between the type (in the notation of Kodaira) of a singular fiber of an elliptic fibration and an $ADE$-type is shown in Table 2.8. We have the

**Table 2.8** Singular Fibers of Elliptic Fibration

| Singular Fiber | $ADE$-Type | Euler Number | Possible Torsion Parts |
|---|---|---|---|
| $\mathrm{I}_0$ | regular | 0 | all |
| $\mathrm{I}_1$ | irreducible | 1 | $\left.\right\} \diamondsuit$ |
| $\mathrm{I}_b \ (b \geq 2)$ | $A_{b-1}$ | $b$ | |
| $\mathrm{I}_b^* \ (b \geq 0)$ | $D_{4+b}$ | $6+b$ | $\begin{cases} [1], [2], [2,2] & \text{if } b \text{ is even} \\ [1], [2], [4] & \text{if } b \text{ is odd} \end{cases}$ |
| II | irreducible | 2 | [1] |
| II$^*$ | $E_8$ | 10 | [1] |
| III | $A_1$ | 3 | [1], [2] |
| III$^*$ | $E_7$ | 9 | [1], [2] |
| IV | $A_2$ | 4 | [1], [3] |
| IV$^*$ | $E_6$ | 8 | [1], [3] |

$$\diamondsuit \begin{cases} [a] \text{ is possible for } a = 1, \dots, 8, \\ [2a, 2] \text{ is possible for } a = 1, \dots, 3 \text{ if and only if } b = 0 \bmod 2, \\ [3, 3] \text{ is possible if and only if } b = 0 \bmod 3, \\ [4, 4] \text{ is possible if and only if } b = 0 \bmod 4. \end{cases}$$

following ambiguities in recovering the configurations of singular fibers from its $ADE$-type:

(a) an irreducible singular fiber is of type either $\mathrm{I}_1$ or II;
(b) a singular fiber of $ADE$-type $A_1$ is of type either $\mathrm{I}_2$ or III;
(c) a singular fiber of $ADE$-type $A_2$ is of type either $\mathrm{I}_3$ or IV.

We next present some restrictions on the possibilities of configuration of singular fibers of an elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with a given $ADE$-type.

Let $i_b$ be the number of singular fibers of $f$ of type $\mathrm{I}_b$. We define similarly $i_b^*$, $ii$, $ii^*$, $iii$, $iii^*$, $iv$, and $iv^*$. Miranda and Persson gave a formula for the degree of the modulus function $J_f : \mathbb{P}^1 \to \mathbb{P}^1 := \overline{\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})}$ associated with $f : X \to \mathbb{P}^1$:

$$\deg J_f := \sum_{b \geq 1} b(i_b + i_b^*).$$

By the Hurwitz formula, they obtained the following necessary condition for configurations: If $\deg J_f > 0$, then

$$\deg J_f \leq 6 \sum_{b \geq 1} (i_b + i_b^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - 12.$$

See [9, Sec. 3] for the proof.

The euler number 24 of the $K3$ surface $X$ is equal to the sum of euler numbers of singular fibers of $f$. The third column of Table 2.8 shows the euler number of a singular fiber of each type. We define the euler number $\mathrm{euler}(\Sigma)$ of an $ADE$-type $\Sigma := \sum a_l A_l + \sum d_m D_m + \sum e_n E_n$ by

$$\mathrm{euler}(\Sigma) := \sum a_l \cdot (l+1) + \sum d_m \cdot (m+2) + \sum e_n \cdot (n+2).$$

Then $\mathrm{euler}(\Sigma_f)$ is less than or equal to the sum of euler numbers of reducible singular fibers. Hence we always have

$$\mathrm{euler}(\Sigma_f) \le 24.$$

For example, we can deduce from Table 2.8 that, if $\mathrm{euler}(\Sigma_f) = 24$, then $f\colon X \to \mathbb{P}^1$ has neither irreducible singular fibers nor any fibers of type III or IV.

When $G_f$ is nontrivial, certain types of singular fibers cannot appear. Let $g\colon S \to \Delta$ be an elliptic fibration over an open unit disk $\Delta$ such that $g$ is smooth over $\Delta^\times := \Delta \setminus \{0\}$, and let $E := g^{-1}(p)$ be the fiber over a point $p \in \Delta^\times$. Looking at the monodromy action of $\pi_1(\Delta^\times, p)$ on the set of torsion points of $E$, we can determine whether a finite abelian group can be embedded into the Mordell–Weil group of $g$. The fourth column of Table 2.8 shows the groups among the list (2.1) that can be isomorphic to the torsion part of the Mordell–Weil group of an elliptic surface having the singular fiber. We see, for example, that if $G_f$ is nontrivial then every irreducible singular fiber must be of type $\mathrm{I}_1$.

### *2.4. Miscellaneous Facts*

For an integer $r$ with $1 \le r \le 18$, we set:

$$\mathcal{R}_r := \{\, \Sigma \mid \Sigma \text{ is an } ADE\text{-type with } \mathrm{rank}(\Sigma) = r \,\};$$

$$\mathcal{E}_r := \{\, \Sigma \in \mathcal{R}_r \mid \mathrm{euler}(\Sigma) \le 24 \,\};$$

$$\mathcal{P}_r := \{\, \Sigma \in \mathcal{E}_r \mid \text{there exists an elliptic } K3 \text{ surface } f\colon X \to \mathbb{P}^1 \text{ with } \Sigma_f = \Sigma \,\} = \bigcup_G \mathcal{P}_r^G.$$

For $\Sigma \in \bigcup_{r=1}^{18} \mathcal{P}_r$, we denote by $\mathcal{G}(\Sigma)$ the set of isomorphism classes of finite abelian groups $G$ such that $(\Sigma, G) \in \mathcal{P}$. For each $r$, we denote by $\mathcal{T}_r$ the set of $\Sigma \in \mathcal{P}_r$ such that $\mathcal{G}(\Sigma)$ consists of only the trivial group [1]. The cardinalities of these sets are given in Table 2.9. Note that, if $\mathrm{rank}(\Sigma) \le 12$, then $\mathrm{euler}(\Sigma) \le 24$ holds automatically.

THEOREM 2.10. *Let $\Sigma$ be an $ADE$-type with* $\mathrm{euler}(\Sigma) \le 24$. *Suppose that* $\mathrm{rank}(\Sigma) \le 13$. *Then there exists an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$ with $\Sigma_f = \Sigma$.*

REMARK 2.11. The complement of $\mathcal{P}_{14}$ in $\mathcal{E}_{14}$ consists of a single element $E_6 + 8A_1$. Hence, when $\mathrm{euler}(\Sigma) \le 24$ and $\mathrm{rank}(\Sigma) = 14$, there exists an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$ with $\Sigma_f = \Sigma$ if and only if $\Sigma \ne E_6 + 8A_1$.

THEOREM 2.12. *Suppose that* $\mathrm{rank}(\Sigma) \le 10$. *Then there exists an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$ with $G_f = [1]$ and $\Sigma_f = \Sigma$.*

**Table 2.9** Cardinalities of $\mathcal{R}_r$, $\mathcal{E}_r$, and $\mathcal{P}_r$

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{R}_r\rvert$ | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 39 | 57 | 88 | 128 | 193 | 276 | 403 | 570 | 815 | 1137 | 1599 | 5366 |
| $\lvert\mathcal{E}_r\rvert$ | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 39 | 57 | 88 | 128 | 193 | 274 | 393 | 531 | 688 | 773 | 712 | 3937 |
| $\lvert\mathcal{P}_r\rvert$ | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 39 | 57 | 88 | 128 | 193 | 274 | 392 | 518 | 624 | 580 | 325 | 3279 |
| $\lvert\mathcal{T}_r\rvert$ | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 38 | 55 | 82 | 115 | 162 | 217 | 289 | 362 | 419 | 372 | 188 | 2360 |

REMARK 2.13. The complement $\mathcal{P}_{11} \setminus \mathcal{P}_{11}^{[1]}$ consists of a single element $11A_1$. We have $\mathcal{G}(11A_1) = \{[2]\}$.

THEOREM 2.14. *Let $f: X \to \mathbb{P}^1$ be an elliptic $K3$ surface. If* $\operatorname{rank}(\Sigma_f) \leq 7$, *then $G_f$ must be trivial.*

REMARK 2.15. The complement $\mathcal{P}_8^{[1]} \setminus \mathcal{T}_8$ consists of a single element $8A_1$, and the complement $\mathcal{P}_9^{[1]} \setminus \mathcal{T}_9$ consists of two elements $9A_1$ and $A_3 + 6A_1$. We have

$$\mathcal{G}(8A_1) = \mathcal{G}(9A_1) = \mathcal{G}(A_3 + 6A_1) = \{[1], [2]\}.$$

REMARK 2.16. There are several $ADE$-types $\Sigma$ with $|\mathcal{G}(\Sigma)| \geq 3$. For example,

$$\mathcal{G}(2A_5 + 2A_2 + 2A_1) = \mathcal{G}(A_{11} + 2A_2 + 2A_1) = \{[1], [2], [3], [6]\}.$$

## 3. Local Invariants of Lattices

First we fix some terminologies about lattices.

Let $R$ be either $\mathbb{Z}$ or $\mathbb{Z}_p$. A *lattice* over $R$ is, by definition, a free $R$-module $L$ of finite rank equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot): L \times L \to R$. For $\alpha \in R \setminus \{0\}$, let $\alpha L$ denote the lattice obtained from $L$ by multiplying the symmetric bilinear form by $\alpha$. We will use $L^-$ to denote $(-1)L$. We often express a lattice by the intersection matrix with respect to a certain basis of $L$. For example, $(a)$ is the lattice of rank 1 generated by a vector $e$ such that $(e, e) = a$. A sublattice $N$ of $L$ is said to be *primitive* if $L/N$ is torsion free. A lattice $L$ over $R$ is said to be *even* if $(v, v) \in 2R$ holds for any $v \in L$. Note that, when $R$ is $\mathbb{Z}_p$ with $p$ an odd prime, every lattice over $R$ is even. The *discriminant* $\operatorname{disc}(L)$ of a lattice $L$ is considered as an element of $(R \setminus \{0\})/(R^\times)^2$. A lattice $L$ is said to be *unimodular* if $\operatorname{disc}(L) \in R^\times/(R^\times)^2$.

Suppose that $R = \mathbb{Z}_p$. Then we have $\operatorname{disc}(L) = p^\nu u$ for some $\nu \geq 0$, where $u \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. We denote the element $u$ by $\operatorname{reddisc}(L)$ and call it the *reduced discriminant* of $L$.

Let $k$ be the quotient field of $R$. The $k$-vector space $L \otimes_R k$ has a natural symmetric bilinear form with values in $k$. We denote by $L^\vee$ the $R$-submodule of $L \otimes_R k$ consisting of all vectors $v$ such that $(v, w) \in R$ holds for every $w \in L$; we call it the *dual lattice* of $L$. An $R$-submodule $M$ of $L^\vee$ is said to be an *overlattice* of $L$ if $M$ contains $L$ and the symmetric bilinear form restricted to $M$ takes values in $R$. Two lattices $L$ and $M$ over $R$ are said to be *$k$-equivalent* if $L \otimes_R k$ and $M \otimes_R k$, together with their symmetric $k$-valued bilinear forms, are isomorphic.

For a detailed account of the following definitions and theorems, see Conway and Sloane [6, Chap. 15] and Cassels [5, Chaps. 8–9].

### *3.1. Local Invariants*

Let $\Lambda$ be a lattice over $\mathbb{Z}_p$. Then $\Lambda$ is decomposed into the orthogonal direct sum $\Lambda = \bigoplus_{\nu \geq 0} p^\nu \Lambda_\nu$, where each $\Lambda_\nu$ is unimodular. This decomposition is called a *Jordan decomposition* of $\Lambda$, and each $p^\nu \Lambda_\nu$ is called a *Jordan component* of $\Lambda$. Note that the reduced discriminant of $\Lambda$ is the product of the discriminants of $\Lambda_\nu$.

Suppose that $p$ is odd. Then a lattice $\Lambda$ over $\mathbb{Z}_p$ is isomorphic to an orthogonal direct sum $\bigoplus_i p^{\nu_i}(a_i)$, where $a_i \in \mathbb{Z}_p^\times$. The *$p$-excess* of $\Lambda$ is defined to be

$$-\mathrm{rank}(\Lambda) + 4m + \sum_i p^{\nu_i} \in \mathbb{Z}/(8),$$

where $m$ is the number of orthogonal direct summands $p^{\nu_i}(a_i)$ such that $\nu_i$ is odd and $a_i$ is not square in $\mathbb{Z}_p^\times$. It is known that the $p$-excess is a well-defined invariant of $\mathbb{Q}_p$-equivalence classes of lattices over $\mathbb{Z}_p$.

Suppose that $p = 2$. We put

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

both of which are unimodular lattices of rank 2 over $\mathbb{Z}_2$. Then a lattice over $\mathbb{Z}_2$ is decomposed into the orthogonal direct sum of lattices such that each direct summand is isomorphic to $2^\nu(a)$ $(a \in \mathbb{Z}_2^\times)$, $2^\nu U$, or $2^\nu V$. We define the 2-excesses of these lattices by:

$$2\text{-excess}(2^\nu(a)) = \begin{cases} 1 - a \bmod 8 & \text{if } \nu \text{ is even or } a = \pm 1 \bmod 8, \\ 5 - a \bmod 8 & \text{if } \nu \text{ is odd and } a = \pm 3 \bmod 8; \end{cases}$$

$$2\text{-excess}(2^\nu U) = 2 \bmod 8;$$

$$2\text{-excess}(2^\nu V) = \begin{cases} 2 \bmod 8 & \text{if } \nu \text{ is even,} \\ 6 \bmod 8 & \text{if } \nu \text{ is odd.} \end{cases}$$

Then we define the 2-excess of

$$\Lambda \cong \bigoplus_i 2^{\nu_i}(a_i) \oplus \bigoplus_j 2^{\nu_j} U \oplus \bigoplus_k 2^{\nu_k} V \tag{3.1}$$

to be the sum of the 2-excesses of direct summands in the decomposition (3.1). Even though the decomposition (3.1) is not unique in general, it turns out that the 2-excess is a well-defined invariant of $\mathbb{Q}_2$-equivalence classes of lattices over $\mathbb{Z}_2$. (Note that $U$ and $V$ are $\mathbb{Q}_2$-equivalent to $2(1) \oplus 2(7)$ and $2(1) \oplus 2(3)$, respectively.)

### *3.2. Existence of Lattices over $\mathbb{Z}$ with Given Local Data*

By combining [6, Chap. 15, Thm. 5] and [5, Chap. 9, Thm. 1.2], we obtain the following.

THEOREM 3.1. *Let $d$ be a nonzero integer, and let $(r, s)$ be a pair of nonnegative integers such that $n := r + s$ is positive and $d = (-1)^s |d|$ holds. Suppose that, for each prime divisor $p$ of $2d$, a lattice $\Lambda^{(p)}$ of rank $n$ over $\mathbb{Z}_p$ is given. Then there exists a lattice $L$ over $\mathbb{Z}$ with discriminant $d$ and signature $(r, s)$ such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic to $\Lambda^{(p)}$ for each $p$ if and only if the following two conditions are satisfied*:

(i) $\operatorname{disc}(\Lambda^{(p)})$ *is equal to* $d \cdot (\mathbb{Z}_p^{\times})^2$ *for each* $p$; *and*
(ii) $r - s + \sum_{p|2d} p\text{-excess}(\Lambda^{(p)}) = n \bmod 8$ *holds.*

## 4. Theory of Discriminant Forms

### *4.1. Definitions*

Let $R$ and $k$ be as before, and let $D$ be a finite abelian group. A finite symmetric bilinear form on $D$ with values in $k/R$ is, by definition, a homomorphism $b\colon D \times D \to k/R$ such that $b(x, y) = b(y, x)$ holds for any $x, y \in D$. A finite quadratic form on $D$ with values in $k/2R$ is a map $q\colon D \to k/2R$ having the following properties:

(i) $q(nx) = n^2 q(x)$ for $n \in \mathbb{Z}$ and $x \in D$; and
(ii) the map $b[q]\colon D \times D \to k/R$ defined by $(x, y) \mapsto (q(x+y) - q(x) - q(y))/2$ is a finite symmetric bilinear form.

Let $H$ be a subgroup of $D$. The orthogonal complement $H^{\perp}$ of $H$ with respect to $q$ is the subgroup of $D$ consisting of elements $y$ such that $b[q](x, y) = 0$ holds for any $x \in H$. We say that $q$ is *nondegenerate* if $D^{\perp} = (0)$. Note that, if $D = H \oplus H^{\perp}$, then $q$ is written as $q\big|_H \oplus q\big|_{H^{\perp}}$, because the homomorphism $a \mapsto a/2$ from $k/2R$ to $k/R$ is injective.

The length of $D$ is, by definition, the minimal number of generators of $D$. A subset $\{\gamma_1, \dots, \gamma_l\}$ of $D$ is said to be a *reduced set of generators* of $D$ if $l$ is the length of $D$ and $D = \langle\gamma_1\rangle \times \cdots \times \langle\gamma_l\rangle$ holds. Let $\{\gamma_1, \dots, \gamma_l\}$ be a reduced set of generators of $D$. Then a finite quadratic form $q$ on $D$ is expressed by a symmetric $l \times l$ matrix whose diagonal entries are $q(\gamma_i) \in k/2R$ and whose off-diagonal entries are $b[q](\gamma_i, \gamma_j) \in k/R$.

Let $L$ be a lattice over $R$. The discriminant group $D_L$ of $L$ is, by definition, the quotient group $L^{\vee}/L$. We denote by $\Psi_L\colon L^{\vee} \to D_L$ the natural projection. Suppose that $L$ is even. Then we can define a finite quadratic form $q_L$ on $D_L$ with values in $k/2R$ by $q_L(x) := (x', x') \bmod 2R$, where $x'$ is a vector of $L^{\vee}$ such that $\Psi_L(x') = x$. We call $q_L$ the *discriminant form* of $L$. Because $L$ is nondegenerate, $q_L$ is also nondegenerate. By definition, we have $(D_{L\oplus M}, q_{L\oplus M}) = (D_L, q_L) \oplus (D_M, q_M)$.

**Table 4.1** Discriminant Forms of Lattices over $\mathbb{Z}_p$

| $\Lambda$ | $p^\nu(a)$ | $2^\nu U$ | $2^\nu V$ |
|---|---|---|---|
| $D_\Lambda$ | $\mathbb{Z}/(p^\nu)$ | $(\mathbb{Z}/(2^\nu))^{\oplus 2}$ | $(\mathbb{Z}/(2^\nu))^{\oplus 2}$ |
| $q_\Lambda$ | $\begin{bmatrix} \dfrac{a}{p^\nu} \end{bmatrix}$ | $\dfrac{1}{2^\nu}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\dfrac{1}{2^\nu}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ |

### *4.2. Discriminant Forms and Overlattices*

The following two propositions, due to Nikulin, play a central role in making the list $\mathcal{P}$.

PROPOSITION 4.1 [11, Prop. 1.4.1]. *Let $L$ be an even lattice over $\mathbb{Z}$.*

(1) *If $H \subset D_L$ is a subgroup that is isotopic with respect to $q_L$, then $M := \Psi_L^{-1}(H)$ is an even overlattice of $L$ and the discriminant form of $M$ is isomorphic to $\left(H^\perp/H, q_L|_{H^\perp/H}\right)$.*

(2) *The map $H \mapsto \Psi_L^{-1}(H)$ establishes a bijection between the set of isotopic subgroups of $(D_L, q_L)$ and the set of even overlattices of $L$.*

PROPOSITION 4.2 [11, Prop. 1.6.1]. *Let $L$ and $M$ be even lattices over $\mathbb{Z}$. Then the following statements are equivalent*:

(i) *the two finite quadratic forms $(D_L, q_L)$ and $(D_M, -q_M)$ are isomorphic*;

(ii) *there exists an even unimodular overlattice of $L \oplus M$ into which $L$ and $M$ are embedded primitively.*

### *4.3. Localization and Discriminant Form*

Let $L$ be an even lattice over $\mathbb{Z}$. We decompose $D_L$ into the direct sum of its $p$-Sylow subgroups $D_L^{(p)}$, where $p$ runs through the set of prime divisors of $|D_L| = |\mathrm{disc}(L)|$. These $p$-parts are orthogonal to each other with respect to $q_L$ and hence $q_L$ is also decomposed into the $p$-parts; $q_L = \bigoplus_p q_L^{(p)}$, where $q_L^{(p)}$ is the restriction of $q_L$ to $D_L^{(p)}$. By our definition of the discriminant form, we can easily prove the following lemma.

LEMMA 4.3. *The image of $q_L^{(p)}$ is contained in $2\mathbb{Z}[1/p]/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$. The natural inclusion $2\mathbb{Z}[1/p] \hookrightarrow \mathbb{Q}_p$ induces an isomorphism $2\mathbb{Z}[1/p]/2\mathbb{Z} \cong \mathbb{Q}_p/2\mathbb{Z}_p$. Under this identification, $(D_L^{(p)}, q_L^{(p)})$ is isomorphic to $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$.*

The discriminant form of an even lattice $\Lambda$ over $\mathbb{Z}_p$ is calculated by Table 4.1. In particular, $D_\Lambda$ is a $p$-group of length equal to $\mathrm{rank}(\Lambda) - \mathrm{rank}(\Lambda_0)$, where $\Lambda_0$ is the first Jordan component of $\Lambda$. We also have $\mathrm{disc}(\Lambda) = |D_\Lambda| \cdot \mathrm{reddisc}(\Lambda)$.

## 5. Existence of Lattices with a Given Discriminant Form

### *5.1. Over $\mathbb{Z}_p$*

Suppose that a finite abelian $p$-group $D$ and a nondegenerate finite quadratic form $q\colon D \to \mathbb{Q}_p/2\mathbb{Z}_p$ are given. It is known that, if $n \geq \operatorname{length}(D)$, then there exists an even lattice $\Lambda$ of rank $n$ over $\mathbb{Z}_p$ such that $(D_\Lambda, q_\Lambda)$ is isomorphic to $(D, q)$. Our purpose here is to describe a method for determining the set $\mathcal{L}^{(p)}(n, D, q)$ of all $[\sigma, u] \in \mathbb{Z}/(8) \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ such that there exists an even lattice $\Lambda$ of rank $n$ over $\mathbb{Z}_p$ with $(D_\Lambda, q_\Lambda) \cong (D, q)$, $p$-excess$(\Lambda) = \sigma$, and reddisc$(\Lambda) = u$.

Note that

$$p\text{-excess}(\Lambda_1 \oplus \Lambda_2) = p\text{-excess}(\Lambda_1) + p\text{-excess}(\Lambda_2)$$

and

$$\operatorname{reddisc}(\Lambda_1 \oplus \Lambda_2) = \operatorname{reddisc}(\Lambda_1) \cdot \operatorname{reddisc}(\Lambda_2).$$

Taking these equalities into account, for sets $\mathcal{L}$ and $\mathcal{L}'$ of elements of $\mathbb{Z}/(8) \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ we define $\mathcal{L} * \mathcal{L}'$ to be the set

$$\{\, [\sigma + \sigma', uu'] \mid [\sigma, u] \in \mathcal{L},\ [\sigma', u'] \in \mathcal{L}' \,\}.$$

We also put $\mathcal{L}_0^{(p)} := \{[0, 1]\}$. Then $\mathcal{L} * \mathcal{L}_0^{(p)} = \mathcal{L}$ holds for any $\mathcal{L}$.

LEMMA 5.1. *Let $l$ be the length of $D$. Then we have*

$$\mathcal{L}^{(p)}(n, D, q) = \mathcal{L}^{(p)}(n - l, (0), [0]) * \mathcal{L}^{(p)}(l, D, q). \tag{5.1}$$

*If $p$ is odd, then*

$$\mathcal{L}^{(p)}(n - l, (0), [0]) = \begin{cases} \emptyset & \text{if } n < l, \\ \mathcal{L}_0^{(p)} & \text{if } n = l, \\ \{[0, 1], [0, v_p]\} & \text{if } n > l, \end{cases} \tag{5.2}$$

*where $v_p$ is the unique nontrivial element of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. If $p = 2$, then*

$$\mathcal{L}^{(2)}(n - l, (0), [0]) = \begin{cases} \emptyset & \text{if } n < l \text{ or } n - l \bmod 2 = 1, \\ \mathcal{L}_0^{(2)} & \text{if } n = l, \\ \{[n - l, 1], [n - l, 5]\} & \text{if } n > l \text{ and } n - l \bmod 4 = 0, \\ \{[n - l, 3], [n - l, 7]\} & \text{if } n > l \text{ and } n - l \bmod 4 = 2. \end{cases}$$

*Proof*. Let $\Lambda = \Lambda_0 \oplus \bigoplus_{\nu > 0} p^\nu \Lambda_\nu$ be a Jordan decomposition of a lattice $\Lambda$ over $\mathbb{Z}_p$ with $(D_\Lambda, q_\Lambda) \cong (D, q)$. We put $\Lambda_{>0} := \Lambda_0^\perp = \bigoplus_{\nu > 0} p^\nu \Lambda_\nu$. Then we have $\operatorname{rank}(\Lambda_{>0}) = l$, $(D_{\Lambda_0}, q_{\Lambda_0}) = ((0), [0])$, and $(D_{\Lambda_{>0}}, q_{\Lambda_{>0}}) = (D_\Lambda, q_\Lambda) \cong (D, q)$. Hence (5.1) holds. The statement (5.2) is obvious. A lattice $\Lambda$ over $\mathbb{Z}_2$ is even if and only if $\Lambda_0$ is of even rank and is isomorphic to an orthogonal direct sum of copies of $U$ and $V$. Because

$$[2\text{-excess}(U), \operatorname{reddisc}(U)] = [2, 7] \text{ and } [2\text{-excess}(V), \operatorname{reddisc}(V)] = [2, 3],$$

we can easily prove the last statement. □

LEMMA 5.2. *Suppose that $\nu$ is a positive integer, and let $u$ be an integer prime to $p$. Then*

$$\mathcal{L}^{(p)}\left(1, \mathbb{Z}/(p^\nu), \left[\frac{u}{p^\nu}\right]\right) = \begin{cases} \{[p^\nu - 1, u]\} & \text{if } p \text{ is odd,} \\ \{[1-u, u]\} & \text{if } p = 2 \text{ and } \nu > 1, \\ \{[1-u, u], [1-u, 5u]\} & \text{if } p = 2 \text{ and } \nu = 1. \end{cases}$$

*Let $u$, $v$, $w$ be integers with $v$ odd. Then*

$$\mathcal{L}^{(2)}\left(2, (\mathbb{Z}/(2^\nu))^{\oplus 2}, \frac{1}{2^\nu}\begin{bmatrix} 2u & v \\ v & 2w \end{bmatrix}\right) = \begin{cases} \{[2, 7]\} & \text{if } uw \text{ is even,} \\ \{[2, 3]\} & \text{if } uw \text{ is odd.} \end{cases}$$

*Proof.* Two nondegenerate quadratic forms $[u/p^\nu]$ and $[u'/p^\nu]$ on $\mathbb{Z}/(p^\nu)$ with values in $\mathbb{Q}_p/2\mathbb{Z}_p$ are isomorphic if and only if

$$uu' \in (\mathbb{Z}_p^\times)^2 \quad \text{or} \quad (p = 2,\ \nu = 1, \text{ and } u = u' \bmod 4)$$

is satisfied. On the other hand, two lattices $p^\nu(u)$ and $p^\nu(u')$ with $u, u' \in \mathbb{Z}_p^\times$ of rank 1 over $\mathbb{Z}_p$ are isomorphic if and only if $uu' \in (\mathbb{Z}_p^\times)^2$ holds. Therefore, the first statement follows. The finite quadratic form

$$q = \frac{1}{2^\nu}\begin{bmatrix} 2u & v \\ v & 2w \end{bmatrix} \quad (v \text{ odd})$$

on $(\mathbb{Z}/(2^\nu))^{\oplus 2}$ with values in $\mathbb{Q}_2/2\mathbb{Z}_2$ is isomorphic to $q_{2^\nu U}$ (resp., $q_{2^\nu V}$) if and only if $uw \bmod 2 = 0$ (resp., $uw \bmod 2 = 1$). These two forms can never be isomorphic to $[u'/2^\nu] \oplus [w'/2^\nu]$ when $u'$ and $w'$ are odd. Thus the second statement follows. □

Now we state an algorithm to calculate $\mathcal{L}^{(p)}(n, D, q)$. By Lemma 5.1, it is enough to determine $\mathcal{L}^{(p)}(l, D, q)$. Let $\{\gamma_1, \dots, \gamma_l\}$ be a reduced set of generators of $D$. We denote the order of $\gamma_i$ by $p^{\nu_i}$, and we arrange the generators in such a way that $\nu_1 \geq \cdots \geq \nu_l$ holds. For an element $\alpha \in \mathbb{Q}_p/\mathbb{Z}_p$, we define $\phi_p(\alpha)$ to be the integer such that the order of $\alpha$ is $p^{\phi_p(\alpha)}$. Note that $\phi_p(b[q](\gamma_i, \gamma_j)) \leq \min(\nu_i, \nu_j)$ holds for any $\gamma_i$ and $\gamma_j$.

If $l = 1$, then $\mathcal{L}^{(p)}(l, D, q)$ is given by Lemma 5.2. Suppose then that $l > 1$.

*Case 1.* Suppose there is a generator $\gamma_i$ such that $\phi_p(b[q](\gamma_i, \gamma_i)) = \nu_1$; then $\nu_i = \nu_1$. Interchanging $\gamma_1$ and $\gamma_i$, we will assume that $\phi_p(b[q](\gamma_1, \gamma_1)) = \nu_1$. Let $u$ be an integer such that $b[q](\gamma_1, \gamma_1) = u/p^{\nu_1} \bmod \mathbb{Z}_p$. Then $u$ is prime to $p$, and hence there is an integer $v$ such that $uv = 1 \bmod p^{\nu_1}$ holds. Since $\phi_p(b[q](\gamma_j, \gamma_1)) \leq \min(\nu_j, \nu_1) = \nu_j$, we can write $b[q](\gamma_j, \gamma_1)$ in the form $w_j/p^{\nu_1} \bmod \mathbb{Z}_p$ by some integer $w_j$ that is divisible by $p^{\nu_1 - \nu_j}$. For $j \geq 2$, we put $\gamma_j' := \gamma_j - v w_j \gamma_1$. Because $\gamma_1$ is of order $p^{\nu_1}$ in $D$, it follows that $\gamma_j'$ is independent of the choice of $u$, $v$, and $w_j$. Moreover, $\gamma_j'$ is of order $p^{\nu_j}$, and $\{\gamma_1, \gamma_2', \dots, \gamma_l'\}$

is again a reduced set of generators. By definition, we have $b[q](\gamma_j', \gamma_1) = 0$ for any $j \geq 2$. We put

$$(D_1, q_1) := \left(\langle\gamma_1\rangle, q\big|_{\langle\gamma_1\rangle}\right) \cong (\mathbb{Z}/(p^{\nu_1}), [u/p^{\nu_1}])$$

and $(D_2, q_2) := \left(\langle\gamma_2', \dots, \gamma_l'\rangle, q\big|_{\langle\gamma_2', \dots, \gamma_l'\rangle}\right)$. Then $(D, q)$ is decomposed into the orthogonal direct sum of $(D_1, q_1)$ and $(D_2, q_2)$.

Let $\Lambda$ be a lattice of rank $l$ over $\mathbb{Z}_p$ such that there exists an isomorphism $h\colon (D_\Lambda, q_\Lambda) \xrightarrow{\sim} (D, q)$. Let $e^* \in \Lambda^\vee$ be a vector such that $h \circ \Psi_\Lambda(e^*) = \gamma_1$, and let $\Lambda_1' \subset \Lambda^\vee$ be the $\mathbb{Z}_p$-submodule generated by $e^*$. Then $\Lambda_1 := \Lambda_1' \cap \Lambda$ is a sublattice of rank 1 generated by $e := p^{\nu_1} e^*$. Let $x$ be an arbitrary vector of $\Lambda$. Because $\mathrm{ord}_p((x, e)) \geq \nu_1 = \mathrm{ord}_p((e, e))$, the vector

$$x' := x - \frac{(x, e)}{(e, e)} e$$

is in $\Lambda$ and orthogonal to $\Lambda_1$. Hence we obtain an orthogonal decomposition $\Lambda = \Lambda_1 \oplus \Lambda_1^\perp$. The homomorphism $h \circ \Psi_\Lambda\colon \Lambda^\vee = \Lambda_1^\vee \oplus (\Lambda_1^\perp)^\vee \to D$ induces isomorphisms $(D_{\Lambda_1}, q_{\Lambda_1}) \cong (D_1, q_1)$ and $(D_{\Lambda_1^\perp}, q_{\Lambda_1^\perp}) \cong (D_2, q_2)$. It follows that

$$\mathcal{L}^{(p)}(l, D, q) = \mathcal{L}^{(p)}(1, D_1, q_1) * \mathcal{L}^{(p)}(l-1, D_2, q_2).$$

Thus $\mathcal{L}^{(p)}(l, D, q)$ is calculated by Lemma 5.2 and the induction hypothesis on $l$.

*Case 2.* Suppose that $\phi_p(b[q](\gamma_i, \gamma_i)) < \nu_1$ holds for any generator $\gamma_i$. Since $q$ is nondegenerate, there exists at least one generator $\gamma_k$ that satisfies $\phi_p(b[q](\gamma_1, \gamma_k)) = \nu_1$. Because $\phi_p(b[q](\gamma_1, \gamma_k)) \leq \nu_k$, we have $\nu_k = \nu_1$.

*Case 2.1.* Suppose that $p$ is odd. We replace $\gamma_1$ by $\gamma_1' := \gamma_1 + \gamma_k$, which is an element of order $p^{\nu_1}$. It is obvious that $\{\gamma_1', \gamma_2, \dots, \gamma_l\}$ is again a reduced set of generators of $D$. Moreover, we have $\phi_p(b[q](\gamma_1', \gamma_1')) = \nu_1$ and we are thus led to Case 1.

*Case 2.2.* Suppose that $p = 2$. We replace $\gamma_2$ by $\gamma_k$. There exist integers $u$, $v$, $w$ ($v$ odd) such that

$$b[q](\gamma_1, \gamma_1) = \frac{2u}{2^{\nu_1}}, \quad b[q](\gamma_1, \gamma_2) = \frac{v}{2^{\nu_1}}, \quad b[q](\gamma_2, \gamma_2) = \frac{2w}{2^{\nu_1}}$$

hold modulo $\mathbb{Z}_2$. Note that $q(\gamma_1) = 2\tilde{u}/2^{\nu_1}$ and $q(\gamma_2) = 2\tilde{w}/2^{\nu_1}$ hold modulo $2\mathbb{Z}_2$ for some integers $\tilde{u}$ and $\tilde{w}$ with $u = \tilde{u} \bmod 2^{\nu_1 - 1}$ and $w = \tilde{w} \bmod 2^{\nu_1 - 1}$.

If $l = 2$, then $\mathcal{L}^{(2)}(l, D, q)$ is determined by Lemma 5.2. Suppose that $l \geq 3$. There exists an integer $t$ such that $(v^2 - uw)t = 1 \bmod 2^{\nu_1}$ holds. For each $j \geq 3$, we choose integers $s_{j1}$ and $s_{j2}$ such that $b[q](\gamma_j, \gamma_1) = s_{j1}/2^{\nu_1} \bmod \mathbb{Z}_2$ and $b[q](\gamma_j, \gamma_2) = s_{j2}/2^{\nu_1} \bmod \mathbb{Z}_2$ hold, and we calculate

$$\begin{pmatrix} \beta_{j1} \\ \beta_{j2} \end{pmatrix} := t \cdot \begin{pmatrix} 2w & -v \\ -v & 2u \end{pmatrix} \cdot \begin{pmatrix} s_{j1} \\ s_{j2} \end{pmatrix}.$$

Then $\beta_{j1}$ and $\beta_{j2}$ are divisible by $2^{\nu_1 - \nu_j}$. Hence $\gamma_j' := \gamma_j - \beta_{j1}\gamma_1 - \beta_{j2}\gamma_2$ is an element of order $2^{\nu_j}$ that is independent of the choice of the integers. The set

$\{\gamma_1, \gamma_2, \gamma_3', \dots, \gamma_l'\}$ is again a reduced set of generators of $D$, and the two subgroups $\langle \gamma_1, \gamma_2 \rangle$ and $\langle \gamma_3', \dots, \gamma_l' \rangle$ of $D$ are orthogonal with respect to $q$. Therefore, putting

$$(D_1, q_1) := \left(\langle \gamma_1, \gamma_2 \rangle, q\big|_{\langle \gamma_1, \gamma_2 \rangle}\right) \cong \left((\mathbb{Z}/(p^{\nu_1}))^{\oplus 2}, \frac{1}{2^{\nu_1}} \begin{bmatrix} 2\tilde{u} & v \\ v & 2\tilde{w} \end{bmatrix}\right)$$

and $(D_2, q_2) := \left(\langle \gamma_3', \dots, \gamma_l' \rangle, q\big|_{\langle \gamma_3', \dots, \gamma_l' \rangle}\right)$, we obtain an orthogonal decomposition $(D, q) = (D_1, q_1) \oplus (D_2, q_2)$.

Let $\Lambda$ be a lattice of rank $l$ over $\mathbb{Z}_2$ such that there exists an isomorphism $h\colon (D_\Lambda, q_\Lambda) \xrightarrow{\sim} (D, q)$. We pick up two vectors $e_1^*, e_2^* \in \Lambda^\vee$ such that $h \circ \Psi_\Lambda(e_1^*) = \gamma_1$ and $h \circ \Psi_\Lambda(e_2^*) = \gamma_2$. Let $\Lambda_1' \subset \Lambda^\vee$ be the $\mathbb{Z}_2$-submodule of $\Lambda^\vee$ generated by $e_1^*$ and $e_2^*$. Then $\Lambda_1 := \Lambda_1' \cap \Lambda$ is a sublattice of $\Lambda$ generated by $e_1 := 2^{\nu_1} e_1^*$ and $e_2 := 2^{\nu_1} e_2^*$. The intersection matrix $M_1$ of $\Lambda_1$ with respect to $e_1$ and $e_2$ satisfies $\mathrm{ord}_2(\det M_1^{-1}) = -\nu_1$. Because $\mathrm{ord}_2((x, e_1)) \geq \nu_1$ and $\mathrm{ord}_2((x, e_2)) \geq \nu_1$ hold for any vector $x \in \Lambda$, we have $((x, e_1), (x, e_2)) \cdot M_1^{-1} \in \mathbb{Z}_2^{\oplus 2}$. Therefore, $\Lambda$ is decomposed into the orthogonal direct sum of $\Lambda_1$ and $\Lambda_1^\perp$. The homomorphism $h \circ \Psi_\Lambda$ induces isomorphisms $(D_{\Lambda_1}, q_{\Lambda_1}) \cong (D_1, q_1)$ and $(D_{\Lambda_1^\perp}, q_{\Lambda_1^\perp}) \cong (D_2, q_2)$. It follows that

$$\mathcal{L}^{(2)}(l, D, q) = \mathcal{L}^{(2)}(2, D_1, q_1) * \mathcal{L}^{(2)}(l-2, D_2, q_2).$$

Thus $\mathcal{L}^{(2)}(l, D, q)$ is calculated by Lemma 5.2 and the induction hypothesis on $l$.

## *5.2. Over $\mathbb{Z}$*

Let $D$ be a finite abelian group and $q\colon D \to \mathbb{Q}/2\mathbb{Z}$ a nondegenerate finite quadratic form. Let $(r, s)$ be a pair of nonnegative integers such that $n := r + s > 0$. We will describe a criterion to determine whether there exists a lattice $L$ over $\mathbb{Z}$ with signature $(r, s)$ such that $(D_L, q_L)$ is isomorphic to the given finite quadratic form $(D, q)$.

We put $d := (-1)^s |D|$. Let $P$ be the set of prime divisors of $2d$, and let $(D, q) = \bigoplus_{p \in P} (D^{(p)}, q^{(p)})$ be the orthogonal decomposition of $(D, q)$ into the $p$-parts. If $d$ is odd then we put $(D^{(2)}, q^{(2)}) = ((0), [0])$. By Lemma 4.3 and Theorem 3.1, a lattice $L$ over $\mathbb{Z}$ with signature $(r, s)$ and $(D_L, q_L) \cong (D, q)$ exists if and only if the following claim is verified:

($\sharp$) For each $p \in P$, there exists a lattice $\Lambda^{(p)}$ of rank $n$ over $\mathbb{Z}_p$ such that
(i) $\mathrm{disc}(\Lambda^{(p)}) = d \cdot (\mathbb{Z}_p^\times)^2$ and
(ii) $(D_{\Lambda^{(p)}}, q_{\Lambda^{(p)}}) \cong (D^{(p)}, q^{(p)})$ hold,
and they satisfy

$$r - s + \sum_{p \in P} p\text{-excess}(\Lambda^{(p)}) = n \bmod 8.$$

We put $\delta_p := d/p^{\mathrm{ord}_p(d)} \in \mathbb{Z}$. Under condition (ii), which implies that $|D_{\Lambda^{(p)}}| = d/\delta_p$, condition (i) is equivalent to

$$\mathrm{reddisc}(\Lambda^{(p)}) = \delta_p \cdot (\mathbb{Z}_p^\times)^2.$$

We can therefore check the claim ($\sharp$) by the following method. First calculate $\mathcal{L}^{(p)}(n, D^{(p)}, q^{(p)})$ for each $p \in P$; then search for an element $([\sigma_p, u_p] \mid p \in P)$ of the Cartesian product of the sets $\mathcal{L}^{(p)}(n, D^{(p)}, q^{(p)})$ that satisfies $u_p = \delta_p \cdot (\mathbb{Z}_p^\times)^2$ for each $p \in P$ and $r - s + \sum \sigma_p = n \bmod 8$. The claim ($\sharp$) is true if and only if we find such an element.

## 6. Roots

For the following, we refer to [3, Chaps. IV–VI], [6, Chap. 4], or [12].

### *6.1. Root System of a Positive Definite Even Lattice over $\mathbb{Z}$*

Let $L$ be a positive definite even lattice over $\mathbb{Z}$. A vector of $L$ is said to be a *root* if its norm is 2. We denote by $L_{\text{root}}$ the sublattice of $L$ generated by roots. A lattice $L$ is said to be a *root lattice* if $L = L_{\text{root}}$ holds. Let $\text{Roots}(L)$ be the set of roots of $L$. We define $\sim$ to be the finest equivalence relation on $\text{Roots}(L)$ that satisfies $(v, w) \neq 0 \implies v \sim w$. Let $I_1, \ldots, I_k$ be the equivalence classes of roots under the relation $\sim$, and let $L_i$ be the sublattice of $L_{\text{root}}$ generated by $I_i$. There exists a basis $B_i \subset I_i$ such that the intersection matrix of $L_i$ with respect to $B_i$ is the Cartan matrix corresponding to a Dynkin diagram of type $A_l$, $D_m$, or $E_n$ (see Figure 6.1). Let $\tau_i$ be the type of the Dynkin diagram of the intersection matrix of $L_i$. We define the root type of $L$ to be $\sum_{i=1}^{k} \tau_i$. Conversely, for an $ADE$-type $\Sigma$, there exists a root lattice $L(\Sigma)$, unique up to isomorphism, whose root type is $\Sigma$.

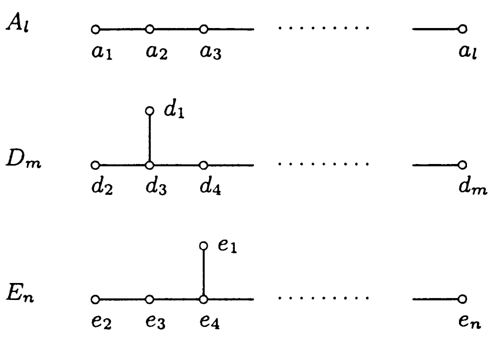


**Figure 6.1** Dynkin diagram

The root type of a positive definite even lattice $L$ over $\mathbb{Z}$ is thus determined by the following procedure:

(1) create the list $\text{Roots}(L)$, and decompose it into $I_1, \ldots, I_k$;
(2) calculate the rank of $L_i$ for $i = 1, \ldots, k$;
(3) determine the type $\tau_i$ from $\text{rank}(L_i)$ and $|I_i|$ by using Table 6.1.

**Table 6.1** Number of Roots and Discriminant Forms of Root Lattices

| $\tau$ | $\lvert\mathrm{Roots}(L(\tau))\rvert$ | $D_{L(\tau)}$ | $q_{L(\tau)}$ |
|---|---|---|---|
| $A_l$ | $l(l+1)$ | $\langle \bar{a}_l^* \rangle \cong \mathbb{Z}/(l+1)$ | $\left[ \dfrac{l}{l+1} \right]$ |
| $D_m$ $(m$ even$)$ | $2m(m-1)$ | $\langle \bar{d}_1^* \rangle \oplus \langle \bar{d}_m^* \rangle \cong (\mathbb{Z}/(2))^{\oplus 2}$ | $\begin{bmatrix} m/4 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ |
| $D_m$ $(m$ odd$)$ | $2m(m-1)$ | $\langle \bar{d}_1^* \rangle \cong \mathbb{Z}/(4)$ | $[m/4]$ |
| $E_6$ | 72 | $\langle \bar{e}_6^* \rangle \cong \mathbb{Z}/(3)$ | $[4/3]$ |
| $E_7$ | 126 | $\langle \bar{e}_7^* \rangle \cong \mathbb{Z}/(2)$ | $[3/2]$ |
| $E_8$ | 240 | $(0)$ | $[0]$ |

### *6.2. Discriminant Forms of Root Lattices*

The discriminant form $(D_{L(\tau)}, q_{L(\tau)})$, where $\tau$ is $A_l$, $D_m$, or $E_n$, is indicated in Table 6.1. In this table, for example, $\{a_1^*, \dots, a_l^*\}$ is the basis of $L(A_l)^\vee$ dual to the basis $\{a_1, \dots, a_l\}$ of $L(A_l)$ given in Figure 6.1, and $\bar{a}_i^* \in D_{L(A_l)}$ is the image of $a_i^*$ by the homomorphism $\Psi_{L(A_l)} \colon L(A_l)^\vee \to D_{L(A_l)}$.

Let $\Gamma(\tau)$ denote the image of the natural homomorphism from the orthogonal group $O(L(\tau))$ of the lattice $L(\tau)$ to $\mathrm{Aut}(D_{L(\tau)}, q_{L(\tau)})$. The structure of $\Gamma(\tau)$ is given as follows.

(a) If $\tau = A_1$ or $\tau = E_7$, then $\Gamma(\tau)$ is trivial.
(b) If $\tau = A_l$ $(l > 1)$ or $\tau = D_m$ $(m$ odd$)$ or $\tau = E_6$, then $\Gamma(\tau)$ is isomorphic to $\mathbb{Z}/(2)$ generated by the multiplication by $-1$.
(c) If $\tau = D_m$ $(m > 4$ and even$)$, then $\Gamma(\tau)$ is isomorphic to $\mathbb{Z}/(2)$ generated by

$$\bar{d}_1^* \mapsto \bar{d}_1^* + \bar{d}_m^*, \qquad \bar{d}_m^* \mapsto \bar{d}_m^*.$$

(d) If $\tau = D_4$, then $\Gamma(\tau)$ is isomorphic to the full symmetric group acting on the set $\{\bar{d}_1^*, \bar{d}_4^*, \bar{d}_1^* + \bar{d}_4^*\}$ of nontrivial elements of $D_{L(\tau)}$.

## 7. Existence of an Elliptic $K3$ Surface with Given Data

THEOREM 7.1. *Let $\Sigma$ be an $ADE$-type with* $\mathrm{rank}(\Sigma) \le 18$, *and let $G$ be a finite abelian group. Then there exists an elliptic $K3$ surface $f \colon X \to \mathbb{P}^1$ with $\Sigma_f = \Sigma$ and $G_f \cong G$ if and only if the root lattice $L(\Sigma)$ has an even overlattice $M$ with the following properties*:

(i) $M/L(\Sigma) \cong G$;
(ii) *there exists an even lattice $N$ of signature* $(2, 18 - \mathrm{rank}(\Sigma))$ *such that* $(D_N, q_N)$ *is isomorphic to* $(D_M, q_M)$; *and*
(iii) *the sublattice $M_{\mathrm{root}}$ of $M$ coincides with $L(\Sigma)$.*

*Proof*. Suppose that a pair $(\Sigma, G)$ satisfies the conditions of the theorem. By Proposition 4.2, property (ii) implies that there exists an even unimodular overlattice $K'$ of $M^- \oplus N$ into which $M^-$ and $N$ are primitively embedded. Let $H$ denote the hyperbolic lattice,

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $K := K' \oplus H$ is an even unimodular lattice with signature $(3, 19)$. Hence $K$ is isomorphic to the $K3$ lattice $L(2E_8)^- \oplus H^{\oplus 3}$ by Milnor's structure theorem (see [15]). There exists a 2-dimensional linear subspace $V$ of $N \otimes_{\mathbb{Z}} \mathbb{R}$ such that (a) the bilinear form is positive definite on $V$ and (b) if $N' \subset N$ is a sublattice such that $N' \otimes_{\mathbb{Z}} \mathbb{R}$ contains $V$, then $N'$ coincides with $N$. By the surjectivity of the period map on the moduli of $K3$ surfaces, there exist a complex $K3$ surface $X$ and an isomorphism $\alpha\colon H^2(X; \mathbb{Z}) \xrightarrow{\sim} K$ of lattices such that

$$\alpha_{\mathbb{R}}^{-1}(V) = (H^{0,2}(X) \oplus H^{2,0}(X)) \cap H^2(X; \mathbb{R})$$

holds, where $\alpha_{\mathbb{R}} := \alpha \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have

$$\alpha^{-1}(M^- \oplus H) = \mathrm{NS}_X. \tag{7.1}$$

By Kondo's lemma [8, Lemma 2.1], there exists a structure of the elliptic fibration $f\colon X \to \mathbb{P}^1$ with a section $O\colon \mathbb{P}^1 \to X$ such that, if $F$ denotes the cohomology class of a general fiber of $f$, then

$$\mathbb{Z}[F]^{\perp}/\mathbb{Z}[F] \cong M^- \tag{7.2}$$

holds, where $\mathbb{Z}[F]^{\perp}$ is the orthogonal complement of $\mathbb{Z}[F]$ in the Néron–Severi lattice $\mathrm{NS}_X$ of $X$. Let $H_f$ be the sublattice of $\mathrm{NS}_X$ spanned by the cohomology classes of the zero section and a general fiber of $f$, let $S_f$ be the sublattice of $\mathrm{NS}_X$ defined in Section 1, and let $W_f$ be the orthogonal complement of $H_f$ in $\mathrm{NS}_X$. The lattice $H_f$ is isomorphic to the hyperbolic lattice $H$ and is orthogonal to $S_f$. By an abuse of notation, we denote by $(W_f)_{\mathrm{root}}$ the sublattice of $W_f$ generated by the vectors of norm $-2$. From (7.2), we have

$$W_f \cong M^-. \tag{7.3}$$

On the other hand, by Nishiyama's lemma [12, Lemma 6.1] we have

$$W_f/(W_f)_{\mathrm{root}} \cong \mathrm{MW}_f, \tag{7.4}$$

$$S_f = (W_f)_{\mathrm{root}}. \tag{7.5}$$

Combining these with the properties (i) and (iii) of $M$ and the isomorphism (7.3), we have $S_f \cong L(\Sigma)^-$ and $\mathrm{MW}_f \cong G$. Hence $\Sigma = \Sigma_f$ and $G \cong G_f$ hold.

Conversely, suppose that there exists an elliptic $K3$ surface $f\colon X \to \mathbb{P}^1$ with $\Sigma_f = \Sigma$ and $G_f \cong G$. Using Nishiyama's lemma again, we see that the primitive closure $\overline{S_f}$ of $S_f$ in $\mathrm{NS}_X$ satisfies $\overline{S_f}/S_f \cong G$ and $(\overline{S_f})_{\mathrm{root}} = S_f$. We have an isomorphism $S_f \cong L(\Sigma)^-$; let $M^-$ be the overlattice of $L(\Sigma)^-$ corresponding to

$\overline{S_f}$ via this isomorphism. Then $M := (M^-)^-$ is an overlattice of $L(\Sigma)$ that possesses the properties (i) and (iii). Moreover, $\overline{S_f} \oplus H_f$ is primitive in the even unimodular lattice $H^2(X;\mathbb{Z})$ and hence Proposition 4.2 implies that the orthogonal complement $N_f$ of $\overline{S_f} \oplus H_f$ in $H^2(X;\mathbb{Z})$ satisfies $(D_{N_f}, q_{N_f}) \cong (D_{\overline{S_f}}, -q_{\overline{S_f}}) \cong (D_M, q_M)$. Because the signature of $N_f$ is $(2, 18 - \operatorname{rank}(\Sigma))$, the overlattice $M$ has the property (ii). $\square$

## 8. Making the List

Recall that, in order for an $ADE$-type $\Sigma$ to be an $ADE$-type of an elliptic $K3$ surface, it is necessary that $\operatorname{rank}(\Sigma) \leq 18$ and $\operatorname{euler}(\Sigma) \leq 24$. It is obvious that the torsion part of the Mordell–Weil group of an elliptic surface is of length $\leq 2$.

First we list all $ADE$-types $\Sigma$ with $\operatorname{rank}(\Sigma) \leq 18$ and $\operatorname{euler}(\Sigma) \leq 24$; there are 3937 such $ADE$-types. For each

$$\Sigma := \sum a_l A_l + \sum d_m D_m + \sum e_n E_n$$

in this list, we carry out the following calculation.

*Step 1.* We calculate the discriminant form $(D_{L(\Sigma)}, q_{L(\Sigma)})$ using Table 6.1. Note that the product of the wreath products

$$\prod_{a_l>0} (\Gamma(A_l) \wr \mathfrak{S}_{a_l}) \times \prod_{d_m>0} (\Gamma(D_m) \wr \mathfrak{S}_{d_m}) \times \prod_{e_n>0} (\Gamma(E_n) \wr \mathfrak{S}_{e_n})$$

acts on $(D_{L(\Sigma)}, q_{L(\Sigma)})$. Here, for example, the full symmetric group $\mathfrak{S}_{a_l}$ acts on $D_{L(\Sigma)}$ as the permutation group on the $a_l$ components of $D_{L(\Sigma)}$ that are isomorphic to $D_{L(A_l)}$. We denote this group by $\Gamma(\Sigma)$.

*Step 2.* We make a complete list of representatives of the quotient set $D_{L(\Sigma)}/\Gamma(\Sigma)$ and pick up from this list elements that are isotopic with respect to $q_{L(\Sigma)}$. Let $\mathcal{V}_\Sigma = \{\bar{v}_1, \dots, \bar{v}_N\}$ be the list of isotopic elements of $D_{L(\Sigma)}$ modulo $\Gamma(\Sigma)$. For each $\bar{v}_i \in \mathcal{V}_\Sigma$, we calculate the stabilizer subgroup $\operatorname{St}(\Gamma(\Sigma), \bar{v}_i)$ of $\bar{v}_i$ in $\Gamma(\Sigma)$. Then we make a complete list of representatives of $D_{L(\Sigma)}/\operatorname{St}(\Gamma(\Sigma), \bar{v}_i)$, and we pick up from this list elements that are isotopic with respect to $q_{L(\Sigma)}$ and orthogonal to $\bar{v}_i$ with respect to $b[q_{L(\Sigma)}]$. Let $\mathcal{W}_{\Sigma,i}$ be the list of isotopic elements orthogonal to $\bar{v}_i$ modulo $\operatorname{St}(\Gamma(\Sigma), \bar{v}_i)$.

Next we make the list $\mathcal{G}'_\Sigma$ of all pairs $[\bar{v}_i, \bar{w}_j]$ of $\bar{v}_i \in \mathcal{V}_\Sigma$ and $\bar{w}_j \in \mathcal{W}_{\Sigma,i}$. Then every isotopic subgroup of $(D_{L(\Sigma)}, q_{L(\Sigma)})$ with length $\leq 2$ is conjugate under the action of $\Gamma(\Sigma)$ to a subgroup $\langle \bar{v}_i, \bar{w}_j \rangle$ generated by $\bar{v}_i$ and $\bar{w}_j$ for some $[\bar{v}_i, \bar{w}_j] \in \mathcal{G}'_\Sigma$. Of course, there are several different pairs that can generate a given subgroup. We remove this redundancy from $\mathcal{G}'_\Sigma$, and make a list $\mathcal{G}_\Sigma$.

*Step 3.* For each $[\bar{v}, \bar{w}] \in \mathcal{G}_\Sigma$, we calculate the subgroup $G := \langle \bar{v}, \bar{w} \rangle$ of $D_{L(\Sigma)}$, its orthogonal complement $G^\perp$ in $(D_{L(\Sigma)}, q_{L(\Sigma)})$, and the finite quadratic form $(D_G, q_G) := \bigl(G^\perp/G, q_{L(\Sigma)}\big|_{G^\perp/G}\bigr)$.

*Step 3.1.* By the algorithm described in Section 5, we determine whether there exists an even lattice $N$ over $\mathbb{Z}$ of signature $(2, 18-\operatorname{rank}(\Sigma))$ such that $(D_N, q_N) \cong (D_G, q_G)$. If the answer is affirmative, we go to the next step.

*Step 3.2.* We calculate the intersection matrix of the even overlattice $M_G$ of $L(\Sigma)$ generated by $L(\Sigma)$ and $v$, $w$ in $L(\Sigma)^\vee$, where $v$ and $w$ are vectors of $L(\Sigma)^\vee$ such that $\Psi_{L(\Sigma)}(v) = \bar{v}$ and $\Psi_{L(\Sigma)}(w) = \bar{w}$. Then we calculate the root type of $M_G$ by the algorithm described in Section 6. If this root type coincides with the initial $ADE$-type $\Sigma$, then we let the pair $(\Sigma, G)$ be a member of the list $\mathcal{P}$.

By Theorem 7.1, the list $\mathcal{P}$ so made is the complete list of the data of elliptic $K3$ surfaces.

The following remarks are useful in checking the program.

REMARK 8.1. Note that neither $\mathrm{euler}(\Sigma) \leq 24$ nor $\mathrm{length}(G) \leq 2$ is contained in the conditions of Theorem 7.1. Therefore, if we input $\Sigma$ with $\mathrm{euler}(\Sigma) > 24$ into the program, it should return no subgroups $G$ of $D_{L(\Sigma)}$ such that $(\Sigma, G)$ can be a member of the list $\mathcal{P}$. If we change Step 2 of the program so that it lists all isotopic subgroups of length $\geq 3$, then the result should also be an empty set.

REMARK 8.2. Suppose that the root type $\Sigma'$ of $M_G$ is not equal to $\Sigma$ in Step 3.2 of the program. Let $G'$ be the finite abelian group $M_G/(M_G)_{\mathrm{root}}$. Then $(\Sigma', G')$ appears in $\mathcal{P}$.

REMARK 8.3. For each $(\Sigma, G) \in \mathcal{P}$, there should be at least one configuration that satisfies the conditions given in Section 2.3.

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