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Gehring's Lemma for Nondoubling Measures

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1. Introduction

Let $Q_0 \subset R^n$ be a fixed cube with sides parallel to the coordinate axes, let w be a strictly positive integrable function on Q_0, and let $1 < p < \infty$. We shall say that a positive function $g \in L^p_w(Q_0)$ belongs to $\mathrm{RH}_p(w)$ (i.e., that g satisfies a *reverse Hölder inequality*) if there exists a $C \geq 1$ such that, for every cube $Q \subset Q_0$ with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{w(Q)}\int_Q g(x)^p w(x)\,dx\right)^{1/p} \leq \frac{C}{w(Q)}\int_Q g(x)w(x)\,dx,$$

with $w(Q) = \int_Q w(x)\,dx$. If the underlying measure $\mu := w(x)\,dx$ satisfies the doubling condition—that is, if there exists a constant $c > 0$ such that $\mu(B(x,2r)) \leq c\mu(B(x,r))$—then by Gehring's lemma [7] there exists an $\varepsilon > 0$ such that $g \in \mathrm{RH}_{p+\varepsilon}(w)$. For excellent accounts of the role that reverse Hölder inequalities play in PDEs, we refer to [9] and [11].

Recently there has been interest in extending the Calderón–Zygmund program to the context of nondoubling measures (cf. [1; 13; 14; 16; 20; 21] and the references therein). The purpose of this note is to prove Gehring's lemma for nondoubling measures of the form $\mu := w(x)\,dx$. Our main results are given in the next two theorems; for proofs, see Section 4. (When preparing the final version of this paper for publication we realized that Theorem 1 can be also obtained by a different method by means of combining Lemma 2.3 and Corollary 2.4 of [16] with Exercise 6.6 of [18].)

THEOREM 1. *Let $1 < p < \infty$, and let w be a positive integrable function on Q_0. Suppose that $g \in \mathrm{RH}_p(w)$. Then there exists an $\varepsilon > 0$ such that $g \in \mathrm{RH}_{p+\varepsilon}(w)$.*

THEOREM 2 (see [13] for the corresponding R^n version of this result; see [9] and the references therein for the doubling case). *Let g, h be positive functions in $L^p_w(Q_0)$ and suppose that there exists $c > 1$ such that, for all cubes $Q \subset Q_0$ with sides parallel to the coordinate axes, we have*

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$$\left(\frac{1}{w(Q)}\int_Q g(x)^p w(x)\,dx\right)^{1/p} \le c\frac{1}{w(Q)}\int_Q g(x)w(x)\,dx + c\left(\frac{1}{w(Q)}\int_Q h(x)^p w(x)\,dx\right)^{1/p}. \tag{1.1}$$

Then there exist $q > p$ and $C = C(c,q) > 0$ such that if $g, h \in L^q_w(Q_0)$ then, for every cube $Q \subset Q_0$ with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{w(Q)}\int_Q g(x)^q w(x)\,dx\right)^{1/q} \le C\left(\frac{1}{w(Q)}\int_Q g(x)^p w(x)\,dx\right)^{1/p} + C\left(\frac{1}{w(Q)}\int_Q h(x)^q w(x)\,dx\right)^{1/q}. \tag{1.2}$$

Our methods are based on covering lemmas and interpolation theory. For doubling measures $d\mu := w(x)\,dx$, the connection with interpolation is given by the fact that the maximal operator of Hardy and Littlewood associated with $d\mu$,

$$M_\mu f(x) = \sup_{Q_0 \supset Q \ni x} \frac{1}{\mu(Q)}\int_Q |f(x)|w(x)\,dx,$$

satisfies

$$(M_\mu f)^*_w(t) \approx \frac{1}{t}\int_0^t f^*_w(s)\,ds = f^{**}_w(t) \tag{1.3}$$

(see Section 2) while—independently of doubling conditions—we always have

$$f^{**}_w(t) = \frac{K(t, f; L^1_w, L^\infty)}{t} \tag{1.4}$$

(see [19, pp. 213–214]). In this case the inverse reiteration theorem of [5] (cf. our Theorem 4) immediately proves Theorem 1. If w is not doubling then (1.3) may not hold (see [1]); in fact, the maximal operator may not be bounded on L^p_w (see [8]), although doubling conditions do not, of course, alter the interpolation theory of L^p spaces. Therefore, dealing with nondoubling measures using the K-method requires a different maximal operator. It turns out that a suitable maximal operator can be obtained through the use of packings [1]. (The idea of maximal operators associated with packings can be traced at least as far back as the classical paper of John and Nirenberg [10].)

In order to explain in more detail what we do in this paper, let us start by recalling that a *packing* in Q_0 is simply a *finite or countably infinite collection* of nonoverlapping cubes with sides parallel to the coordinate axes contained in Q_0. For a given packing $\pi = \{Q_i\}_{i=1}^{|\pi|}$ in Q_0, we associate a linear operator S_π defined by:

$$S_\pi(f)(x) = \sum_{i=1}^{|\pi|}\left(\frac{1}{w(Q_i)}\int_{Q_i} f(y)w(y)\,dy\right)\chi_{Q_i}(x), \quad f \in L^1_w(Q_0) + L^\infty(Q_0).$$

(Here $|\pi| = \infty$ if the packing has infinitely many cubes.) We consider the maximal operator defined by

$$(F_f)_w(t) = \sup_\pi (S_\pi(|f|))^*_w(t),$$

where g^*_w denotes the nonincreasing rearrangement of g with respect to the measure $w(y)\,dy$, and the supremum is taken over all packings.

A characterization of the K-functional for the pair $(L^1_w(R^n), L^\infty(R^n))$ in terms of the maximal operator $(F_f)_w(t)$ was given in [1]. This characterization was exploited in [13] to prove the nonlocal version of Theorem 2. In order to prove local self-improving results of Gehring type using the K-method, we show the following complement to the global computations of [1].

THEOREM 3. *Let $f \in L^1_w(Q_0) + L^\infty(Q_0)$; then, for $0 < t < w(Q_0) = \int_{Q_0} w(y)\,dy$,*

$$K(t, f; L^1_w(Q_0), L^\infty(Q_0)) \approx t(F_f)_w(t),$$

with constants of equivalence that are independent of f.

The proof of this characterization relies on a modification of the Calderón–Zygmund decomposition for nondoubling measures that was recently obtained in [14] and [16].

The paper is organized as follows. In Section 2 we provide a rather concise review (but with detailed references) of the parts of the real method of interpolation we shall use in this paper. In Section 3 we give a brief but self-contained account of the Calderón–Zygmund decomposition for nondoubling measures obtained in [14] and [16], conveniently modified for our purposes, and then use it to prove the equivalence between maximal operators associated with packings and K-functionals. Then, in Section 4, we provide the proofs of Theorem 1 and Theorem 2.

Finally, it is important to note here our belief that the methods we are developing are more interesting than the particular results obtained so far. For example, the interpolation method can be used to study Gehring-type self-improving results in a geometry-free context (see [12]). Moreover, our method can be used to study self-improving inequalities where the qualitative property whose improvement is sought is not necessarily integrability. We hope to return to this point elsewhere.

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2. Background

The main tools from real interpolation that we use are the K-functional and the reiteration theorem. Our main references will be [4; 5; 19], to which the reader is referred for further information.

We work with *pairs* $\vec{X} = (X_0, X_1)$ of quasi-normed spaces that are continuously embedded into a common Hausdorff topological vector space. For a given pair $\vec{X}$, we can thus form the sum space $\Sigma(\vec{X}) = X_0 + X_1$ and define for $x \in \Sigma(\vec{X})$, $t > 0$, the "K-functional"

$$K(t, x; \vec{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i,\ i = 0, 1\}.$$

In the context of the pair $(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$, the splitting implicit in the computation of the corresponding K-functionals is closely related to the Calderón–Zygmund decomposition. In fact, following [17, p. 1 (4); 17, Thm. 3.2], if we split f using a Calderón–Zygmund decomposition $f = b_\alpha + g_\alpha$—where b_α is the "bad" part, g_α is the "good" part, and the usual parameter α of the Calderón–Zygmund decomposition is chosen to be $(Mf)^*(t)$, where $(Mf)^*$ is the nondecreasing rearrangement of the maximal operator of Hardy–Littlewood—then we have

$$K(t, f; L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \approx \|b_{(Mf)^*(t)}\|_{L^1(\mathbb{R}^n)} + t\|g_{(Mf)^*(t)}\|_{L^\infty(\mathbb{R}^n)} \approx t(Mf)^*(t) \tag{2.1}$$

(cf. [4, p. 123]). The following elementary formula also holds [19, pp. 213–214]:

$$K(t, f; L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) = \int_0^t f^*(s)\,ds. \tag{2.2}$$

Comparing (2.1) and (2.2), we see that

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)\,ds. \tag{2.3}$$

The equivalences (2.1) and (2.3) fail, in general, if we replace the Lebesgue measure dx by a nondoubling measure $d\mu(x)$ (see [1]). For more examples of computations of K-functionals, see [4].

For a given pair $\vec{X}$, a ("real") scale of interpolation spaces between them can be constructed as follows. Given $0 < \theta < 1 \le q \le \infty$, define

$$\vec{X}_{\theta,q} = \left\{x \in \Sigma(\vec{X}) : \left\{\int_0^\infty (t^{-\theta}K(t, x; \vec{X}))^q \,\frac{dt}{t}\right\}^{1/q} < \infty\right\}.$$

It turns out that many of the familiar scales of spaces used in analysis can be identified with suitable real interpolation scales. The process of identification of concrete spaces as interpolation spaces for a given pair hinges upon the computation of K-functionals, and it is usually greatly simplified by the following reiteration (or iteration) property (cf. [4, Thm. 2.4; 5, Thm. 3.5.3]):

$$(\vec{X}_{\theta_0,q_0}, \vec{X}_{\theta_1,q_1})_{\theta,q} = \vec{X}_{\tau,q}, \tag{2.4}$$

where $\tau = (1-\theta)\theta_0 + \theta\theta_1$. A quantitative form of the reiteration formula (2.4) is given by Holmstedt's formula [4, Thm. 2.1; 5, Thm. 3.6.1]:

$$K(t, f; \vec{X}_{\theta_0,q_0}, \vec{X}_{\theta_1,q_1}) \approx \left\{\int_0^{t^{1/(\theta_1-\theta_0)}} (s^{-\theta_0}K(s, f; \vec{X}))^{q_0}\,\frac{ds}{s}\right\}^{1/q_0} + \left\{\int_{t^{1/(\theta_1-\theta_0)}}^\infty (s^{-\theta_1}K(s, f; \vec{X}))^{q_1}\,\frac{ds}{s}\right\}^{1/q_1}.$$

The following endpoint version of Holmstedt's formula (cf. [4, Cor. 2.3; 5, Cor. 3.6.2]) will be particularly useful here:

$$K(t, f; \vec{X}_{\theta_0,q_0}, \vec{X}_1) \approx \left\{ \int_0^{t^{1/(1-\theta_0)}} (s^{-\theta_0} K(s, f; \vec{X}))^{q_0} \frac{ds}{s} \right\}^{1/q_0}. \tag{2.5}$$

Reverse Hölder inequalities were formulated as "inverse" reiteration theorems in [15], where the following abstract form of Gehring's lemma was obtained.

THEOREM 4. *Let (A_0, A_1) be an ordered pair of Banach spaces (i.e. $A_1 \subset A_0$), and suppose that $f \in A_0$ is such that there exist some constant $c > 1$, $\theta_0 \in (0, 1)$, and $1 \leq p < \infty$ such that, for every $t \in (0, 1)$,*

$$K(t, f; A_{\theta_0,p;K}, A_1) \leq ct \frac{K(t^{1/(1-\theta_0)}, f; A_0, A_1)}{t^{1/(1-\theta_0)}}. \tag{2.6}$$

Then there exists a $\theta_1 > \theta_0$ such that, for $q \geq p$ and $0 < t < 1$, we have

$$K(t, A_{\theta_1,q;K}, A_1) \approx t \frac{K(t^{1/(1-\theta_1)}, f; A_0, A_1)}{t^{1/(1-\theta_1)}}.$$

The connection with the classical Gehring's lemma can be seen from the following facts:

$$L^p = (L^1, L^\infty)_{1/p',p}, \quad 1 < p < \infty; \tag{2.7}$$

$$\frac{K(t^{1/p}, f; L^p, L^\infty)}{t^{1/p}} \approx \left(\frac{1}{t} \int_0^t f^*(s)^p \, ds \right)^{1/p}, \quad 1 \leq p < \infty, \tag{2.8}$$

$$(M_p f)^*(t) \approx \frac{K(t^{1/p}, f; L^p, L^\infty)}{t^{1/p}}. \tag{2.9}$$

For (2.7) and (2.8), see [5, Thm. 5.2.1]; while (2.9) follows from (2.1), the fact that $M_p(f) = (M(|f|^p))^{1/p}$ by definition, and (2.8).

For a more detailed discussion on the connection with Gehring's lemma, see Section 4 and [15].

3. The *K*-Functional for the Pair $(L^1_w(Q_0), L^\infty(Q_0))$

We start with the Calderón–Zygmund decomposition for nondoubling measures obtained in [14] and [16]. We briefly indicate a proof of a version that is convenient for our development in this note.

Let Q_0 be a fixed cube in R^n with sides of length L that are parallel to the coordinate axes. For each x in the interior of Q_0 we define the basis

$$C_{Q_0}(x) = \{Q_x(r)\},$$

where $Q_x(r)$ is the unique cube with side r that minimizes the distance from x of the center of $Q_x(r)$ $(0 < r < L :=$ side of $Q_0)$, so that $Q_x(r) \subset Q_0$. We consider

$$M_Q(g)(x) = \sup_{Q \in C_{Q_0}(x)} \frac{1}{w(Q)} \int_Q |f(y)| w(y)\, dy.$$

LEMMA 1. *Let $g \in L^1_w(Q_0)$ be a nonnegative function. Let λ be a positive number such that $\lambda > 1/w(Q_0) \int_{Q_0} g(y)w(y)\, dy$ and the level set $\Omega_\lambda = \{x \in Q_0 : M_Q(g)(x) > \lambda\}$ is not empty. Then there exists a quasi-disjoint family of cubes $\{Q_j\}$ contained in Q_0 such that, for each j,*

$$\lambda < \frac{1}{w(Q_j)} \int_{Q_j} g(y)w(y)\, dy \le 2\lambda \tag{3.1}$$

and, moreover,

$$g(x) \le \lambda \quad \textit{for } x \in Q_0 \backslash \bigcup_j Q_j \textit{ a.e.} \tag{3.2}$$

In fact, we can write

$$\bigcup_j Q_j = \bigcup_{k=1}^{B(n)} \bigcup_{i \in \mathcal{F}_k} Q_i, \tag{3.3}$$

where each family $\{Q_i\}_{i \in \mathcal{F}_k}$, $k = 1, \dots, B(n)$, is formed by pairwise disjoint cubes.

NOTES. Recall that a family of cubes $\{Q_j\}$ is *quasi-disjoint* if there exists a universal constant C such that $\sum_j \chi_{Q_j}(x) \le C$; $B(n)$ is usually called the Besicovitch constant.

Proof of Lemma 1. Since Ω_λ is not empty, it follows that for any $x \in \Omega_\lambda$ we can find a cube $A_x \in C_{Q_0}(x)$ such that

$$\lambda < \frac{1}{w(A_x)} \int_{A_x} g(y)w(y)\, dy.$$

Therefore, since the function $h_x(r) = 1/\mu(Q_x(r)) \int_{Q_x(r)} |f(y)|\, d\mu(y)$ is continuous for each $x \in \operatorname{int}(Q_0)$, we can select a cube $Q_x \in C_{Q_0}(x)$ satisfying

$$\lambda < \frac{1}{w(Q_x)} \int_{Q_x} g(y)w(y)\, dy \le 2\lambda$$

with $Q_x \subsetneq Q_0$. The family $\{Q_x\}$ selected in this fashion covers Ω_λ. From now on we follow verbatim the argument in [14; 16]. Thus, for any cube Q_x we define the rectangle R_x in R^n as the unique rectangle in R^n centered at x such that $R_x \cap Q_0 = Q_x$. It follows that the ratio of any two side lengths of R_x is bounded by 2, and thus by Besicovitch's covering lemma we can select a countable collection $\{R_j\}$ of rectangles covering Ω_λ and such that every point of Ω_λ belongs to at most $B(n)$ rectangles R_j. Replacing each R_j by its corresponding cube Q_j, we obtain a family of cubes $\{Q_j\}$ with the properties that we need. Finally, (3.2) follows (as usual) by Lebesgue's differentiation theorem. □

The connection between packings and interpolation theory is given by the following local version of a result originally proved in [1] for R^n.

THEOREM 5. *Let* $f \in L^1_w(Q_0) + L^\infty(Q_0)$. *Then, for* $0 < t < w(Q_0)$,

$$K(t, f; L^1_w(Q_0), L^\infty(Q_0)) \approx t(F_f)_w(t),$$

with constants of equivalence independent of f.

Proof. For a given packing π, the operator $f \to S_\pi(|f|)$ is obviously a norm-1 sublinear operator acting on the pair $(L^1_w(Q_0), L^\infty(Q_0))$; therefore,

$$K(t, S_\pi(|f|); L^1_w(Q_0), L^\infty(Q_0)) \le K(t, f; L^1_w(Q_0), L^\infty(Q_0)).$$

Combining this inequality with

$$t(S_\pi(|f|))^*_w(t) \le K(t, S_\pi(|f|); L^1_w(Q_0), L^\infty(Q_0))$$

and taking the supremum over all packings, we obtain

$$t(F_f)_w(t) \le K(t, f; L^1_w(Q_0), L^\infty(Q_0)).$$

To prove the converse we fix a constant $1 < c < 2$ and, for a given $0 < t < w(Q_0)$, consider the set

$$\Omega_0(t) = \left\{x \in Q_0 : \sup_{r>0} \frac{1}{w(Q_x(r))} \int_{Q_x(r)} |f(y)|w(y)\,dy > c(F_f)_w(t)\right\}.$$

If $\Omega_0(t)$ is empty then by Lebesgue's theorem we have that $f \in L^\infty$; in fact,

$$|f| \le c(F_f)_w(t) \text{ a.e.}$$

Consequently, using the decomposition $f = 0 + f$ yields

$$K(t, f; L^1_w(Q_0), L^\infty(Q_0)) \le t\|f\|_\infty \le ct(F_f)_w(t),$$

as we wanted to show.

Suppose now that $\Omega_0(t)$ is not empty. Note that, for $0 < t < w(Q_0)$,

$$\begin{aligned} c(F_f)_w(t) > (F_f)_w(t) &= \sup_\pi (S_\pi(|f|))^*_w(t) \\ &\ge \left\{\left(\frac{1}{w(Q_0)} \int_{Q_0} |f(y)|w(y)\,dy\right)\chi_{Q_0}(x)\right\}^*_w(t) \\ &= \left(\frac{1}{w(Q_0)} \int_{Q_0} |f(y)|w(y)\,dy\right)\chi_{[0,w(Q_0))}(t) \\ &= \frac{1}{w(Q_0)} \int_{Q_0} |f(y)|w(y)\,dy. \end{aligned}$$

We can therefore apply Lemma 1 (with $\lambda = c(F_f)_w(t)$) to obtain a family of cubes $\mathcal{F} = \{Q_j\}$ such that

$$c(F_f)_w(t) < \frac{1}{w(Q_j)} \int_{Q_j} |f(y)|w(y)\,dy \le 2c(F_f)_w(t) \text{ on } Q_j \tag{3.4}$$

and

$$|f| \leq c(F_f)_w(t) \text{ on } Q_0 \setminus \bigcup Q_j \tag{3.5}$$

with

$$\bigcup_j Q_j = \bigcup_{k=1}^{B(n)} \bigcup_{i \in \mathcal{F}_k} Q_i = \bigcup_{k=1}^{B(n)} \pi_k, \tag{3.6}$$

where each $\pi_k = \{Q_i : i \in \mathcal{F}_k\}$, $k = 1, \ldots, B(n)$, is a packing.

The nearly optimal decomposition we need will then be given by

$$f = f\chi_{\bigcup Q_j} + f\chi_{Q_0 \setminus \bigcup Q_j}.$$

In fact,

$$\begin{aligned} \|f\chi_{\bigcup Q_j}\|_{L^1} &= \|f\chi_{\bigcup_{k=1}^{B(n)} \bigcup_{i \in \mathcal{F}_k} Q_i}\|_{L^1_w(Q_0)} \\ &\leq \sum_{k=1}^{B(n)} \sum_{i \in \mathcal{F}_k} \left(\frac{w(Q_i)}{w(Q_i)} \int_{Q_i} |f| w \, dx \right) \\ &\leq 2c(F_f)_w(t) \sum_{k=1}^{B(n)} \left(\sum_{i \in \mathcal{F}_k} w(Q_i) \right) \quad \text{(by (3.4))}. \end{aligned}$$

We shall show in a moment that

$$\sum_{i \in \mathcal{F}_k} w(Q_i) \leq t. \tag{3.7}$$

Assuming for now the validity of (3.7), we obtain

$$\begin{aligned} \|f\chi_{\bigcup Q_j}\|_{L^1} &\leq 2c(F_f)_w(t) \sum_{k=1}^{B(n)} \left(\sum_{i \in \mathcal{F}_k} w(Q_i) \right) \\ &\leq 2c(F_f)_w(t) \sum_{k=1}^{B(n)} t \\ &\leq 2cB(n)t(F_f)_w(t). \end{aligned}$$

Moreover, by (3.5) we have

$$t\|f\chi_{Q_0 \setminus \bigcup Q_j}\|_{L^\infty} \leq ct(F_f)_w(t).$$

Collecting estimates, we finally arrive at

$$K(t, f; L^1_w(Q_0), L^\infty(Q_0)) \leq 3cB(n)t(F_f)_w(t).$$

In order to prove (3.7) we must show that, given $\pi = \{Q_i\}$ an arbitrary packing from the family $\mathcal{F}$, we have

$$\sum_{Q_i \in \pi} w(Q_i) \leq t.$$

Indeed, if $\sum_{Q_i \in \pi} w(Q_i) > t$ then using (3.4) yields

$$S_\pi(|f|)(z) = \sum_{Q_i\in\pi}\left(\frac{1}{w(Q_i)}\int_{Q_i}|f(y)|w(y)\,dy\right)\chi_{Q_i}(z)$$
$$> c(F_f)_w(t)\left(\sum_{Q_i\in\pi}\chi_{Q_i}(z)\right)$$
$$> (F_f)_w(t)\left(\sum_{Q_i\in\pi}\chi_{Q_i}(z)\right).$$

Therefore, for any $z \in \bigcup_{Q_i\in\pi} Q_i$ we have $S_\pi(|f|)(z) > (F_f)_w(t)$, and since $S_\pi(|f|)(z) = 0$ on $\left(\bigcup_{Q_i\in\pi} Q_i\right)^c$ we see that

$$S_\pi(|f|)^*_w(t) > (F_f)_w(t) \quad \text{for } t < \sum_{Q_i\in\pi} w(Q_i),$$

contradicting the definition of $(F_f)_w(t)$. □

4. Proof of Theorems 1 and 2

In preparation for the proof of Theorems 1 and 2, let us introduce (following [1]) the functionals

$$S_{\pi,p}(f)(x) = \sum_{i=1}^{|\pi|}\left(\frac{1}{\mu(Q_i)}\int_{Q_i}|f(y)|^p w(y)\,dy\right)^{1/p}\chi_{Q_i}(x),$$

which are associated with "packings" $\pi = \{Q_i\}_{i=1}^{|\pi|} \subset Q_0$. If $p = 1$ then $S_{\pi,1}$ coincides with S_π, as defined in Section 2.

4.1. Proof of Theorem 1

Since we are dealing with families of disjoint cubes it follows readily that $g \in \mathrm{RH}_p(w)$ implies that, for any packing π, we have

$$S_{\pi,p}(g)(x) \le cS_\pi(g)(x).$$

Taking nondecreasing rearrangements in the previous inequality with respect to the measure $d\mu = w(x)\,dx$, we obtain

$$(S_{\pi,p}(g))^*_w(t) \le c(S_\pi(g))^*_w(t), \quad t > 0 \ \ (\pi \text{ any packing}).$$

Therefore, taking the supremum over all packings yields

$$\sup_\pi (S_{\pi,p}(g))^*_w(t) \le c\sup_\pi (S_\pi(g))^*_w(t). \tag{4.1}$$

By Theorem 5, we know that

$$K(t, g; L^1_w(Q_0), L^\infty(Q_0)) \approx t\sup_\pi (S_\pi(g))^*_w(t). \tag{4.2}$$

On the other hand (cf. [1]), by well-known general considerations it follows from (4.2) that

$$\begin{aligned} K(t^{1/p}, g; L^p_w(Q_0), L^\infty(Q_0)) &\approx (K(t, |g|^p; L^1_w(Q_0), L^\infty(Q_0))^{1/p} \\ &\approx \Big(t \sup_\pi (S_\pi(g^p))^*_w(t)\Big)^{1/p} \\ &= t^{1/p} \sup_\pi (S_{\pi,p}(g))^*_w(t). \end{aligned} \tag{4.3}$$

Multiplying (4.1) by $t^{1/p}$, we have

$$t^{1/p} \sup_\pi (S_{\pi,p}(g))^*_w(t) \le ct^{1/p} \sup_\pi (S_\pi(g))^*(t)$$

and thus arrive at the K-functional estimate

$$K(t^{1/p}, g; L^p_w(Q_0), L^\infty(Q_0)) \le ct^{-1/p'} K(t, g; L^1_w(Q_0), L^\infty(Q_0))$$

or, equivalently,

$$K(t, g; L^p_w(Q_0), L^\infty(Q_0)) \le ct^{1-p} K(t^p, g; L^1_w(Q_0), L^\infty(Q_0)). \tag{4.4}$$

Now we can apply Theorem 4 (cf. [15, Thm. 1]) to conclude that there exists a $q > p$ such that

$$K(t, g; L^q_w(Q_0), L^\infty(Q_0)) \le ct^{1-q/p} K(t^q, g; L^p_w(Q_0), L^\infty(Q_0)). \tag{4.5}$$

Thus, in view of the well-known formula

$$K(t, h; L^r_w(Q_0), L^\infty(Q_0)) \simeq \left(\int_0^{t^r} h^*_w(s)^r\, ds\right)^{1/r}$$

(cf. (2.8)), we have that (4.5) is equivalent to

$$\left(\frac{1}{t}\int_0^t g^*_w(s)^q\, ds\right)^{1/q} \le C\left(\frac{1}{t}\int_0^t g^*_w(s)^p\, ds\right)^{1/p} \tag{4.6}$$

for $0 < t < w(Q_0)$.

Observe that

$$\left(\frac{1}{w(Q_0)}\int_{Q_0} g(x)^r w(x)\, dx\right)^{1/r} = \left(\frac{1}{w(Q_0)}\int_0^{w(Q_0)} g^*_w(s)^r\, ds\right)^{1/r};$$

therefore we see that (4.6) gives

$$\left(\frac{1}{w(Q_0)}\int_{Q_0} g(x)^q w(x)\, dx\right)^{1/q} \le C\left(\frac{1}{w(Q_0)}\int_{Q_0} g(x)^p w(x)\, dx\right)^{1/p}. \qquad \square$$

4.2. Proof of Theorem 2

Let $1 \le p < \infty$ and denote by $P_p f$ the Hardy operator defined on locally integrable functions by

$$P_p f(t) = \left(\frac{1}{t}\int_0^t |f(s)|^p\, ds\right)^{1/p}, \quad t > 0.$$

If $p = 1$, we let $P_p f = Pf$.

We shall need the following lemma from [3], which we prove here for the sake of completeness.

LEMMA 2 (cf. [3, Prop. 2.1]). *Let f be a nonincreasing function, and let $1 < p < \infty$. Then*

$$\left(\int_0^t f(s)^p\,ds\right)^{1/p} \le \left(\frac{p-1}{p}\right)^{1/p}\left(\int_0^t Pf(s)^p\,ds\right)^{1/p} + \left(\frac{1}{p}\right)^{1/p} t^{(1-p)/p}\int_0^t f(s)\,ds. \tag{4.7}$$

Proof. Because f is decreasing,

$$\int_0^x f(s)\,ds \ge xf(x).$$

It follows that

$$\frac{d}{dx}\left(\int_0^x f(s)\,ds\right)^p = pf(x)\left(\int_0^x f(s)\,ds\right)^{p-1} \ge px^{p-1}f(x)^p;$$

integrating, we have

$$(Pf(x))^p \ge \frac{1}{x^p}\int_0^x ps^{p-1}f(s)^p\,ds.$$

Further integration and Fubini yield

$$\int_0^t Pf(s)^p\,ds \ge \frac{p}{p-1}\left(\int_0^t f(s)^p\,ds - t^{1-p}\int_0^t s^{p-1}f(s)^p\,ds\right),$$

which implies that

$$\left(\int_0^t f(y)^p\,dy\right)^{1/p} \le \left(\frac{p}{p-1}\right)^{-1/p}\left(\int_0^t Pf(s)^p\,ds\right)^{1/p} + t^{(1-p)/p}\left(\int_0^t s^{p-1}f(s)^p\,ds\right)^{1/p}. \tag{4.8}$$

Finally, since $\left(\int_0^t s^{p-1}f(s)^p\,ds\right)^{1/p} \le (1/p)^{1/p}\left(\int_0^t f(s)\,ds\right)$ (cf. [19, Thm. 3.11]), we obtain (4.7). □

Now let us proceed with the proof of Theorem 2. We shall follow closely the argument in [13].

As in the proof of Theorem 1, we see that (1.1) implies for any packing π that

$$S_{\pi,p}(g)(x) \le c(S_\pi(g)(x) + S_{\pi,p}(h)(x)).$$

Taking nondecreasing rearrangements with respect to the measure $d\mu = w(x)\,dx$ and taking the supremum over all packings yields

$$\sup_\pi (S_{\pi,p}(g))^*(2t) \le c\Big(\sup_\pi (S_\pi(g))^*(t) + \sup_\pi (S_{\pi,p}(h))^*(t)\Big).$$

Multiplying this inequality by $t^{1/p}$ and then using (4.3) and Theorem 5, we arrive at the K-functional estimate

$$K(t^{1/p}, g; L^p_w(Q_0), L^\infty) \le c\Big(t^{-1/p'}K(t, g; L^1_w(Q_0), L^\infty) + K(t^{1/p}, h; L^p_w(Q_0), L^\infty)\Big). \tag{4.9}$$

Using Holmstedt's reiteration formula (cf. (2.5)), we write

$$K(t^{1/p}, g; L^p_w(Q_0), L^\infty) \simeq \left(\int_0^t \left(s^{-1/p'} K(s, g, L^1_w(Q_0), L^\infty)\right)^p \frac{ds}{s}\right)^{1/p}$$
$$\simeq \left(\int_0^t \left(\frac{K(s, g, L^1_w(Q_0), L^\infty)}{s}\right)^p ds\right)^{1/p}.$$

Inserting this expression into (4.9) gives

$$\left(\int_0^t \left(\frac{K(s, g; L^1_w(Q_0), L^\infty)}{s}\right)^p ds\right)^{1/p}$$
$$\leq ct^{1/p} \frac{K(t, g; L^1_w(Q_0), L^\infty)}{t}$$
$$+ ct^{1/p}\left(\frac{1}{t}\int_0^t \left(\frac{K(s, h; L^1_w(Q_0), L^\infty)}{s}\right)^p ds\right)^{1/p}.$$

In terms of Hardy operators, we have

$$P_p\left(\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)^p\right)(t) \leq c\frac{K(t, g; L^1_w(Q_0), L^\infty)}{t}$$
$$+ cP_p\left(\frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot}\right)(t).$$

Applying $L^q(0, w(Q_0))$-norms to the previous inequality then yields

$$\left\|P_p\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)\right\|_q \leq c\left\|\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right\|_q$$
$$+ c\left\|P_p\left(\frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot}\right)\right\|_q.$$

Now, since

$$\left\|P_p\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)\right\|_q^p = \left\|P\left(\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)^p\right)\right\|_{q/p} \tag{4.10}$$

and since $K(s, f, L^1_w(Q_0), L^\infty)/s$ is decreasing, we can apply Lemma 2 to obtain

$$\left\|\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)^p\right\|_{q/p}$$
$$\leq \left(\frac{\frac{q}{p} - 1}{\frac{q}{p}}\right)^{p/q} \left\|P\left(\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)^p\right)\right\|_{q/p}$$
$$+ \left(\frac{p}{q}\right)^{p/q} w(Q_0)^{(1-q/p)/(q/p)} \left\|\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot}\right)^p\right\|_1.$$

Combining this last inequality, (4.10), and the well-known inequality $(x + y)^\alpha \leq x^\alpha + y^\alpha$ if $0 < \alpha \leq 1$, we arrive at

$$\left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q$$
$$= \left\| \left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right)^p \right\|_{q/p}^{1/p}$$
$$\leq \left(\frac{\frac{q}{p} - 1}{\frac{q}{p}} \right)^{1/q} \left\| P_p\left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right) \right\|_q$$
$$+ \left(\frac{p}{q} \right)^{1/q} (w(Q_0)^{(1-q/p)/(q/p)})^{1/p} \left\| \left(\frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right)^p \right\|_1^{1/p}.$$

On the other hand, by the classical Hardy inequality we have

$$\left\| P_p\left(\frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot} \right) \right\|_q \leq \left(\frac{\frac{q}{p}}{\frac{q}{p} - 1} \right)^{1/p} \left\| \frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q.$$

Collecting these estimates gives

$$\left(\frac{\frac{q}{p}}{\frac{q}{p} - 1} \right)^{1/q} \left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q$$
$$\leq c \left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q$$
$$+ \left(\frac{1}{\frac{q}{p} - 1} \right)^{1/q} (w(Q_0)^{(1-q/p)/(q/p)})^{1/p} \left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_p$$
$$+ c \left(\frac{\frac{q}{p}}{\frac{q}{p} - 1} \right)^{1/p} \left\| \frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q.$$

We can therefore choose $q > p$, with q sufficiently close to p, such that

$$\left(\frac{\frac{q}{p}}{\frac{q}{p} - 1} \right)^{1/q} - c > 0,$$

and thus we can write

$$\left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q \leq C(w(Q_0)^{(p-q)/q})^{1/p} \left\| \frac{K(\cdot, g, L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q + C \left\| \frac{K(\cdot, h; L^1_w(Q_0), L^\infty)}{\cdot} \right\|_q. \tag{4.11}$$

Now, using (2.7) (see also [5, Thm. 5.5.1]),

$$(L^1_w(Q_0), L^\infty)_{1/q',q} = L^q_w(Q_0), \qquad (L^1_w(Q_0), L^\infty)_{1/p',p} = L^p_w(Q_0);$$

hence (4.11) can be rewritten as

$$\|g\|_{L^q_w(Q_0)} \le C\big((w(Q_0)^{(p-q)/q})^{1/p}\|g\|_{L^p_w(Q_0)} + \|h\|_{L^q_w(Q_0)}\big). \tag{4.12}$$

Dividing by $w(Q_0)^{1/q}$, we thus obtain

$$\left(\frac{1}{w(Q_0)}\int_{Q_0} g(x)^q w(x)\,dx\right)^{1/q} \le C\left(\frac{1}{w(Q_0)}\int_{Q_0} g(x)^p w(x)\,dx\right)^{1/p} + C\left(\frac{1}{w(Q_0)}\int_{Q_0} h(x)^q w(x)\,dx\right)^{1/q}.$$

Obviously this argument can be repeated for every $Q \subset Q_0$ in order to obtain inequality (1.2). □

REMARK 1. If we work in R^n instead of a cube Q_0 and if $w(R^n) = \infty$, then the term $(w(Q_0)^{(p-q)/q})^{1/p}$ that appears in (4.12) is equal to 0 and we obtain

$$\left(\int_{R^n} g(x)^q w(x)\,dx\right)^{1/q} \le C\left(\int_{R^n} h(x)^q w(x)\,dx\right)^{1/q}$$

(cf. [13, Thm. 1]).

REMARK 2. Using inequality (4.8) instead of (4.7) and the fact that

$$(L^1_w(Q_0), L^\infty)_{1/p',q} = L^{p,q}_w(Q_0),$$

we can show that

$$\|g\|_{L^q_w(Q_0)} \le C\big((w(Q_0)^{(p-q)/q})^{1/p}\|g\|_{L^{p,q}_w(Q_0)} + \|h\|_{L^q_w(Q_0)}\big),$$

which implies (4.12) because $L^p_w(Q_0) \subset L^{p,q}_w(Q_0)$ if $q > p$.

REMARK 3. Our proof of Theorem 1, combined with the argument in [2, Thm. 3.1], can be used to derive—for nondoubling measures—the endpoint version of Gehring's lemma originally obtained by Fefferman [6].

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