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Global Plurisubharmonic Defining Functions

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1. Introduction

Given a bounded pseudoconvex domain with smooth (i.e., infinitely differentiable) boundary in $\mathbf{C}^n$, the Levi condition ensures that, for every defining function, the Levi form restricted to complex tangent vectors is positive semi-definite at each boundary point. Here we study the stronger condition that there exists a defining function plurisubharmonic on the boundary in the sense that, at each boundary point, the Levi form is positive semi-definite on *all* complex vectors. For strongly pseudoconvex domains (i.e., when the Levi form is positive definite on complex tangent vectors) it is elementary that there is a strongly plurisubharmonic defining function. For weakly pseudoconvex domains, a defining function plurisubharmonic on the boundary need not exist. The well-known "worm" domain introduced by Diederich and Fornæss in [7] has no such defining function, nor does the version of that domain with real-analytic boundary defined by Fornæss in [8]. In fact, a defining function plurisubharmonic on the boundary need not exist even locally. Fornæss gave such an example with only smooth boundary in [8], and later Behrens [1] gave a simpler example: There is a domain with a polynomial defining function such that (a) the Levi form degenerates at only one boundary point but (b) near this point, there is no local defining function plurisubharmonic on the boundary. Her example shows that any sufficient condition for the existence of a local defining function plurisubharmonic on the boundary must involve more than merely the structure of the degeneracy set of the Levi form.

Here we are interested in conditions under which the existence at each boundary point of a local defining function plurisubharmonic on the boundary implies the existence of a global defining function with this property. In this setting the structure of the degeneracy set enters naturally, since an obstruction can exist in the attempt to patch local defining functions along this set. This situation is illustrated by the real-analytic version of the "worm" domain constructed by Fornæss in [8], which does have local defining functions of the desired type even though (as just mentioned) there is no such global function. In this example the degeneracy set is a curve whose tangent space at each point is contained in the null space of the Levi form restricted to complex tangent vectors. Our condition that the domain be linearly regular (as defined in Section 2) rules out the existence of such curves, and this condition, along with the existence of local defining functions

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plurisubharmonic on the boundary, turns out to be sufficient for the existence of a global function on bounded domains with real-analytic boundary. Thus we have the following result.

THEOREM. *Let D be a linearly regular domain with real-analytic boundary in $\mathbf{C}^n$, and suppose that for each $p \in \partial D$ there is a neighborhood U_p of p on which D has a smooth defining function that is plurisubharmonic on $\partial D \cap U_p$. Then D has a global smooth defining function plurisubharmonic on ∂D.*

For the case $n = 2$ this theorem was proved in [11]. The higher-dimensional case we treat is considerably more complicated owing to the CR geometry that comes into play. We need the stratification theorem from [3] to describe the degeneracy set of the Levi form in terms of this geometry, and we use the real Frobenius theorem to find nonholomorphic coordinates that simplify the description of how to patch local defining functions. In Section 2 we describe this stratification along with other background material, and we prove the preliminary patching results in Section 3. In Section 4 we prove our main result.

2. Background

If ϕ is a smooth function defined near $p \in \mathbf{C}^n(z_1, \dots, z_n)$, then for $t = (t_1, \dots, t_n) \in \mathbf{C}^n$ we write $\partial\phi_p(t)$ for

$$\sum_{j=1}^{n} \frac{\partial \phi}{\partial z_j}(p) t_j$$

and write $L_p(\phi, t)$ for

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k,$$

the Levi form of ϕ at p applied to t.

Let S be a smooth submanifold of $\mathbf{C}^n$ and let $p \in S$. We write $T_p(S)$ for the (real) tangent space to S at p. The complex tangent space to S at p is, by definition, the maximal complex subspace of $T_p(S)$; it is denoted $T_p^{\mathbf{C}}(S)$. We say that S is a CR manifold if the dimension of $T_p^{\mathbf{C}}(S)$ is independent of p.

Let D denote a bounded pseudoconvex domain with smooth boundary in $\mathbf{C}^n$, and let r be a local defining function near $p \in \partial D$. Then it is easy to see that

$$T_p^{\mathbf{C}}(\partial D) = \{t \in \mathbf{C}^n : \partial r_p(t) = 0\}.$$

We write $N(p)$ for the null space in the complex tangent space of the Levi form at p, so

$$N(p) = \{t \in T_p^{\mathbf{C}}(\partial D) : L_p(r, t) = 0\}.$$

We let $w(\partial D)$ denote the set of weakly pseudoconvex boundary points of D, that is, the set of all $p \in \partial D$ such that $N(p) \neq \{0\}$. We say that D is linearly regular if there does not exist a nontrivial smooth curve γ in ∂D such that $\gamma'(t)$ lies in $N(\gamma(t))$ for all t.

Let D be a bounded domain with smooth boundary in $\mathbf{C}^n$ and let S be a smooth submanifold of ∂D. We say that S is complex-tangential at a point $p \in S$ if $T_p(S) \subset T_p^{\mathbf{C}}(\partial D)$. If L is a smooth vector field near a point $p \in \partial D$, we say that L is complex-tangential at p if its value there belongs to $T_p^{\mathbf{C}}(\partial D)$.

The following theorem is proved in [3, Thm. 5]. It shows that, for linearly regular domains, the degeneracy set of the Levi form can be locally stratified by CR submanifolds whose tangent directions lying in the complex-tangent space to the boundary are positive for the Levi form of the boundary. (In [3] the domain is assumed to be convex, but, as noted in [3], the proof shows that linear regularity suffices. Also, the statement about the dimension of the null space in part (b) of the Stratification Theorem that follows is implicit in the proof in [3].)

STRATIFICATION THEOREM. *Let D be a bounded pseudoconvex domain with real-analytic boundary in $\mathbf{C}^n$, and assume that D is linearly regular. Then each point in $w(\partial D)$ has a neighborhood U such that the following statements hold.*

(a) *We have $w(\partial D) \cap U = \bigcup_{j=0}^{2n-3} S_j$, where each S_j is a finite disjoint union of j-dimensional real-analytic CR submanifolds of $\partial D \cap U$. Furthermore, for all $p \in S_j$ we have $T_p(S_j) \cap N(p) = \{0\}$.*

(b) *If S is a component of some S_j, then the null space N has constant dimension along S. Also, if S is complex-tangential at some point then it is complex-tangential at every point.*

(c) *Each S_k is closed in $(\partial D \cap U) \setminus \left(\bigcup_{j=0}^{k-1} S_j\right)$.*

We remark that the concept of linear regularity was defined (under a different name) for domains with real-analytic boundary in $\mathbf{C}^2$ in [9]. The concept was extended to domains with smooth boundary in $\mathbf{C}^n$ in [10], where it was proved that bounded convex domains with real-analytic boundary in $\mathbf{C}^n$ are linearly regular. As a consequence, a smooth bounded domain is linearly regular if it has a proper holomorphic embedding into a bounded convex domain with real-analytic boundary, assuming the embedding extends smoothly to the boundary. In fact, the context of [10] was the study of passing from local to global maps into convex domains. The original setting in [9] was patching local peak functions on linearly regular domains in $\mathbf{C}^2$. This study of local peak functions was extended to $\mathbf{C}^3$ in [3] with the aid of the Stratification Theorem. This stratification was also used in [2] to study compact subsets of peak sets in $\mathbf{C}^n$.

3. Patching Functions along Strata

The key part of the proof of our main result is to patch local defining functions near a given stratum from the Stratification Theorem. To control the Levi form we require that the cutoff functions be constant in weakly pseudoconvex directions. The following proposition shows how to accomplish this in terms of a smooth (but not holomorphic) diffeomorphism. The result itself does not refer to any stratification and may be of general interest.

PROPOSITION. *Let D be a bounded pseudoconvex domain with smooth boundary in $\mathbf{C}^n$, and fix $p \in \partial D$. Write the complex dimension of $N(p)$ as $n-m-1$. Then there exist: a neighborhood W of p; a subbundle $\mathbf{E}$ of $T(\mathbf{C}^n)$ over W of real dimension $2m$ such that the fiber of $\mathbf{E}$ over each point $q \in \partial D \cap W$ is contained in $T_q^{\mathbf{C}}(\partial D)$; a smooth real vector field τ on W that, at each point of $\partial D \cap W$, is tangential to ∂D but not complex-tangential; and a smooth diffeomorphism Φ on W with the properties in* (a) *and* (b) *below. In stating these properties we write $\mathbf{F}$ for the real subbundle of $T(\mathbf{C}^n)$ over W generated by $\mathbf{E}$ and the vector field τ, and we denote with a subscript q the fiber of a bundle over a point $q \in W$.*

(a) *For all $q \in \partial D \cap W$ we have that $N(q)$ is contained in the orthogonal complement of $\mathbf{E}_q$ relative to $T_q^{\mathbf{C}}(\partial D)$.*
(b) *If we write $\Phi = (\phi_1, \dots, \phi_{2n})$, then $L\phi_j = 0$ on W if $L \in \mathbf{F}$ and $2m+1 < j \leq 2n$.*

Proof. We work in a neighborhood W of p, which we will shrink without comment. We define the real vector fields we seek in terms of the complex vector fields L_j and T that arise naturally in the study of the CR geometry of real hypersurfaces (see e.g. [6, Sec. 3.1]). Let r be a local defining function for D near p. After an orthogonal change of coordinates, we may assume that $\operatorname{grad} r(p) = (0, \dots, 0, 1)$ and that $N(p)$ is the span of

$$\frac{\partial}{\partial z_{m+1}}, \dots, \frac{\partial}{\partial z_{n-1}}$$

at p. For $j = 1, \dots, n-1$, define near p the smooth complex vector fields L_j by

$$L_j = \frac{\partial}{\partial z_j} - \frac{r_j}{r_n}\frac{\partial}{\partial z_n},$$

where we write r_j for the partial derivative of r with respect to z_j. Note that these are complex-tangential along ∂D and in fact span $T^{\mathbf{C}}(\partial D)$. Define the pure imaginary vector field T by

$$T = \frac{1}{r_n}\frac{\partial}{\partial z_n} - \frac{1}{r_{\bar{n}}}\frac{\partial}{\partial \bar{z}_n},$$

where $r_{\bar{n}}$ denotes the partial derivative of r with respect to $\bar{z}_n$.

Now we use $L_1, \dots, L_m$ to find a complex vector bundle $\mathbf{G}$ of dimension $n-m-1$ so that $N(q) \subset \mathbf{G}_q \subset T_q^{\mathbf{C}}(\partial D)$ if $q \in \partial D \cap W$. (See the beginning of the construction in [5, Sec. 2].) Toward this end, if $q \in W$ then we identify a vector at q of the form $s = \sum_{j=1}^n s_j(\partial/\partial z_j)$ with $(s_1, \dots, s_n)$. If also $t = \sum_{j=1}^n t_j(\partial/\partial z_j)$ is a vector at q, we define

$$\mathcal{L}_q(s,t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(q) s_j \bar{t}_k.$$

With this notation we define the bundle $\mathbf{G}$ by requiring that its fiber over a point $q \in W$ satisfy

$$\mathbf{G}_q = \{s : \partial r_q(s) = 0 \text{ and } \mathcal{L}_q(s, L_j) = 0 \text{ for } 1 \leq j \leq m\}.$$

It remains to check that $N(q) \subset \mathbf{G}_q$ for $q \in \partial D \cap W$. For such q, the Levi form of r restricted to $T_q^{\mathbf{C}}(\partial D)$ is positive semi-definite, from which it follows that

$$N(q) = \{s \in T_q^{\mathbf{C}}(\partial D) : \mathcal{L}_q(s,t) = 0 \text{ whenever } t \in T_q^{\mathbf{C}}(\partial D)\}.$$

We conclude that $\mathbf{G}$ has the desired property.

Now we define the bundle $\mathbf{E}$ by the requirement that its fiber over each point $q \in W$ be the orthogonal complement of $\mathbf{G}_q$ relative to $\{t : \partial r_q(t) = 0\}$. Then it follows easily that part (a) is true.

Next we put

$$\tau = \frac{i}{2}T.$$

We will apply the Frobenius theorem (see e.g. [4, Sec. 4.1]) to obtain a smooth diffeomorphism Φ satisfying part (b). Let $[\cdot,\cdot]$ denote the Lie bracket of vector fields, so $[X,Y](f) = X(Y(f)) - Y(X(f))$. Standard computations show that, for $1 \le j,k \le n-1$,

$$[L_j, L_k] = 0, \quad [L_j, \bar{L}_k] = b_{jk}T, \quad \text{and} \quad [L_j, T] = c_j T$$

for some smooth functions b_{jk} and c_j. (Note that these are identities; they do not hold merely modulo complex-tangential terms.) Since $L_1, \ldots, L_{n-1}$ span $T^{\mathbf{C}}(\partial D)$ along ∂D, it follows easily that the subbundle $\mathbf{F}$ is involutive in the sense that $[X,Y]$ belongs to $\mathbf{F}$ whenever X and Y belong to $\mathbf{F}$. Thus by the real Frobenius theorem there exists a smooth diffeomorphism Φ on W such that part (b) holds. □

In the following lemma we employ the Proposition to patch local defining functions. Using the notation of the Stratification Theorem, we fix k with $1 \le k \le 2n-3$ and patch along S_k away from the lower-dimensional strata. Here the crucial property of S_k is that $T_p(S_k) \cap N(p) = \{0\}$ for all $p \in S_k$.

PATCHING LEMMA. *Let D be a linearly regular domain with real-analytic boundary in $\mathbf{C}^n$, let V be a neighborhood of $\bigcup_{j=0}^{k-1} S_j$, and put $K = S_k \setminus V$. Suppose that for each $p \in K$ there is a neighborhood U_p of p on which D has a smooth defining function that is plurisubharmonic on $\partial D \cap U_p$. Then there exists a defining function on a neighborhood of K that is plurisubharmonic on the boundary.*

Proof. We will apply the Proposition at each point of K and then use the diffeomorphisms to obtain the desired defining function near K. At the outset we note that the local bundle $\mathbf{E}$ can be globalized over a neighborhood of K since (by part (b) of the Stratification Theorem) the null space has constant dimension along each component of S_k.

First we fix $p \in K$. We will shrink the neighborhood U_p without comment. We apply the proposition to obtain a smooth diffeomorphism Φ on U_p. We want to patch defining functions near p using a function χ defined in terms of this diffeomorphism. Toward this end, assume that $\Phi(p) = 0$ and let ψ be a smooth non-negative function with compact support on a small neighborhood of 0 in $\mathbf{R}^{2m+1}$ so that $\psi \equiv 1$ near 0. Define $\chi = \psi \circ (\phi_1, \ldots, \phi_{2m+1})$, and let $U_p' \subset U_p$ be a small neighborhood of p on which $\chi \equiv 1$. We observe that χ restricted to $K \cap U_p$ has compact support relative to that set. This is so because the Proposition and the condition $T_p(S_k) \cap N(p) = \{0\}$ together imply that the orthogonal projection from $T_p(S_k) \cap T_p^{\mathbf{C}}(\partial D)$ to $\mathbf{E}_p$ is injective.

This choice of χ enables us to control the Levi form in complex-tangential directions. To gain control in the complex-normal direction, we modify the given defining function. Let r be a smooth defining function that is plurisubharmonic on $\partial D \cap U_p$. Define, for $M > 0$ to be chosen, the modified function $\sigma = r + Mr^2$. A simple computation shows that

$$L_q(\sigma, t) = L_q(r, t) + 2M|\partial r_q(t)|^2 \tag{1}$$

if $q \in \partial D \cap U_p$. This equation certainly shows that the smooth local defining function σ is plurisubharmonic on $\partial D \cap U_p$.

Now choose from the collection of all U_p' with $p \in K$ a finite subcover of K, say $\{V_j\}_{j=1}^J$, and for each j let χ_j be the function corresponding to V_j. Let λ be the local defining function $\lambda = \sum_{j=1}^J \chi_j \sigma_j$, where σ_j is the modified defining function for V_j as before. This is well-defined on a thin neighborhood Ω of K by our choice of χ_j.

We claim that λ is plurisubharmonic on the boundary. In order to see this, first compute that for $q \in \partial D \cap \Omega$ and $t \in \mathbf{C}^n$ we have

$$L(\lambda, t) = \sum_{j=1}^{J} \{\chi_j(q) L(\sigma_j, t) + 2\,\mathrm{Re}[\partial \chi_j(t) \overline{\partial \sigma_j(t)}]\}, \tag{2}$$

where, for ease of notation, we have suppressed the subscripts involving q. We now fix ℓ (with $1 \le \ell \le J$) and prove that $L(\lambda, t) \ge 0$ on $\partial D \cap V_\ell$. This will complete the proof. For a given $q \in \partial D \cap V_\ell$ and $t \in \mathbf{C}^n$ we write t' for the orthogonal projection of t onto $\mathbf{E}_q$. It follows from part (a) of the Proposition that there exists a constant $c > 0$ independent of M such that, for all $q \in \partial D \cap V_\ell$ and $t \in \mathbf{C}^n$,

$$L_q(r_\ell, t) \ge c\|t'\|^2;$$

here, of course, r_ℓ is the defining function on which σ_ℓ is based. Using this in (1) yields

$$L_q(\sigma_\ell, t) \ge c\|t'\|^2 + M|\partial r_\ell(t)|^2 \tag{3}$$

for $q \in \partial D \cap V_\ell$ and $t \in \mathbf{C}^n$. Also, by our definition of χ_j and the Proposition there exists a constant $C > 0$ independent of M such that, if $1 \le j \le J$, $q \in \partial D \cap V_\ell$, and $t \in \mathbf{C}^n$, then

$$|\partial \chi_j(t) \partial \sigma_j(t)| \le C(\|t'\||\partial r_\ell(t)| + |\partial r_\ell(t)|^2). \tag{4}$$

We combine (2), (3), (4), and the fact that each σ_j is plurisubharmonic on the boundary to get that, for all $q \in \partial D \cap V_\ell$ and $t \in \mathbf{C}^n$,

$$L(\lambda, t) \ge c\|t'\|^2 + M|\partial r_\ell(t)|^2 - 2CJ(\|t'\||\partial r_\ell(t)| + |\partial r_\ell(t)|^2). \tag{5}$$

Now we use the elementary inequality

$$\alpha\beta \le A\alpha^2 + \frac{1}{4A}\beta^2$$

with $\alpha = \|t'\|$, $\beta = |\partial r_\ell(t)|$, and A small to see that the quantity on the right in (5) is nonnegative if M is sufficiently large. □

4. Proof of the Theorem

In this section we prove our main result.

THEOREM. *Let D be a linearly regular domain with real-analytic boundary in $\mathbf{C}^n$, and suppose that for each $p \in \partial D$ there is a neighborhood U_p of p on which D has a smooth defining function that is plurisubharmonic on $\partial D \cap U_p$. Then D has a global smooth defining function plurisubharmonic on ∂D.*

Proof. We fix a point in $w(\partial D)$ and apply the Stratification Theorem to get a neighborhood U and strata $\{S_j\}_{j=0}^{2n-3}$. We will show how to obtain a defining function on U that is plurisubharmonic on the boundary. The procedure for finding such a function on the entire boundary is similar.

We patch defining functions inductively using the Patching Lemma. Trivially there exists a smooth defining function that is plurisubharmonic on a neighborhood of the finite set S_0. Assume that $k \ge 1$ and that a neighborhood V_k of $\bigcup_{j=0}^{k-1} S_j$ exists on which there is a smooth defining function plurisubharmonic on the boundary. Apply the Patching Lemma to obtain such a function on a neighborhood of $S_k \setminus V_k$. Then patch together these two functions along S_k using the method of proof in the Patching Lemma to obtain a smooth defining function on a neighborhood of $\bigcup_{j=0}^{k} S_j$ that is plurisubharmonic on the boundary. By induction we obtain a smooth defining function on a neighborhood of $w(\partial D)$ in U that is plurisubharmonic on the boundary. Now it is a standard fact that, for any bounded pseudoconvex domain with smooth boundary, there exists a strictly plurisubharmonic defining function away from a neighborhood of $w(\partial D)$; a moment's reflection on equation (1) in the proof of the Patching Lemma suggests the proof of this fact. Given this fact, it is a simple matter to patch these two functions to get one on all of U. □

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