On Boundary Regularity of Analytic Discs

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1. Introduction

In this paper we study the boundary behavior of analytic discs near the zero set of a nonnegative plurisubharmonic function or a totally real submanifold of $\mathbb{C}^n$.

Our main result is the following.

Theorem 1.1. *Let Ω be a complex manifold, ρ a plurisubharmonic function in Ω, and $f: \Delta \to \Omega$ a holomorphic map of the unit disc $\Delta \subset \mathbb{C}$ into Ω such that $\rho \circ f \geq 0$ and $\rho \circ f(\zeta) \to 0$ as $\zeta \in \Delta$ tends to an open arc $\gamma \subset \partial\Delta$. Assume that, for a certain point $a \in \gamma$, the cluster set $C(f, a)$ contains a point $p \in \Omega$ such that ρ is strictly plurisubharmonic in a neighborhood of p. Then f extends to a Hölder $1/2$-continuous mapping in a neighborhood of a on $\Delta \cup \gamma$. If, moreover, $\rho \geq 0$ and the function ρ^θ is plurisubharmonic in a neighborhood of p for some $\theta \in [1/2, 1]$, then f is Hölder $1/2\theta$-continuous (Lipschitz, if $\theta = 1/2$) in a neighborhood of a on $\Delta \cup \gamma$.*

Although this result is new even in the case when the function ρ is of class C^∞, we note that ρ is supposed only to be upper semicontinuous. In what follows we write p.s.h. for plurisubharmonic. A function ρ is called *strictly* p.s.h. in a neighborhood of p with local coordinates z if, for some $\varepsilon > 0$, the function $\rho - \varepsilon|z|^2$ is p.s.h. in a neighborhood of p; ρ is called strictly p.s.h. in Ω if it is strictly p.s.h. at each point of Ω.

It seems that the assertion of Theorem 1.1 is new even in the case when Ω is a domain in the complex plane $\mathbb{C}$ (i.e., f is a usual holomorphic function in Δ). Some comments on the conditions of the theorem may be listed as follows.

(1) The manifold Ω cannot be arbitrary because of the existence condition of the described function ρ. For instance, it implies that all the manifolds $\Omega \cap \{\rho < c\}$, $c > 0$, are hyperbolic at the point p by a theorem of Sibony [14].

(2) It is enough to assume that ρ is p.s.h. in a neighborhood of its zero set only. Then, replacing Ω by this neighborhood and f by $f \circ \phi$ where $\phi: \Delta \to V \cap \Delta$ is a conformal mapping for a suitable neighborhood $V \supset \gamma$, we are in the setting of the theorem.

(3) If f is known to be continuous at the point a, then the situation becomes purely local and we can work with Ω as a domain in $\mathbb{C}^n$. But one of the essential

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difficulties is that $C(f, a)$ can be *a priori* unbounded in Ω, and we can localize the situation at the very end of the proof. Thus, the important special case when Ω is a domain in $\mathbb{C}^n$ is not simpler than the general case.

Nevertheless, a typical situation in the theorem is when Ω is a domain in $\mathbb{C}^n$ and f is a bounded holomorphic map with cluster set contained in the zero set of ρ. In this case we have the following.

COROLLARY 1.2. *Let D be a domain in $\mathbb{C}^n$, ρ a p.s.h. function in D with the zero set $X = \rho^{-1}(0)$, and $f : \Delta \to D^+ := \{\rho \geq 0\}$ a bounded analytic disc such that the cluster set $C(f, \gamma)$ on an open arc $\gamma \subset \partial\Delta$ is contained in X. Assume that, for a certain point $a \in \gamma$, the cluster set $C(f, a)$ contains a point $p \in X$ such that, for some $\varepsilon > 0$, the function $\rho(z) - \varepsilon|z|^2$ is p.s.h. in a neighborhood of p. Then f extends to a Hölder $1/2$-continuous mapping in a neighborhood of a on $\Delta \cup \gamma$. If, moreover, $\rho \geq 0$ and ρ^θ is p.s.h. in a neighborhood of p for some $\theta \in [1/2, 1]$, then f is Hölder $1/2\theta$-continuous in a neighborhood of a on $\Delta \cup \gamma$.*

Indeed, it is sufficient to note that, for any closed subarc $\gamma' \subset \gamma$, the cluster set is a compact set contained in X; since ρ is upper semicontinuous, we obtain that $\rho(z) \to 0$ as $z \to \gamma'$.

We emphasize that there is no assumption of boundedness type in our main theorem.

The regularity of analytic discs was studied by many authors. Our approach is quite elementary and is partially inspired by some ideas of [12] and [14]; it is based on estimates of the Kobayashi–Royden infinitesimal metric in a "tube" neighborhood of a maximal totally real manifold and on the technique of boundary continuous extension of holomorphic mappings between domains in $\mathbb{C}^n$ which we adapt to our case. From this point of view one can consider our main result as an analog of the Forstneric–Rosay theorem [6] on the boundary continuity of holomorphic mappings between strictly pseudoconvex domains in $\mathbb{C}^n$.

In formulating Theorem 1.1 we had in mind two important special cases. The first one concerns analytic discs in the complement of a strictly pseudoconvex domain. In this case we have the following.

COROLLARY 1.3. *Let ρ be a strictly p.s.h. function in Ω, and let $f : \Delta \to \Omega^+ := \{\rho \geq 0\}$ be an analytic disc such that $\rho \circ f(\zeta) \to 0$ as $\zeta \to \gamma$ and the intersection $C(f, a) \cap \Omega$ is not empty for each $a \in \gamma$. Then f extends to a Hölder $1/2$-continuous map on $\Delta \cup \gamma$.*

This result is new even for $\Omega = \mathbb{C}^n$ and $\rho(z) = |z|^2 - 1$, that is, for analytic discs in the exterior of the unit ball in $\mathbb{C}^n$. (Note the deep contrast with the boundary behavior of analytic discs properly embedded in the ball, which can be wild in general.)

The second main case is about the behavior of analytic discs near totally real manifolds and their generalizations, which we introduce as follows. A closed subset X in a complex manifold Ω is called *totally real* if there exists a strictly p.s.h.

nonnegative function ρ in a neighborhood U of X such that $X = U \cap \rho^{-1}(0)$. This definition is justified by the following well-known assertion about strictly p.s.h. functions of class C^2.

(i) Let $X = \rho^{-1}(0)$ be the zero set of a nonnegative strictly p.s.h. function of class C^2 in a complex manifold. Then X is locally contained in a maximal totally real manifold of class C^1.
(ii) Conversely, if M is a totally real submanifold of class C^1 in Ω, then M can be represented as the zero set of a certain C^2 strictly p.s.h. nonnegative function ρ of class C^2 in a neighborhood of M. Moreover, for every $\theta : 1/2 < \theta \leq 1$ there exists a neighborhood U of M in Ω such that ρ^θ is p.s.h. in U.

See [7] for the proof of (i) and [4; 9] for (ii), where the statements are proved for domains in $\mathbb{C}^n$; the general case follows in an obvious way by a partition of unity.

As another corollary of Theorem 1.1, we obtain the following statement.

COROLLARY 1.4. *Let M be a totally real C^1-submanifold of Ω, and let $f : \Delta \to \Omega$ be an analytic disc such that $f(\Delta) \subset\subset \Omega$ and the cluster set $C(f, \gamma)$ is contained in M. Then f is Hölder α-continuous on $\Delta \cup \gamma$ for any $\alpha < 1$.*

In $\mathbb{C}^n$, a somewhat less restrictive condition can be assumed as follows.

COROLLARY 1.5. *Let M be a totally real C^1-submanifold of a domain $\Omega \subset \mathbb{C}^n$, and let $f : \Delta \to \Omega$ be a bounded analytic disc such that $C(f, \gamma)$ is contained in M. Then f is Hölder α-continuous on $\Delta \cup \gamma$ for any $\alpha < 1$.*

Classical examples from the one-variable theory show that this result is precise in terms of Hölder classes (in general, f is not Lipschitz). For the case where M is of smoothness > 1, similar results were obtained in [1; 4].

We note also that our method allows one to control the Hölder constants and to obtain compactness theorems for families of analytic discs. One can control the constants under perturbations of M as well.

If the defining strictly p.s.h. function ρ is not C^2-smooth, then the structure of the totally real set $X : \rho = 0$ can be more complicated. Even for a function ρ with Lipschitz first partial derivatives, the zero set X can have corners. For instance, it was shown in [5] that the union of two real rays in $\mathbb{C}$ issued from the origin is a totally real set iff the angle between them is strictly larger than $\pi/2$ (rays can be replaced by smooth curves). Nevertheless, in these cases also, Theorem 1.1 guarantees the $C^{1/2}$-smoothness up to γ.

If Ω is a Runge domain in $\mathbb{C}^n$ and X is a compact subset of Ω, then the condition $X = \rho^{-1}(0)$ for some nonnegative p.s.h. function in Ω is equivalent to the polynomial convexity of X (see [8]). Hence, the structure of the zero set of a nonnegative p.s.h. function (not necessarily strictly p.s.h.) can be rather complicated, and we need additional assumptions on ρ in order to have the boundary regularity of attached discs.

2. An Estimate of the Kobayashi Metric

In what follows we need estimates of the Kobayashi metric in a "tube" neighborhood of a totally real set. Our approach is based on the technique of Sibony [14], who proved a global assertion that we localize here with uniform estimates. We remark also that our proof is partially inspired by some ideas of [3; 16; 17].

In what follows, $K_D(z, \xi)$ denotes the value of the Kobayashi–Royden infinitesimal metric in a domain $D \subset \Omega$ on the pair (z, ξ), where $z \in D$ and $\xi \in T_z\Omega$. Denote by $\mathbb{B}$ the Euclidean unit ball in $\mathbb{C}^n$. The following estimate of the Kobayashi metric is of crucial importance for our approach.

PROPOSITION 2.1. *Let D be a domain in Ω, let $z\colon U \to 3\mathbb{B}$ be a coordinate neighborhood in Ω with the center at $p \in D$ $(z(p) = 0)$, and let $|\xi|$ be the norm of a vector in $T\Omega|_U$ induced by the Euclidean norm in $\mathbb{C}^n$. Let u be a negative p.s.h. function in D such that the function $u - \varepsilon|z|^2$ is p.s.h. in $D \cap U$ and $|u| \le B$ in $D \cap z^{-1}(2\mathbb{B})$ for some constants $\varepsilon, B > 0$. Then there exists a positive constant $M = M(\varepsilon, B)$ (independent of u) such that*

$$K_D(w, \xi) \ge M|\xi| \cdot |u(w)|^{-1/2}$$

for each $w \in D \cap z^{-1}(\mathbb{B})$ and $\xi \in T_w\Omega$.

The coordinate neighborhood U is not assumed to be contained in D; the main point here is just the behavior of K_D near the boundary of D in Ω. Note also that we do not assume any condition of the boundedness or hyperbolicity type.

Proof. We begin with an estimate of the Kobayashi metric that is not precise enough but does allow us to localize the metric. Let $\psi(x)$ be a smooth nondecreasing function on $\mathbb{R}_+$ such that $\psi(x) = x$ for $0 \le x \le 1/2$ and $\psi(x) = 1$ for $x \ge 3/4$. For any point q with $|z(q)| < 2$, we define the function $\Psi_q = \psi(|z - z(q)|^2)e^{\lambda u}$ in $D \cap U$ and $\Psi_q = e^{\lambda u}$ in $D \setminus U$; the positive constant λ will be chosen later. Then the function $\log \Psi_q = \log \psi(|z - z(q)|^2) + \lambda u$ is p.s.h. in $D \setminus \{|z - z(q)|^2 \le 3/4\}$. There exists a constant $A > 0$ depending only on the function ψ such that the function $\log \psi(|z - z(q)|^2) + A|z|^2$ is p.s.h. in U. On the other hand, it follows by the assumption on u that the function $u - \varepsilon|z|^2$ is p.s.h. on $D \cap \{|z - z(q)| \le 1\}$. Hence, taking $\lambda = A/\varepsilon$ we obtain the function $\log \Psi_q$, which is p.s.h. on $D \cap \{|z - z(q)| \le 1\}$ and therefore everywhere in D.

Let now $g\colon \Delta \to D$ be a holomorphic map such that $g(0) = q \in U$ with $|z(q)| < 2$. Then the function $v(\zeta) = \Psi_q(g(\zeta))/|\zeta|^2$ is defined in the punctured unit disc $\Delta\setminus\{0\}$ and bounded from above by 1 as ζ tends to the unit circle. It is subharmonic on $\Delta\setminus\{0\}$, and $\limsup_{\zeta\to 0} v(\zeta) = |g'(0)|^2 \exp(Au(q)/\varepsilon)$ (as usual, we denote by $g'(0)$ the image $dg_0(\bar{1})$ of the unit vector $\bar{1} = 1$ in $T_0\Delta \simeq \mathbb{C}$). Hence, v is subharmonic in Δ and it follows by the maximum principle that $|g'(0)|^2 \le \exp(-Au(q)/\varepsilon)$. By the definition of the Kobayashi metric it follows that, for any q in $D \cap z^{-1}(2\mathbb{B})$ and ξ in $T_q\Omega$, one has

$$K_D(q,\xi) \geq \exp(Au(q)/2\varepsilon)|\xi| \geq N(\varepsilon,B)|\xi|, \tag{1}$$

where $N = N(\varepsilon,B) = \exp(-AB/2\varepsilon)$.

Let d_D be the Kobayashi pseudodistance in D and let $B_D(q,\delta) = \{z \in D : d_D(q,z) < \delta\}$ be the Kobayashi ball of radius δ in D centered in q. We will use the estimate (1) in order to compare the Kobayashi ball with a suitable Euclidean ball; this allows us to control the distortion of holomorphic discs centered near p.

LEMMA 2.2. *For any point q in $D \cap z^{-1}(\mathbb{B})$ and any $\delta \leq N$, the Kobayashi ball $B_D(q,\delta)$ is contained in $D \cap \{|z - z(q)| < \delta/N\}$.*

Proof. Let w be a point of D and let $\Gamma(q,w)$ be the set of all differentiable paths $\gamma\colon [0,1] \to D$ joining q and w, with $\gamma(0) = q$ and $\gamma(1) = w$. Recall that, by [13], the Kobayashi infinitesimal metric $K_D(w,\xi)$ is upper semicontinuous on the holomorphic tangent bundle of D and

$$d_D(q,w) = \inf_{\gamma\in\Gamma(q,w)} \int_0^1 K_D(\gamma(t),\gamma'(t))\,dt. \tag{2}$$

Setting $G = \{w' \in U : |z(w') - z(q)| < 1\}$, we have (using (1) and (2)):

$$d_D(q,w) \geq \inf_{\gamma\in\Gamma(q,w)} \int_{\gamma^{-1}(G)} K_D(\gamma(t),\gamma'(t))\,dt \geq N \inf_{\gamma\in\Gamma(q,w)} \int_{\gamma^{-1}(G)} |\gamma'(t)|\,dt.$$

For any $\gamma \in \Gamma(q,w)$, the last integral represents the Euclidean length of the part of γ contained in G. Hence, if w is in G then the last inf is not less than $|z(w) - z(q)|$. Indeed, if the path γ is contained in G then its length is $\geq |z(w) - z(q)|$; if γ intersects the boundary of this ball then the length of its connected component joining q and a boundary point of G is $\geq 1 \geq |z(w)-z(q)|$. If w is not in G (in particular, if w is not in U), then the length can be simply estimated from below by 1.

Thus, we have

$$\begin{aligned} d_D(w,q) &\geq N\min\{1,|z(w) - z(q)|\}, \quad w \in D \cap U,\\ \text{and} \qquad d_D(w,q) &\geq N, \quad w \notin U. \end{aligned} \tag{3}$$

In view of (3), the relation $w \in B_D(q,\delta)$ implies that $w \in U$ and $|z(w) - z(q)| < \delta/N$. Note also that, for any $0 < \delta \leq N$, the Kobayashi ball $B_D(q,\delta)$ is nonempty because we have the trivial upper estimate of the Kobayshi distance in D by the Kobayashi distance in a Euclidean ball centered at q and contained in $D \cap U$. □

Now we continue the proof of Proposition 2.1. Let ψ be as before, with $\psi(x) = x$ for $x \leq 1/2$ and $\psi(x) = 1$ for any $x \geq 1$. For $w \in D \cap z^{-1}(\mathbb{B})$ and $\lambda, \beta > 0$ we set $\Phi_{\lambda,\beta,w} = \psi(|z - z(w)|^2/\beta^2)e^{\lambda u}$ in $D \cap U$. The function $\Phi_{\lambda,\beta,w}$ is well-defined in $D \cap U$ and takes its values in $[0,1]$. There exists a constant $C > 0$ depending only on the function ψ such that the function $\log\Phi_{\lambda,\beta,w} + (C/\beta^2 - \lambda\varepsilon)|z|^2$ is p.s.h. in $D \cap U$. Setting $\lambda = 1/|u(w)|$ and $\beta^2 = C|u(w)|/\varepsilon$, we obtain a function Φ_w such that $\log\Phi_w$ is p.s.h. in $D \cap U$.

Set $s = (e^{2N} - 1)/(e^{2N} + 1)$ (so the Poincaré radius of the disc $\{|\zeta| < s\}$ in Δ is equal to N). It follows by Lemma 2.2 that, for any holomorphic mapping $g\colon \Delta \to D$ such that $g(0) = w$ is in $D \cap z^{-1}(\mathbb{B})$, one has the inclusion $g(s\Delta) \subset D \cap z^{-1}(2\mathbb{B})$. Let $f\colon \Delta \to D$ be a holomorphic map with $f(0) = w$ and $f'(0) = \xi/\alpha$ for $\xi \in T_w\Omega$. Then $v(\zeta) = \Phi_w(f(\zeta))/|\zeta|^2$ is a well-defined subharmonic function on $s\Delta\backslash\{0\}$, and $\limsup_{\zeta\to 0} v(\zeta) = \varepsilon|\xi|^2/eC|u(w)|\alpha^2$. Hence, v is subharmonic in $s\Delta$ and the maximum principle gives $\alpha \geq \varepsilon^{1/2}s|\xi|(eC|u(w)|)^{-1/2}$. By the definition of the Kobayashi metric, it follows that

$$K_D(w, \xi) \geq \varepsilon^{1/2}s|\xi|(eC|u(w)|)^{-1/2}.$$

This estimate completes the proof of Proposition 2.1. □

Note the difference of the obtained estimate from standard estimates of the Kobayashi metric near boundary points of pseudoconvex domains. In our case, we do the estimates in interior points but the uniformity of constants allows to “move” a domain; we will use this feature in the next section.

3. Boundary Continuity and Regularity

This section is devoted to the proof of Theorem 1.1, so we assume that we are in the setting of this theorem. We begin with the following lemma, which is well known.

Lemma 3.1. *Let ϕ be a positive subharmonic function in Δ such that $\phi(\zeta) \to 0$ as ζ tends to an arc $\gamma \subset \partial\Delta$. Then, for every compact subset $K \subset \Delta \cup \gamma$, there exists a constant C_K such that $\phi(\zeta) < C_K(1 - |\zeta|)$ for any $\zeta \in K \cap \Delta$.*

Proof. Let V be a neighborhood of $\gamma \cap K$ such that $W = V \cap \Delta$ is simply connected and $\phi < 1$ in W, and let $g\colon \Delta \to W$ be a conformal mapping. Then, by the reflection principle, g^{-1} extends holomorphically across γ. Hence, replacing ϕ by $\phi \circ g$, we reduce the question to the case of a function that is uniformly bounded in Δ. But then the assertion follows by an obvious estimate of the Poisson kernel. □

Fix a constant $\delta > 0$ small enough so that the intersection $\gamma \cap (a + \delta\bar{\Delta})$ is compact in γ; we denote by Ω_δ the intersection $\Delta \cap (a + \delta\Delta)$. By Lemma 3.1, there exists a constant $C > 0$ such that, for any z in Ω_δ, one has

$$\rho \circ f(\zeta) \leq C(1 - |\zeta|). \tag{4}$$

By hypothesis, the function ρ is strictly p.s.h. in a neighborhood of p; hence we can assume there are local coordinates $z\colon U \to 3\mathbb{B}$ centered at p and a constant $\varepsilon > 0$ such that the function $\rho - \varepsilon|z|^2$ is p.s.h. in $D \cap U$.

Lemma 3.2. *There exists a constant $A > 0$ with the following property: If ζ is an arbitrary point of $\Omega_{\delta/2}$ such that $f(\zeta)$ is in $D \cap z^{-1}(\mathbb{B})$, then*

$$|f'(\zeta)| \leq A(1 - |\zeta|)^{-1/2}.$$

Proof. Set $d = 1 - |\zeta|$. Then the disc $\zeta + d\Delta$ is contained in Ω_δ. Define the domain $D_d = \{w \in D : \rho(w) < 2Cd\}$. Then it follows by (4) that the image $f(\zeta + d\Delta)$ is contained in D_d, where the p.s.h. function $u_d(w) = \rho(w) - 2Cd$ is negative. By Proposition 2.1, there exists a constant $M > 0$ (independent of d) such that, for any w in $D \cap z^{-1}(\mathbb{B})$ and any ξ in $T_w\Omega$, one has $K_{D_d}(w, \xi) \geq M|\xi| \cdot |u_d(w)|^{-1/2}$. On another hand, for the Poincaré metric in the disc $\zeta + d\Delta$, we have $K_{\zeta+d\Delta}(\zeta, \tau) = |\tau|/d$ for any τ in $T_\zeta\Delta \sim \mathbb{C}$. By the decreasing property of the Kobayashi metric, for any τ one has

$$M|f'(\zeta)| \cdot |\tau| \cdot |u_d(f(\zeta))|^{-1/2} \leq K_{D_d}(f(\zeta), f'(\zeta)\tau) \leq K_{\zeta+d\Delta}(\zeta, \tau) = |\tau|/d.$$

Therefore, $|f'(\zeta)| \leq M^{-1}|u_d(f(\zeta))|^{1/2}/d$. As $-2Cd \leq u_d(f(\zeta)) < 0$, this implies the desired statement with $A = M^{-1}(2C)^{1/2}$. $\square$

Lemma 3.2 implies that f extends continuously to the point a in view of an integration argument (as in [2]) that is a variation of the classical Hardy–Littlewood theorem.

Indeed, since the cluster set $C(f, a)$ contains p, there exists a sequence of points $a_\nu \in \Delta$ converging to a and such that $f(a_\nu) \to p$. Assume to the contrary that there exists a constant $r > 0$ and a sequence $\{b_\nu\} \subset \Delta$ converging to a such that $d(f(a_\nu), f(b_\nu)) \geq r$ for all ν. (Here $d(\cdot, \cdot)$ is the distance in Ω induced by the metric from Proposition 2.1 which is Euclidean in $U \cap z^{-1}(\mathbb{B})$, where $z\colon U \to 3\mathbb{B}$ is the coordinate neighborhood of p, $z(p) = 0$.) Choose ν so large that $d(f(a_\nu), p) < 1/2$. Consider the piecewise linear path I_ν (oriented from a_ν to b_ν) in Δ formed by three segments: the first one is $[a_\nu, a'_\nu]$, where $a'_\nu \in [0, a_\nu]$ and $|a_\nu - a'_\nu| = |a_\nu - b_\nu|$; the second one is $[a'_\nu, b'_\nu]$, where $b'_\nu \in [0, b_\nu]$ and $|b_\nu - b'_\nu| = |b_\nu - a_\nu|$; and the last one is $[b'_\nu, b_\nu]$. Let $c_\nu \in I_\nu$ be the closest point to a_ν along I_ν such that $d(f(a_\nu), f(c_\nu)) \geq \min(1/2, r)$, and let J_ν be the path in I_ν between a_ν and c_ν. Then $f(J_\nu)$ is contained in $U \cap z^{-1}(\mathbb{B})$. Because the metric in U is Euclidean with respect to the coordinates z, we have $|f'(\zeta)| = |g'(\zeta)| = \left(\sum |g'_j(\zeta)|^2\right)^{1/2}$ for $\zeta \in V = f^{-1}(U)$, where $g = z \circ f\colon V \to \mathbb{C}^n$ and $g = (g_1, \ldots, g_n)$. By Lemma 3.2 and the construction of J_ν, we have $|g'(\zeta)| \leq A(1 - |\zeta|)^{-1/2}$ for all $\zeta \in J_\nu$. Thus, integrating along J_ν, we obtain

$$\begin{aligned}|g(c_\nu) - g(a_\nu)| &\leq A \int_{|a'_\nu|}^{|a_\nu|} \frac{dt}{(1-t)^{1/2}} + A\frac{|a'_\nu - b'_\nu|}{|a_\nu - b_\nu|^{1/2}} + A \int_{|b'_\nu|}^{|b_\nu|} \frac{dt}{(1-t)^{1/2}} \\ &\leq 6A|a_\nu - b_\nu|^{1/2},\end{aligned}$$

a contradiction. Hence, f extends continuously on $\Delta \cup \{a\}$; in particular, there is a neighborhood $V' \ni a$ such that $f(\Delta \cap V') \subset U \cap z^{-1}(\mathbb{B})$.

Choose now $\delta > 0$ so small that the disc $|\zeta - a| < 3\delta$ is contained in V', and set $W = \Delta \cap \{|\zeta - a| < \delta\}$. Then, for arbitrary $\zeta, \eta \in W$, we choose (as before) $\zeta' \in [0, \zeta]$ and $\eta' \in [0, \eta]$ such that $|\zeta - \zeta'| = |\eta - \eta'| = |\zeta - \eta|$; we denote by I the path $[\zeta, \zeta'] \cup [\zeta', \eta'] \cup [\eta', \eta]$. Since $I \subset W$, we have $|g'(\tau)| \leq A(1 - |\tau|)^{-1/2}$ on I. Hence, integrating along I, we obtain as before that $d(f(\zeta), f(\eta)) = |g(\zeta) - g(\eta)| \leq 6A|\zeta - \eta|^{1/2}$. It follows that f extends to a Hölder $1/2$-continuous map on $\Delta \cup \{|\zeta| = 1,\ |\zeta - a| < \delta\}$. This completes the proof of the first part of Theorem 1.1.

For the proof of the regularity part we can assume f to be continuous on $\Delta \cup \gamma$; we then need show only that the $1/2\theta$-Hölderness if ρ^θ is still plurisubharmonic. The composition $\rho^\theta \circ f$ is defined in a neighborhood V (depending on θ) of the arc γ in Δ. Replacing (if necessary) f by the composition with a biholomorphic mapping between $V \cap \Delta$ and Δ, we can assume without loss of generality that the composition $\rho^\theta \circ f$ is defined in Δ. Applying Lemma 3.1 to the function ρ^θ we obtain that, for any ζ in Δ, one has $\rho \circ f(\zeta) \le C(1 - |\zeta|)^{1/\theta}$, where the positive constant C depends on θ.

Now it remains to repeat the former argument. Let ζ be a point in Δ (sufficiently close to a) and let $d = 1 - |\zeta|$. Then the image $f(\zeta + d\Delta)$ is contained in the domain $D_d = \{\, w \in D : u_d(w) \equiv \rho(w) - 2Cd^{1/\theta} < 0 \,\}$. Repeating the proof of Lemma 3.2, we obtain that $|f'(\zeta)| \le M^{-1}|u_d(f(\zeta))|^{1/2}/d$. As $-2Cd^{1/\theta} \le u_d(f(\zeta)) < 0$, this implies the estimate $|f'(\zeta)| \le A(1 - |\zeta|)^{1/2\theta - 1}$ in a neighborhood of a in Δ; hence, f is Hölder $1/2\theta$-continuous on $\Delta \cup \gamma$ near a by the same integration argument as before.

This completes the proof of the theorem. □

In conclusion we would like to indicate two possible applications of our results.

(1) Using Corollary 1.5, we derive that the area of an analytic disc $f(\Delta)$ attached to a C^1 smooth totally real manifold M is finite (for other proofs, see [4; 15]). Moreover, the area of $f(\{1 - \delta < |\zeta| < 1\})$ is estimated by $C(\varepsilon)\delta^{1-\varepsilon}$ when $\delta \to 0$ for arbitrary $\varepsilon > 0$.

(2) Corollary 1.5 and Lempert's theory [10; 11] imply that any extremal disc for the Kobayashi metric of a strongly convex domain with C^2 boundary is Hölder α-continuous up to the boundary for every $\alpha < 1$. (Lempert established the Hölder $1/2$-continuity.)

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