Componentwise Linear Ideals and Golod Rings

J. Herzog, V. Reiner, & V. Welker

Dedicated to Jack Eagon on the occasion of his 65th birthday

1. Introduction

Let $A = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and let $R = A/I$ be the quotient of A by an ideal $I \subset A$ that is homogeneous with respect to the standard grading in which $\deg(x_i) = 1$. When I is generated by square-free monomials, it is traditional to associate with it a certain simplicial complex Δ, for which $I = I_\Delta$ is the *Stanley–Reisner ideal* of Δ and $R = K[\Delta] = A/I_\Delta$ is the *Stanley–Reisner ring* or *face ring.* The definition of Δ as a simplicial complex on vertex set $[n] := \{1, 2, \dots, n\}$ is straightforward: the minimal non-faces of Δ are defined to be the supports of the minimal square-free monomial generators of I.

Many of the ring-theoretic properties of I_Δ then translate into combinatorial and topological properties of Δ (see [14, Chap. II]). In particular, a celebrated formula of Hochster [14, Thm. II.4.8] describes $\mathrm{Tor}_\cdot^A(R, K)$ in terms of the homology of the full subcomplexes of Δ. Here K is considered the trivial A-module $K = A/\mathfrak{m}$ for $\mathfrak{m} = (x_1, \dots, x_n)$. It is well known that the dimensions of these K-vector spaces $\mathrm{Tor}_\cdot^A(R, K)$ give the ranks of the resolvents in the finite minimal free resolution of R as an A-module.

In a series of recent papers, beginning with [8] and subsequently [9; 15; 13], it has been recognized that, for square-free monomial ideals $I = I_\Delta$, there is another simplicial complex Δ^* which can be even more convenient for understanding free A-resolutions of R. The complex Δ^*, which from now on we will call the *Eagon complex* of $I = I_\Delta$, carries equivalent information to Δ and is, in a certain sense, its *canonical Alexander dual*:

$$\Delta^* := \{ F \subseteq [n] : [n] - F \notin \Delta \}.$$

The crucial property of Δ^* that makes it convenient for the study of $\mathrm{Tor}_\cdot^A(R, K)$ is that, instead of the full subcomplexes of Δ that are relevant in Hochster's formula, the relevant subcomplexes of Δ^* are the *links* of its faces. Therefore, various hypotheses on Δ^* which are inherited by the links of faces, or which control the topology of these links, lead to strong consequences for $\mathrm{Tor}_\cdot^A(R, K)$ (see Section 3).

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Our motivation was to understand whether hypotheses on Δ^* might also lead to good consequences for the *infinite* resolution of K as a trivial R-module—that is, for $\mathrm{Tor}_{\cdot}^{R}(K, K)$. There are relatively few classes of rings where one can compute $\mathrm{Tor}_{\cdot}^{R}(K, K)$ (see [12]); however, there is a particularly nice class called *Golod rings* where $\mathrm{Tor}_{\cdot}^{R}(K, K)$ is determined by $\mathrm{Tor}_{\cdot}^{A}(R, K)$ in a simple fashion. Our goal then is to show that, under reasonably simple hypotheses on Δ^*, the ring $R = A/I_\Delta$ is Golod.

It is known [3] that if a homogeneous ideal I has linear resolution as an A-module (defined in the next section), then $R = A/I$ is Golod. Herzog and Hibi [13] generalized the notion of linear resolution to that of *componentwise linearity,* and our main result (Theorem 4) states that, when I is a componentwise linear ideal, the ring $R = A/I$ is Golod. We also observe (Theorem 9) that, for square-free monomial ideals $I = I_\Delta$, componentwise linearity is equivalent to the Eagon complex Δ^* being *sequentially Cohen–Macaulay over* K, a notion introduced by Stanley [14, Sec. III.2.9]. Checking whether Δ^* is sequentially Cohen–Macaulay over K is relatively easy, and sequentially Cohen–Macaulay-ness is implied for all fields K by the hypothesis that Δ is *shellable* in the nonpure sense defined by Björner and Wachs [5]. Thus, shellability is a simple condition on the Eagon complex Δ^* implying that both $\mathrm{Tor}_{\cdot}^{A}(R, K)$ and $\mathrm{Tor}_{\cdot}^{R}(K, K)$ are easy to compute and independent of the field K.

The paper is structured as follows. Section 2 reviews the notions of Golod rings and componentwise linearity and also proves Theorem 4. Then Section 3 gives a "dictionary" summarizing how various conditions on a square-free monomial ideal I_Δ translate into conditions on the Eagon complex Δ^*, including the observation (Theorem 9) that componentwise linearity of I_Δ corresponds to sequentially Cohen–Macaulay-ness of Δ^*.

2. Componentwise Linear Resolution and Golodness

As in the introduction, let $A = K[x_1, \dots, x_n]$, let I be a homogeneous ideal in A, and let $R = A/I$. Any finitely generated graded A-module M has a finite minimal free resolution

$$0 \to A^{\beta_h} \to \cdots \to A^{\beta_1} \to A^{\beta_0} \to M \to 0, \tag{2.1}$$

in which the maps can be made homogeneous by shifting the degrees of the various A-basis elements in the free modules A^{β_i}. It is well known that the number of A-basis elements of A^{β_i} having degree j is the dimension of the jth graded piece $\mathrm{Tor}_i^A(M, K)_j$ of the graded K-vector space $\mathrm{Tor}_i^A(M, K)$.

We say that M *has linear resolution* if all nonzero entries in the matrices $\partial_i : A^{\beta_i} \to A^{\beta_{i-1}}$ for $i \geq 1$ are linear forms in A. It is not hard to see that M has linear resolution if and only if M has a minimal generating set all of the same degree t, and that $\mathrm{Tor}_i^A(M, K)_{i+j} = 0$ for $j \neq t$.

In [13], the authors defined the notion of componentwise linear homogeneous ideals as follows. Given a homogeneous ideal I in A, let $I_{\langle k \rangle}$ denote the ideal generated by all homogeneous polynomials of degree k in I, and let $I_{\leq k}$ denote the ideal generated by the homogeneous polynomials of degree at most k in I. We say that I is *componentwise linear* if $I_{\langle k\rangle}$ has linear resolution for all k. In [13] it is observed that *stable* monomial ideals [10] are componentwise linear, as are ideals that are *Gotzmann* in the sense that every $I_{\langle k\rangle}$ is a Gotzmann ideal.

We next wish to relate componentwise linearity to the (infinite) minimal free resolution of K as an R-module and $\mathrm{Tor}^R_{\cdot}(K, K)$. The *Poincaré series* relevant for the finite and infinite resolutions are defined as follows:

$$\mathrm{Poin}^{\mathrm{fin}}(R, t, x) := \sum_{i,j\geq 0} \dim_K \mathrm{Tor}_i^A(R, K)_j t^i x^j,$$

$$\mathrm{Poin}^{\mathrm{inf}}(R, t, x) := \sum_{i,j\geq 0} \dim_K \mathrm{Tor}_i^R(K, K)_j t^i x^j.$$

In the late 1950s (see [12]), Serre proved by means of a spectral sequence that

$$\mathrm{Poin}^{\mathrm{inf}}(R, t, x) \leq \frac{(1+tx)^n}{1 - t\,\mathrm{Poin}^{\mathrm{fin}}(R, t, x)}, \tag{2.2}$$

where "$\leq$" is used here to mean coefficientwise comparison of power series in t, x. Subsequently, Eagon (see [12, Chap. 4.2.4]) and Golod [11] independently gave a very concrete proof of this result by constructing a certain free (but not necessarily minimal) resolution of K as an R module that interprets the right-hand side of equation (2.2). This Eagon-Golod construction:

(a) starts with the Koszul resolution $\mathbb{K}^A$ for K as an A-module;
(b) tensors with R to obtain a Koszul complex $K^A \otimes R$ whose homology computes $\mathrm{Tor}^A_{\cdot}(K, R) \cong \mathrm{Tor}^A_{\cdot}(R, K)$;
(c) "kills" the homology of the complex $K^A \otimes R$ in a certain fashion to obtain a free R-resolution of K.

Furthermore, Golod [11] was able to characterize equality in Serre's result (2.2) (or, equivalently, characterize minimality in the Eagon–Golod resolution) by the vanishing of all *Massey operations* in the Koszul complex $K \otimes R$ considered as a *differential graded algebra* (DGA). When this vanishing occurs we say that R is *Golod* or the ideal I is Golod, where $R = A/I$. We refer the reader to [12] for full definitions and discussion of Massey operations, but emphasize here the properties that we will use as follows.

(i) The Massey operation $\mu(z_1, \ldots, z_r)$, which is defined only for certain r-tuples $z_1, \ldots, z_r$ of cycles in a DGA $\mathcal{A}$, is a cycle in $\mathcal{A}$. It is defined using the DGA structure, and its homology class depends only upon the homology classes of $z_1, \ldots, z_r$.
(ii) If z_s has homological degree i_s and degree ℓ_s with respect to some extra grading preserved by the multiplication in $\mathcal{A}$, then $\mu(z_1, \ldots, z_r)$ will have homological degree $r - 2 + \sum_s i_s$ and degree $\sum_s \ell_s$ with respect to the extra grading.

We now wish to prove our main result, beginning with two lemmas. Recall that $\mathfrak{m} = (x_1, \ldots, x_n)$ denotes the irrelevant ideal in A.

LEMMA 1. *If I has linear resolution then $\mathfrak{m}I$ also has linear resolution.*

Proof. Assume that I has linear resolution and is generated in degree t, so that $\operatorname{Tor}_i^A(I,K)_j = 0$ for $j > i+t$. The short exact sequence of A-modules

$$0 \to \mathfrak{m}I \to I \to I/\mathfrak{m}I \to 0$$

gives rise to a long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}^A(I/\mathfrak{m}I, K) \to \operatorname{Tor}_i^A(\mathfrak{m}I, K) \to \operatorname{Tor}_i^A(I,K) \to \cdots.$$

Note that $I/\mathfrak{m}I$ is also generated in degree t and isomorphic to a direct sum $I/\mathfrak{m}I \cong \bigoplus_{m=1}^g K(-t)$, where $K(-t)$ denotes the trivial A-module structure on K with generator in degree t and where g is the number of minimal generators of I. Therefore, the minimal free A-resolution of $I/\mathfrak{m}I$ is a direct sum of Koszul resolutions for K, each shifted by degree t. Since Koszul resolutions are linear, $\operatorname{Tor}_i^A(I/\mathfrak{m}I, K)_{i+j} = 0$ for $j \neq t$. It then follows from the displayed portion of the long exact sequence that $\operatorname{Tor}_i^A(\mathfrak{m}I, K)_{i+j} = 0$ for $j \neq t+1$, which means that $\mathfrak{m}I$ has linear resolution since it is generated in degree $t+1$. □

REMARK 2. Note that the only property of the polynomial ring A (and its maximal homogeneous ideal $\mathfrak{m}$) used in the preceding lemma is that the field $K = A/\mathfrak{m}$ has a linear minimal free A-resolution—that is, that A is a *Koszul* ring (see e.g. [3]). Thus the lemma remains valid in all Koszul rings.

LEMMA 3. *If I is componentwise linear then $I_{\leq k}$ is componentwise linear for all k.*

Proof. This follows directly from the definition of componentwise linearity and the previous lemma, since

$$(I_{\leq k})_{\langle j\rangle} = \begin{cases} I_{\langle j\rangle} & \text{for } j \leq k, \\ \mathfrak{m}^{j-k} I_{\langle k\rangle} & \text{for } j > k. \end{cases}$$ □

THEOREM 4. *If I is componentwise linear and contains no linear forms, then I is Golod.*

Proof. Let I be a componentwise linear ideal, with t and T the minimum and maximum degrees of a minimal generator for I. We will prove that I is Golod by induction on the difference $T-t$.

The base case where $t = T$ requires us to show that an ideal I having linear resolution and generated in degree $t \geq 2$ is Golod. This is well known [3], but we include the proof for completeness. We must show that the Massey operations in the Koszul complex $\mathbb{K}^A \otimes R$ that computes $\operatorname{Tor}^A(K,R)$ all vanish. Given $z_1, \dots, z_r \in \mathbb{K}^A \otimes R$ with z_s an i_s-cycle of degree $i_s + j_s$, we may assume without loss of generality that $j_s = t-1$; otherwise,

$$\operatorname{Tor}_{i_s}^A(K,R)_{i_s+j_s} \cong \operatorname{Tor}_{i_s}^A(R,K)_{i_s+j_s} \cong \operatorname{Tor}_{i_s-1}^A(I,K)_{i_s-1+(j_s+1)} = 0$$

by the linearity of the resolution. Therefore, the Massey operation $\mu(z_1, \dots, z_r)$, when it is defined, will be represented by an i-cycle with $i := r-2+\sum_s i_s$ having degree $\sum_s i_s + j_s = i + (2-r) + r(t-1)$. Hence, this Massey operation represents a class in $\operatorname{Tor}_i^A(R,K)_{i+j}$ with

$$j = r(t-2) + 2.$$

By linearity of the resolution, it will vanish unless $j = t - 1$, which one can check is equivalent to $t = 1 + \frac{r-1}{r-2} < 2$. Since I has no linear forms the latter cannot happen, and the Massey operation vanishes.

We now proceed to the inductive step, assuming the result for componentwise linear ideals with $T - t$ smaller. Consider the ideal $J = I_{\le T-1}$, which is componentwise linear by Lemma 3 and hence Golod by induction. If we let $R' := A/J$, then note that the natural surjection $\phi\colon R' \to R$ induces a k-vector space isomorphism $R'_j \to R_j$ in the range $0 \le j \le T-1$. It also induces a surjection of differential graded algebras $\phi_\sharp\colon \mathbb{K}^A \otimes R' \to \mathbb{K}^A \otimes R$, which gives an isomorphism

$$(\mathbb{K}^A \otimes R')_{i+j} \cong (\mathbb{K}^A \otimes R)_{i+j}$$

for $0 \le j \le T-1$ and hence induces an isomorphism

$$\phi_* : \operatorname{Tor}_i^A(K, R')_{i+j} \cong \operatorname{Tor}_i^A(K, R)_{i+j} \tag{2.3}$$

for $0 \le j \le T-2$.

With this information, we can now proceed to show that all Massey operations in $\mathbb{K}^A \otimes R$ vanish. Given $z_1, \dots, z_r \in \mathbb{K}^A \otimes R$ with z_s an i_s-cycle of degree $i_s + j_s$, we have two cases.

Case 1: Each $j_s \le T-2$. In this case we are in the range of the isomorphism ϕ_* for each i_s, j_s. Setting $z'_s = \phi_*^{-1}(z_s) \in \mathbb{K}^A \otimes R'$ for each s, the Massey operation $\mu(z'_1, \dots, z'_s)$, when it is defined, must vanish in $\operatorname{Tor}_\cdot^A(K, R')$ since R' is Golod. Let $c \in \mathbb{K}^A \otimes R'$ be a chain with $\partial c = \mu(z'_1, \dots, z'_s)$. Because $\phi_\sharp$ is a differential graded algebra map, we may conclude that $\partial\phi_\sharp(c) = \mu(z_1, \dots, z_s)$ and hence the Massey operation $\mu(z_1, \dots, z_s)$ vanishes as desired.

Case 2: Some $j_s \ge T-1$. Without loss of generality, say that $j_1 \ge T-1$. Since I contains no linear forms, we have $j_s \ge 1$ for all s and hence the Massey operation $\mu(z_1, \dots, z_s)$, when defined, represents a class in $\operatorname{Tor}_i^A(R, K)_{i+j}$ with

$$\begin{aligned} j &= 2 - r + \sum_s j_s \\ &\ge 2 - r + (T-1) + (r-1)\cdot 1 \\ &\ge T. \end{aligned}$$

However, according to [13, Prop. 1.3], the componentwise linearity of I implies $\operatorname{Tor}_i(K, R)_{i+j} = 0$ for $j \ge T$. Therefore, the Massey operation again vanishes. □

REMARK 5. The converse to Theorem 4 is already false for square-free monomial ideals I generated in a single degree. We have the following more general fact.

PROPOSITION 6. *Let I_Δ be a square-free monomial ideal in $A = K[x_1, \dots, x_n]$ containing no linear forms, and assume that the Eagon complex Δ^* has no two faces F, F' with $F \cup F' = [n]$. Then I_Δ is Golod.*

Proof. When I is a monomial ideal, there is a fine $\mathbb{N}^n$-grading by monomials carried by A, I, $R = A/I$, and $\mathrm{Tor}^A_{\cdot}(K, R)$. According to Hochster's formula [14, Thm. II.4.8], $\mathrm{Tor}^A_{\cdot}(K, R)$ vanishes except in square-free multidegrees. On the other hand, if $\mu(z_1, \dots, z_r)$ is a Massey product of some nonzero cycles in $K^A \otimes R$, then each z_i lives in a multidegree that divides $(x_1 \cdots x_n)/x^{F_i}$ for some face F_i of Δ^*. Since no F_i, F_j satisfy $F_i \cup F_j = [n]$, we conclude that no product of these multidegrees can be square-free, so $\mu(z_1, \dots, z_r)$ must vanish. □

This provides many examples of Golod square-free monomial ideals I_Δ—for example, whenever Δ^* has dimension less than $n/2 - 1$. By Theorem 9, one need only construct a pure but non-Cohen–Macaulay complex Δ^* of low dimension (such as the graph on six vertices consisting of three disjoint edges) in order to obtain a counterexample I_Δ to the converse of Theorem 4.

3. An Eagon Complex Dictionary

In this section we collect some recent and new results on properties of a square-free monomial ideal $I = I_\Delta$ in $A = K[x_1, \dots, x_n]$ that can be phrased conveniently in terms of the Eagon complex Δ^*. The first result appeared as [9, Thm. 3].

THEOREM 7. *I_Δ has linear resolution if and only if Δ^* is Cohen–Macaulay over K.*

We wish to discuss two generalizations of Theorem 7. The first is a beautiful result of Terai [15], related to the notion of *Castelnuovo–Mumford regularity*. Recall that the *regularity* of a graded A-module M is defined by

$$\operatorname{reg} M := \max\{\, j : \mathrm{Tor}_i(M, K)_{i+j} \neq 0 \,\},$$

and its *initial degree* is defined by

$$\operatorname{indeg} M := \min\{\, j : M_j \neq 0 \,\} = \min\{\, j : \mathrm{Tor}_0(M, K)_j \neq 0 \,\}.$$

It is clear that $\operatorname{reg} M \geq \operatorname{indeg} M$, with equality if and only if M has linear resolution.

THEOREM 8.

$$\operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta = \dim K[\Delta^*] - \operatorname{depth}_A K[\Delta^*],$$

where dim *denotes Krull dimension and* $\operatorname{depth}_A M$ *denotes depth of M as an A-module* (*i.e., the length of the longest M-regular sequence in A*).

Our second generalization of Theorem 7 is a new observation linking componentwise linearity for square-free monomial ideals to the notion of sequential Cohen–Macaulay-ness, whose definition we recall from [14, Def. 2.9]. A module graded M over a graded ring R is said to be *sequentially Cohen–Macaulay* if it has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ of graded submodules satisfying two conditions:

(i) each quotient M_i/M_{i-1} is a Cohen–Macaulay R-module;
(ii) $\dim M_1/M_0 < \dim M_2/M_1 < \cdots < \dim M_r/M_{r-1}$, where "dim" denotes Krull dimension.

We say that a simplicial complex Δ is *sequentially Cohen–Macaulay over K* if its Stanley–Reisner ring $K[\Delta] = A/I_\Delta$ is sequentially Cohen–Macaulay. For a simplicial complex Δ and some $k \geq 0$, we denote by $\Delta(k)$ the simplicial complex generated by the k-dimensional faces of Δ.

THEOREM 9. *Let Δ be a simplicial complex. Then I_Δ is componentwise linear over K if and only if its Eagon complex Δ^* is sequentially Cohen–Macaulay over K.*

Proof. Theorem 2.1 of [13] characterizes componentwise linear square-free monomial ideals I_Δ as those for which the pure simplicial complex $\Delta^*(k)$ is a Cohen–Macaulay complex for every k. On the other hand, in Theorem 3.3 of [7], the complex $\Delta^*(k)$ is denoted $\Delta^{*[k]}$ and is called the *pure k-skeleton*; it is proven there that Δ^* is sequentially Cohen–Macaulay over K if and only if $\Delta^{*[k]}$ is Cohen–Macaulay for every k. The theorem follows. □

Theorems 8 and 9 show that the duality operation $I \mapsto I^*$ defined on square-free monomial ideals in A by $I_\Delta \mapsto I_{\Delta^*}$ has two amazing properties:

(i) $\operatorname{reg}(I) - \operatorname{indeg}(I) = \dim A/I^* - \operatorname{depth}_A A/I^*$;
(ii) I is componentwise linear if and only if A/I^* is sequentially Cohen–Macaulay.

QUESTION 10. Can this operation be extended to a natural duality $I \mapsto I^*$ with similar properties for more general ideals $I \subseteq A$, or for some class of A-modules M?

Theorem 9 provides a wealth of new examples of componentwise linear square-free monomial ideals. For example, Δ^* is sequentially Cohen–Macaulay over all fields K (and hence I_Δ for all fields K) whenever Δ^* is *shellable* in the nonpure sense of Björner and Wachs [5]. Recall that shellability of Δ^* means there is an ordering $F_1, F_2, \ldots$ of the facets of Δ^* with the property that, for any $j > 1$, the closure of the facet F_j intersects the subcomplex generated by the previous facets $F_1, F_2, \ldots, F_{j-1}$ in a subcomplex that is pure of codimension 1 inside F_j.

We next discuss another pleasant feature related to Theorem 9: When I_Δ is componentwise linear, the multigraded Betti numbers of I_Δ appearing in the minimal free resolution turn out to encode the exact same information as what Björner and Wachs call the *f-triangle* (or *h-triangle*) of the sequentially Cohen–Macaulay complex Δ^*. For a simplicial complex Δ, define (as in [5]) the f-triangle $(f_{ij})_{i \geq j}$ and the h-triangle $(h_{ij})_{i \geq j}$ as follows:

(a) f_{ij} = number of faces of Δ of dimension j that are contained in a face of dimension j but in no face of higher dimension;
(b) $h_{ij} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{ik}$.

It is shown in [5, Thm. 3.6] that the h-triangle of a shellable complex is nonnegative and may be interpreted in terms of the shelling order. For a simplicial complex

Δ and $k \geq 0$ we write $\Delta(k)'$ for the k-skeleton of $\Delta(k+1)$. Because $\Delta(k)$ and $\Delta(k)'$ are pure complexes, their h-triangles degenerate to the usual h-vector $h_j = h_{kj}$. We may write $h_i(\Delta)$ (resp. $h_{ij}(\Delta)$) to indicate which simplicial complex is meant when discussing the h-vector (resp. h-triangle) if this is not clear from the context; we use analogous conventions for the f-vector (resp. f-triangle).

LEMMA 11. *For all $i \geq j$ we have*

$$h_{ij}(\Delta) = h_j(\Delta(i)) - h_j(\Delta(i)').$$

Proof. By definition of the h-triangle, we have

$$h_j(\Delta(i)) - h_j(\Delta(i)') = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} (f_{ik}(\Delta(i)) - f_{ik}(\Delta(i)')).$$

Clearly,

$$f_{ik}(\Delta) = f_{ik}(\Delta(i)) - f_{ik}(\Delta(i)').$$

Again by the definition of the h-triangle, the assertion follows. □

The difference $h_j(\Delta(i)) - h_j(\Delta(i)')$ was first considered in [13]. There it is shown that, for componentwise linear I_Δ, this difference is nonnegative for the complex Δ^*. Thus, for sequentially Cohen–Macaulay complexes Δ, Theorem 9 and Lemma 11 imply that the h-triangle is nonnegative (a fact first discovered in [7, Thm. 5.1]). Furthermore, it is shown [13, Thm. 2.1(b)] that, for componentwise linear I_Δ and $j \geq 1$,

$$\sum_{i \geq 0} \dim_K \operatorname{Tor}_i^A(I_\Delta, K)_{i+j} t^i = \sum_{i \geq 0} (h_{n-j-1}(\Delta^*(i)) - h_{n-j-1}(\Delta^*(i)'))(t+1)^i.$$

Again by Theorem 9 and Lemma 11, this yields the following result.

PROPOSITION 12. *Let I_Δ be componentwise linear or (equivalently) let Δ^* be sequentially Cohen–Macaulay over K. Then*

$$\sum_{i \geq 0} \dim_K \operatorname{Tor}_i^A(I_\Delta, K)_{i+j} t^i = \sum_{i \geq 0} h_{i,n-j-1}(\Delta^*)(t+1)^i.$$

In particular, the f-triangle and h-triangle encode the same information as the multigraded Betti numbers $\dim_K \operatorname{Tor}_i^A(I_\Delta, K)_{i+j}$.

The remaining properties of square-free monomial ideals that we will discuss relate to stability properties of the monomials with respect to linear orderings of the variables $x_1, x_2, \dots, x_n$, or equivalently of the set of indices $[n] := \{1, 2, \dots, n\}$. Given a square-free monomial m in A, define its *support* as

$$\operatorname{supp}(m) := \{\, i \in [n] : m \text{ is divisible by } x_i \,\}.$$

and let $\max(m)$ be the maximum element of $\operatorname{supp}(m)$. By identifying a square-free monomial with its support, we will intentionally make no distinction between subsets of $[n]$ and square-free monomials in A. Given a linear ordering Λ, define the *lexicographic* order induced by Λ on k-subsets as follows: $S <_{\text{lex}} T$ if S contains the Λ-smallest element of the symmetric difference $S\Delta T := (S - T) \cup (T - S)$. Define the *colexicographic* order by $S <_{\text{colex}} T$ if T contains the Λ-largest element of $S\Delta T$. In the remaining definitions it will be assumed that a fixed linear ordering Λ on $[n]$ has been chosen.

A square-free monomial ideal I is *square-free lexsegment* if the square-free monomials in I of degree k form an initial segment in the lexicographic order on k-subsets of $[n]$. Equivalently, I is square-free lexsegment if, for every minimal generator m of I and every square-free monomial $m' <_{\text{lex}} m$, one has $m' \in I$.

A square-free monomial ideal I is *square-free 0-Borel fixed* [2] if, for every minimal generator m of I and for $j \notin \text{supp}(m)$ and $i \in \text{supp}(m)$ with $j < i$, one has $(x_j/x_i)m \in I$. A square-free monomial ideal I is *square-free stable* [2] if, for every minimal generator m of I and for $j \notin \text{supp}(m)$ with $j < \max(m)$, one has $(x_j/x_{\max(m)})m \in I$. A square-free monomial ideal I is *square-free weakly stable* [1] if, for every minimal generator m of I and for $j \notin \text{supp}(m)$ with $j < \max(\text{supp}(m) - \{\max(m)\})$, there exists $i \in \text{supp}(m)$ such that $i > j$ and $(x_j/x_i)m \in I$.

It is easy to see that, for a square-free monomial ideal I,

square-free lexsegment $\Rightarrow$

square-free 0-Borel fixed $\Rightarrow$

square-free stable $\Rightarrow$

square-free weakly stable.

We wish to relate these to some combinatorial notions about simplicial complexes. Given a linear order Λ on $[n]$, a simplicial complex on a vertex set $[n]$ is said to be *compressed* if, for each i, its faces of dimension i form an initial segment in colex order induced on the $(i + 1)$ subsets of $[n]$. A simplicial complex is *shifted* if, whenever F forms a face and $i \notin F$ but $j <_{\Lambda} i \in F$, one has that $(F - \{i\}) \cup \{j\}$ also forms a face. Given a simplicial complex Δ and face F of Δ, its *link, star* and *deletion* in Δ are defined as follows

$$\text{star}_{\Delta} F := \{ G \in \Delta : G \cup F \in \Delta \};$$

$$\text{link}_{\Delta} F := \{ G \in \Delta : G \cup F \in \Delta, G \cap F = \emptyset \};$$

$$\text{del}_{\Delta} F := \{ G \in \Delta : G \cap F = \emptyset \}.$$

A simplicial complex Δ is a *near-cone over the vertex* $v \in [n]$ if every face F has the property that $F - \{i\}$ lies in $\text{star}_{\Delta} v$ for every $i \in F$. Equivalently, one must check that this properties holds on the maximal faces F of Δ.

A simplicial complex Δ is called *vertex-decomposable* if either (a) Δ is a simplex or $\Delta = \{\emptyset\}$ or (b) there is a vertex v such that $\text{link}_{\Delta}(v)$ and $\text{del}_{\Delta}(v)$ are vertex decomposable and no facet of $\text{link}_{\Delta}(v)$ is a facet of $\text{del}_{\Delta}(v)$. In this case, the vertex v is called a *shedding vertex*, and the sequence of shedding vertices that are deleted in reducing Δ to a simplex or empty face is called a *shedding sequence.*

It is not hard to check (or see [6] for proofs of some of these implications) that, for a simplicial complex Δ on vertex set $[n]$, compressed $\Rightarrow$

shifted $\Rightarrow$

every face $F \in \Delta$ has $\text{link}_\Delta F$ a near-cone on the vertex $\min([n] - F) \Rightarrow$

shellable.

It is also shown in [6, Sec. 11] that Δ being shifted implies that it is vertex-decomposable with shedding order $n, n-1, \dots$. Furthermore, it is shown there that vertex decomposability implies both the lexicographic and colexicographic orders induced from the shedding order given by shelling orders on the facets of Δ.

We can now relate the stability properties of I_Δ to combinatorial properties of the Eagon complex Δ^*.

PROPOSITION 13. *I_Δ is a square-free lexsegment ideal with respect to $x_1 < \cdots < x_n$ if and only if Δ^* is compressed with respect to $n <_\Lambda \cdots <_\Lambda 1$.*

Proof. The definition of Δ^* states that F is a face of Δ^* if and only if $[n] - F$ is the support of a monomial in I_Δ. Therefore, the crucial point (which is easy to check) is that $S <_{\text{lex}} S'$ in the lexicographic order on subsets induced from $1 < \cdots < n$ if and only if $[n] - S <_{\text{colex}} [n] - S'$ in the colexicographic order on subsets induced by $n <_\Lambda \cdots <_\Lambda 1$. □

PROPOSITION 14. *I_Δ is square-free 0-Borel-fixed with respect to $x_1 < \cdots < x_n$ if and only if Δ^* is shifted with respect to $n <_\Lambda \cdots <_\Lambda 1$.*

Proof. Similarly straightforward; the crucial point is that

$$F = \text{supp}(m) \quad \text{and} \quad F' = \text{supp}\left(\frac{x_j}{x_i} m\right) \text{ with } j < i$$

if and only if

$$[n] - F' = ([n] - F') - \{j\}) \cup \{i\} \text{ with } i <_\Lambda j. \qquad \square$$

PROPOSITION 15. *I_Δ is square-free stable with respect to $x_1 < \cdots < x_n$ if and only if Δ^* has $\text{link}_{\Delta^*} F$ a near-cone over $\max([n] - F)$ for each face $F \in \Delta^*$.*

Proof. We begin by proving the forward implication. Assume I_Δ is square-free stable with respect to $x_1 < \cdots < x_n$. This translates into the condition on Δ^* that, for every maximal face F and $j \in F$ with $j < \max([n] - F)$, one has

$$(F - \{j\}) \cup \{\max([n] - F)\} \in \Delta^*.$$

Given any face F of Δ^*, a maximal face G of $\text{link}_{\Delta^*} F$, and $j \in G$, we must now show that $G - \{j\}$ is in $\text{star}_{\text{link}_{\Delta^*} F}(\max([n] - F))$. In other words, we must show that $G - \{j\}$ lies in some face of $\text{link}_{\Delta^*} F$ containing $i := \max([n] - F)$. If $i \in G$, then we are done since G is such a face. If $i \notin G$, then $i = \max([n] - (F \cup G))$. Therefore, since $j \in F \cup G$ and $F \cup G$ is a maximal face of Δ^* (note that G is a maximal face of $\mathrm{link}_{\Delta^*} F$), we conclude from stability that $F \cup (G - \{j\})$ lies in some face F' of Δ^* containing i. Hence $G - \{j\}$ lies in the face $F' - F$ of $\mathrm{link}_{\Delta^*} F$ that contains i, as desired.

For the backward implication, assume $\mathrm{link}_{\Delta^*} F$ a near-cone over $\max([n] - F)$ for each face $F \in \Delta^*$. We need to show that, for every maximal face G and $j \in G$ with $j < \max([n] - G)$, one has $G - \{j\} \cup \{\max([n] - F)\} \in \Delta$. To see this, use the fact that $\mathrm{link}_{\Delta^*}(G - \{j\})$ is a near-cone over the vertex $\max([n] - (G \cup \{j\})) = \max([n] - G) =: i$. Because G is a maximal face of Δ^*, we have that $\{j\}$ is a maximal face of $\mathrm{link}_{\Delta^*}(G - \{j\})$; hence i must also be a face of $\mathrm{link}_{\Delta^*}(G - \{j\})$, since i is the near-cone vertex. Thus $(G - \{j\}) \cup \{i\}$ is a face of Δ^*, as desired. □

Finally, we deal with square-free weakly stable ideals I_Δ.

THEOREM 16. *If I_Δ is square-free weakly stable with respect to $x_1 < \cdots < x_n$, then Δ^* is vertex decomposable with shedding order* $1, 2, \ldots$. *Consequently, Δ^* is shellable and hence I_Δ is componentwise linear independent of the field K.*

Proof. Assume that I_Δ is square-free weakly stable. This translates into the following condition.

($*$) For every maximal face F of Δ^* and $j \in F$ with

$$j < \max\big(([n] - F) - \{\max([n] - F)\}\big),$$

there exists $i \in [n] - F$ such that $i > j$ and $(F - \{j\}) \cup \{i\} \in \Delta^*$.

We will show that a simplicial complex satisfying ($*$) is vertex-decomposable. Clearly, if Δ^* satisfies ($*$) then so do $\mathrm{link}_{\Delta^*}(1)$ and $\mathrm{del}_{\Delta^*}(1)$ as simplicial complexes on the ground set $[n] - \{1\}$. Let F be a facet of $\mathrm{link}_{\Delta^*}(1)$. If $F = [n] - \{1\}$ then Δ^* is the full simplex and so is clearly vertex-decomposable. If $F \neq [n] - \{1\}$ then condition ($*$) is satisfied in Δ^* for $j = 1$ and $F \cup \{1\}$. Hence there is an $i > 1$ such that $F \cup \{i\}$ in Δ^* and F is not a facet of $\mathrm{del}_{\Delta^*}(1)$. □

The implication "I square-free weakly stable implies I componentwise linear for all fields K" from the previous theorem can also be deduced algebraically, but first we require the following result, which characterizes componentwise linear ideals in terms of regularity.

THEOREM 17. *Let I be a monomial ideal. Then I is componentwise linear over K if and only if* $\mathrm{reg}(I_{\le k}) \le k$ *for all k.*

Proof. First, assume that I_Δ is componentwise linear. Fix some k and set $I = I_\Delta$. Since $I_{\le k}$ is componentwise linear, [13, Prop. 1.3] implies

$$\dim_K \mathrm{Tor}_i^A(I, K)_{i+j} = \dim_K \mathrm{Tor}_i^A((I_{\le k})_{\langle j\rangle}, K) - \dim_K \mathrm{Tor}_i^A(\mathfrak{m}(I_{\le k})_{\langle j-1\rangle}, K).$$

If $j > k$, then $(I_{\le k})_{\langle j\rangle} = \mathfrak{m}(I_{\le k})_{\langle j-1\rangle}$. Therefore, $\mathrm{Tor}_i^A(I_{\le k}, K)_j = 0$ for $j > k$. This implies $\mathrm{reg}(I_{\le k}) \le k$.

Now assume that $\mathrm{reg}(I_{\le k}) \le k$ for all k. We show by induction on j that $I_{\langle j\rangle}$ has a linear resolution. From the exact sequence

$$0 \to I_{\le j-1} \to I_{\le j} \to I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle} \to 0$$

we obtain an exact sequence

$$\cdots \to \mathrm{Tor}_i^A(I_{\le j}, K)_{i+r} \to \mathrm{Tor}_i^A(I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle}, K)_{i+r}$$
$$\to \mathrm{Tor}_{i-1}^A(I_{\le j-1}, K)_{i+r} \to \cdots.$$

By hypothesis, the $\mathrm{Tor}_i^A(\cdot, K)_{i+r}$ of $I_{\le j}$ and $I_{\le j-1}$ vanish for $r > j$. Hence,

$$\mathrm{Tor}_i^A(I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle}, K)_{i+j} = 0$$

for $r > j$. Now consider the exact sequence

$$0 \to \mathfrak{m}I_{\langle j-1 \rangle} \to I_{\langle j \rangle} \to I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle} \to 0$$

and the associated long exact sequence

$$\cdot \to \mathrm{Tor}_i^A(\mathfrak{m}I_{\langle j \rangle}, K)_{i+r} \to \mathrm{Tor}_{i-1}^A(I_{\langle j-1 \rangle}, K)_{i+r}$$
$$\to \mathrm{Tor}_i^A(I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle}, K)_{i+r} \to \cdot.$$

By induction, hypothesis $I_{\langle j-1 \rangle}$ has a linear resolution. Then, by Lemma 3, $\mathfrak{m}I_{\langle j-1 \rangle}$ has a linear resolution. Therefore, for $r > j$,

$$\mathrm{Tor}_i(\mathfrak{m}I_{\langle j-1 \rangle}, K)_{i+r} = \mathrm{Tor}_i^A(I_{\langle j \rangle}/\mathfrak{m}I_{\langle j-1 \rangle})_{i+r} = 0.$$

It follows that $\mathrm{Tor}_i^A(I_{\langle j \rangle}, K)_{i+r} = 0$ for $r > j$. For trivial reasons,

$$\mathrm{Tor}_i(I_{\langle j \rangle}, K)_{i+r} = 0 \quad \text{for } r < j.$$

We conclude that $I_{\langle j \rangle}$ has a linear resolution. □

With this result, the proof that a weakly stable ideal I is componentwise linear for all fields K is as follows. First, by Theorem 17, I is componentwise linear if and only if $\mathrm{reg}(I_{\le k}) \le k$ for all k. On the other hand, I weakly stable implies $\mathrm{reg}(I)$ is the same as the degree of a maximal generator for I by [1, Thm. 1.4], and I weakly stable trivially implies $I_{\le k}$ is weakly stable for all k. Hence it implies I is componentwise linear.

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J. Herzog
FB 6, Mathematik und Informatik
Universität GH-Essen
Essen 45117
Germany

juergen.herzog@uni-essen.de

V. Reiner
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

reiner@math.umn.edu

V. Welker
Fachbereich Mathematik
Philipps-Universität Marburg
35032 Marburg
Germany

welker@mathematik.uni-marburg.de