Commuting Toeplitz Operators on the Harmonic Bergman Space

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1. Introduction

For $p \geq 1$, we let $L^p = L^p(D, A)$ denote the usual Lebesgue space of the open unit disk D in the complex plane. Here, the letter A denotes the normalized area measure on D. The harmonic Bergman space b^2 is the subspace of the Lebesgue space L^2 consisting of all complex-valued L^2-harmonic functions on D. One can check the relation $b^2 = L_a^2 + \overline{L_a^2}$, where L_a^2 denotes the holomorphic Bergman space consisting of all L^2-holomorphic functions on D. As is well known, the harmonic Bergman space b^2 is a closed subspace of L^2 and hence is a Hilbert space. We will write Q for the Hilbert space orthogonal projection from L^2 onto b^2. Each point evaluation is easily verified to be a bounded linear functional on b^2. Hence, for each $z \in D$, there exists a unique function R_z—called the harmonic Bergman kernel—in b^2 that has the following reproducing property:

$$u(z) = \langle u, R_z \rangle \tag{1}$$

for every $u \in b^2$. Here and elsewhere, the notation $\langle \cdot, \cdot \rangle$ denotes the usual inner product in L^2. Since $b^2 = L_a^2 + \overline{L_a^2}$, there is a simple relation between the harmonic Bergman kernel R_z and the well-known (holomorphic) Bergman kernel K_z: $R_z = K_z + \overline{K_z} - 1$. Thus, the explicit formula of R_z is given by

$$R_z(w) = \frac{1}{(1 - w\bar{z})^2} + \frac{1}{(1 - \bar{w}z)^2} - 1 \quad (w \in D). \tag{2}$$

The formulas (1) and (2) lead us to the following integral representation of the projection Q:

$$Q\varphi(z) = \int_D \left(\frac{1}{(1 - z\bar{w})^2} + \frac{1}{(1 - \bar{z}w)^2} - 1 \right) \varphi(w)\, dA(w) \quad (z \in D) \tag{3}$$

for functions $\varphi \in L^2$. See [ABR, Chap. 8] for more information and related facts.

For $u \in L^2$, the *Toeplitz operator* T_u with *symbol* u is defined by

$$T_u f = Q(uf)$$

for functions $f \in b^2$. The operator T_u is densely defined and not bounded in general.

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Here, we are concerned with the characterizing problem of harmonic symbols of commuting Toeplitz operators acting on b^2. For the holomorphic Bergman space, the corresponding problem, as well as its essential version, has been studied by several authors [AČ; CL1; CL2; L1; L2; Z] and has been completely resolved in case of holomorphic or harmonic symbols. The present harmonic case was first studied by Ohno, who observed [O, Thm. 2.2] that, for a holomorphic symbol f on D, T_f commutes with T_z if and only if f is a polynomial of degree at most 1. In this paper we obtain complete descriptions for certain types of harmonic symbols of commuting Toeplitz operators. All the results we obtained show that Toeplitz operators can commute only in the obvious cases. However, we do not know whether the same is true for general harmonic symbols.

In Section 2 we characterize holomorphic symbols of commuting Toeplitz operators. Our result (Theorem 5) is:

> *For $f, g \in L_a^2$, T_f and T_g commute on b^2 if and only if a nontrivial linear combination of f and g is constant on D.*

For this result, we consider more general symbols and derive a necessary condition for those symbols to induce commuting Toeplitz operators (Theorem 4):

> *Let $u, v \in b^2$, and suppose that T_u and T_v commute on b^2. If u and v are both not antiholomorphic then there exists a constant α such that $\partial v = \alpha(\partial u)$, where $\partial = \partial/\partial z$.*

This result also plays the key role in proving our results in the next section.

In Section 3 we consider two special types of symbols and prove characterizations for those symbols. We first consider a pair of symbols related to each other by complex conjugation and give a characterization for those symbols. This might be of some independent interest, because they are just symbols of normal Toeplitz operators. Recall that a bounded linear operator on a Hilbert space is called *normal* if it commutes with its adjoint operator. Our characterization shows that only obvious ones are normal. Then we consider a pair of symbols in case one of them is a (harmonic) polynomial. Our results are as follows.

> *For $u \in b^2$, T_u is normal on b^2 if and only if $u(D)$ is contained in a straight line* (Theorem 8).
>
> *Let u be a (harmonic) polynomial. For $v \in b^2$, T_u and T_v commute on b^2 if and only if a nontrivial linear combination of u and v is constant on D* (Theorem 10).

2. Holomorphic Symbols

In this section we give a characterization of holomorphic symbols of commuting Toeplitz operators (Theorem 5). For that purpose, we consider somewhat more general symbols having nonconstant holomorphic parts and derive a necessary condition (Theorem 4). This necessary condition is also the key to the proofs of results in the next section.

Before proceeding, let us recall the well-known Bergman projection P. For each $z \in D$, the explicit formula for the Bergman kernel K_z is given by

$$K_z(w) = \frac{1}{(1 - w\bar{z})^2} \quad (w \in D),$$

and the Bergman projection P is the integral operator

$$Pu(z) = \langle u, K_z \rangle = \int_D \frac{u(w)}{(1 - z\bar{w})^2} \, dA(w),$$

taking L^1-functions u into the space of all holomorphic functions. As is well known (see e.g. [Zh, Chap. 4]), the Bergman projection P, when restricted to L^2, is the Hilbert space orthogonal projection from L^2 onto L^2_a. Moreover, P has the reproducing property

$$Pf = f \quad \text{and} \quad P\bar{f} = \overline{f(0)} \tag{4}$$

for all holomorphic L^1-functions f.

We start with some simple properties of the Bergman projection that will be useful in the proofs.

LEMMA 1. *Let $f \in L^2_a$. Then the following statements hold for all $z \in D$.*

(a) $\frac{d}{dz}\{z^2 P[(1 - |w|^2)f]\} = zf(z)$.

(b) $\frac{d}{dz}\{z^2 P[(1 - |w|^2)\bar{f}]\} = z\overline{f(0)}$.

(c) $P(|w|^2 f)(z) = f(z) - \frac{1}{z^2}\int_0^z \zeta f(\zeta)\, d\zeta$.

(d) $P(|w|^2 \bar{f}) = \overline{P(|w|^2 f)(0)} = \frac{1}{2}\overline{f(0)}$.

Proof. For $\psi = f$ or $\psi = \bar{f}$, let $g = P[(1 - |w|^2)\psi]$. That is,

$$g(z) = \int_D \frac{\psi(w)}{(1 - z\bar{w})^2}(1 - |w|^2)\, dA(w) \quad (z \in D).$$

Then, by differentiating under the integral sign, we have

$$g'(z) = 2\int_D \frac{\psi(w)\bar{w}}{(1 - z\bar{w})^3}(1 - |w|^2)\, dA(w),$$

so that

$$\begin{aligned} \frac{d}{dz}\{z^2 g(z)\} &= 2zg(z) + z^2 g'(z) \\ &= 2z\int_D \frac{\psi(w)(1 - |w|^2)}{(1 - z\bar{w})^3}\, dA(w) \end{aligned}$$

for all $z \in D$. Now, (a) and (b) follow from Theorem 1 of [C].

Note from (4) that

$$z^2 P[(1 - |w|^2)f](z) = z^2 f(z) - z^2 P(|w|^2 f)(z)$$

for all $z \in D$. Thus, integrating both sides of (a), we have

$$z^2 f(z) - z^2 P(|w|^2 f)(z) = \int_0^z \zeta f(\zeta)\, d\zeta \quad (z \in D),$$

which implies (c). Also, by (4) we have

$$z^2P[(1-|w|^2)\bar f] = z^2P(\bar f) - z^2P(|w|^2\bar f) = z^2\overline{f(0)} - z^2P(|w|^2\bar f);$$

thus, integrating both sides of (b) shows that

$$P(|w|^2\bar f) = \tfrac{1}{2}\overline{f(0)}.$$

Since

$$P(|w|^2 f)(0) = \int_D f(w)|w|^2\,dA(w) = f(0)\int_D |w|^2\,dA(w) = \frac{1}{2}f(0),$$

we also have

$$\overline{P(|w|^2 f)(0)} = \tfrac{1}{2}\overline{f(0)}.$$

This proves (d), completing the proof of Lemma 1. □

The next two lemmas will much simplify our arguments in the proofs to follow.

LEMMA 2. *Let $f \in L_a^2$ and suppose $f(0) = 0$. Then the following statements hold for all $z \in D$.*

(a) $P(\bar w f)(z) = \frac{1}{z}f(z) - \frac{1}{z^2}\int_0^z f(\zeta)\,d\zeta$.
(b) $P(w\bar f)(z) = \overline{P(\bar w f)(0)} = \frac{1}{2}\overline{f'(0)}$.

Proof. Since $f(0) = 0$, there is a holomorphic function g on D such that $f(z) = zg(z)$. One can easily check that g is in L_a^2. Now, by Lemma 1, we have

$$P(\bar w f)(z) = P(|w|^2 g)(z) = g(z) - \frac{1}{z^2}\int_0^z \zeta g(\zeta)\,d\zeta \quad (z \in D),$$

which gives (a). Similarly, we have

$$P(w\bar f) = P(|w|^2\bar g) = \overline{P(|w|^2 g)(0)} = \tfrac{1}{2}\overline{g(0)}.$$

Thus (b) holds, and the proof is complete. □

LEMMA 3. *Let $f, g \in L_a^2$. Then the following statements hold.*

(a) $P\big(\bar f P(\bar w g)\big)(z) = P(\bar f \bar w g)(z)$ *for each* $z \in D$.
(b) $P\big(f\overline{P(\bar w g)}\big)(0) = P(fw\bar g)(0)$.

Proof. Since the Bergman projection P is the orthogonal projection from L^2 onto L_a^2, we see from (4) that

$$\begin{aligned} P\big(\bar f P(\bar w g)\big)(z) &= \langle \bar f P(\bar w g), K_z\rangle = \langle P(\bar w g), fK_z\rangle \\ &= \langle \bar w g, fK_z\rangle = P(\bar f \bar w g)(z) \end{aligned}$$

for each $z \in D$, so we have (a). By a similar argument, one can also see that

$$P\big(f\overline{P(\bar w g)}\big)(0) = \int_D f\overline{P(\bar w g)}\,dA = \langle f, P(\bar w g)\rangle = \langle f, \bar w g\rangle = P(fw\bar g)(0),$$

so (b) holds. The proof is complete. □

Because

$$P\varphi(0)=\int_D \varphi\,dA$$

for functions $\varphi\in L^2$, we see from (3) that the projection Q can be rewritten as

$$Q\varphi = P(\varphi)+\overline{P(\bar\varphi)}-P(\varphi)(0) \tag{5}$$

for functions $\varphi\in L^2$. Now, we prove the following.

THEOREM 4. *Let $u, v\in b^2$, and suppose that T_u and T_v commute on b^2. If ∂u and ∂v are both not identically zero, then there exists a constant α such that $\partial v=\alpha(\partial u)$.*

For holomorphic functions f and g, we will use the fact that $f+\bar g\in b^2$ implies $f, g\in L^2_a$. A proof is given here for the reader's convenience. Assume $g(0)=0$ for simplicity. Put $u=f+\bar g$ and let $u_r(z)=u(rz)$ for $z\in D$ and $0<r<1$. By (4), we have $P(u_r)=f_r$. Since P is bounded on L^2, taking the limit $r\to 1$, we have $P(u)=f\in L^2_a$ and thus $g=\bar u-\bar f\in L^2_a$. In fact, a little bit more is true: If f and g are holomorphic functions such that $f+\bar g\in L^p$ for $p\geq 1$, then $f, g\in L^p$. See the proof of Theorem 7.1.5 of [R].

Proof. By the foregoing remark, there are functions f, g, h, k in L^2_a such that $u=f+\bar g$ and $v=h+\bar k$. Without loss of generality, we may assume $f(0)=g(0)=h(0)=k(0)=0$. By assumption, f and h are nonconstant. We need to show $h=\alpha f$ for some constant α.

By (5) and Lemma 2, we have

$$\begin{aligned}T_h(\bar w)&=Q(\bar w h)\\&=P(\bar w h)+\overline{P(w\bar h)}-P(\bar w h)(0)\\&=P(\bar w h)+P(\bar w h)(0)-P(\bar w h)(0)\\&=P(\bar w h)\end{aligned} \tag{6}$$

and hence

$$T_fT_h(\bar w)=fP(\bar w h).$$

Also, since $Q(\bar w\bar k)=\bar w\bar k$, we have

$$\begin{aligned}T_fT_{\bar k}(\bar w)&=Q[fQ(\bar w\bar k)]\\&=Q(f\bar w\bar k)\\&=P(f\bar w\bar k)+\overline{P(\bar f w k)}-P(f\bar w\bar k)(0).\end{aligned}$$

Next, by (6) and Lemma 3, we have

$$\begin{aligned}T_{\bar g}T_h(\bar w)&=Q[\bar g P(\bar w h)]\\&=P[\bar g P(\bar w h)]+\overline{P[g\overline{P(\bar w h)}]}-P[\bar g P(\bar w h)](0)\\&=P(\bar g\bar w h)+\overline{P[g\overline{P(\bar w h)}]}-P(\bar g\bar w h)(0);\end{aligned}$$

finally note that

$$T_{\bar{g}}T_{\bar{k}}(\bar{w}) = \bar{g}\bar{w}\bar{k}.$$

It follows that

$$\begin{aligned} T_{f+\bar{g}}T_{h+\bar{k}}(\bar{w}) &= T_f T_h(\bar{w}) + T_f T_{\bar{k}}(\bar{w}) + T_{\bar{g}}T_h(\bar{w}) + T_{\bar{g}}T_{\bar{k}}(\bar{w}) \\ &= fP(\bar{w}h) + P(f\bar{w}\bar{k}) + \overline{P(\bar{f}wk)} - P(f\bar{w}\bar{k})(0) \\ &\quad + P(\bar{g}\bar{w}h) + \overline{P[g\overline{P(\bar{w}h)}]} - P(\bar{g}\bar{w}h)(0) + \bar{g}\bar{w}\bar{k}. \end{aligned}$$

By exactly the same way,

$$\begin{aligned} T_{h+\bar{k}}T_{f+\bar{g}}(\bar{w}) &= hP(\bar{w}f) + P(h\bar{w}\bar{g}) + \overline{P(\bar{h}wg)} - P(h\bar{w}\bar{g})(0) \\ &\quad + P(\bar{k}\bar{w}f) + \overline{P[k\overline{P(\bar{w}f)}]} - P(\bar{k}\bar{w}f)(0) + \bar{k}\bar{w}\bar{g}. \end{aligned}$$

Because $T_{f+\bar{g}}T_{h+\bar{k}} = T_{h+\bar{k}}T_{f+\bar{g}}$ on b^2 (by assumption), we have $T_{f+\bar{g}}T_{h+\bar{k}}(\bar{w}) = T_{h+\bar{k}}T_{f+\bar{g}}(\bar{w})$ in particular and so

$$fP(\bar{w}h) + \overline{P(\bar{f}wk)} + \overline{P[g\overline{P(\bar{w}h)}]} = hP(\bar{w}f) + \overline{P(\bar{h}wg)} + \overline{P[k\overline{P(\bar{w}f)}]}.$$

Recall that $f(0) = h(0) = 0$. Hence, taking the holomorphic part on both sides of the preceding equation, we obtain

$$fP(\bar{w}h) = hP(\bar{w}f). \tag{7}$$

It follows from Lemma 2 that

$$f(z)\left\{\frac{1}{z}h(z) - \frac{1}{z^2}\int_0^z h(\zeta)\,d\zeta\right\} = h(z)\left\{\frac{1}{z}f(z) - \frac{1}{z^2}\int_0^z f(\zeta)\,d\zeta\right\}$$

for all $z \in D$. Consequently, letting

$$H(z) = \int_0^z h(\zeta)\,d\zeta \quad \text{and} \quad F(z) = \int_0^z f(\zeta)\,d\zeta,$$

we have $fH = hF$ on D. Hence $F'H = H'F$ because $H' = h$ and $F' = f$. Now, since f and h are nonconstant, we see that $H = \alpha F$ for some constant α. Consequently, we have $h = \alpha f$ for some constant α, as desired. This completes the proof. □

As an immediate consequence of Theorem 4, we obtain a complete description of holomorphic symbols of commuting Toeplitz operators.

THEOREM 5. *Let $f, g \in L^2_a$ be nonconstant functions. Then $T_f T_g = T_g T_f$ on b^2 if and only if $g = \alpha f + \beta$ for some constants α and β.*

Proof. This is immediate from Theorem 4. □

3. Two Special Cases

In this section we give characterizations for a pair of symbols of special type to induce commuting Toeplitz operators. We consider two types. One is a pair of symbols related to each other by complex conjugation. The other is a pair of symbols for which at least one is a polynomial. Our results (Theorem 8 and Theorem 10) show that such symbols must be related in an obvious way, as expected.

We begin with an integral identity taken from [C].

LEMMA 6. *For $f, g \in L^2_a$, we have*

$$\int_D f(w)\overline{g(w)}\,dA(w) = \int_D f(w)[2\overline{g(w)} + \overline{wg'(w)}](1-|w|^2)\,dA(w).$$

Proof. See Theorem 12 of [C], where the lemma is stated for slightly different pairs of f and g. □

The following lemma will be useful for our purposes.

LEMMA 7. *Let $f, g \in L^2_a$ and assume $f(0) = g(0) = 0$. If T_f and $T_{\bar g}$ commute on b^2, then*

$$\int_D f(w)\overline{G(w)w^k}(1-|w|^2)\,dA(w) = 0 \quad (k = 0, 1, 2, \ldots),$$

where

$$G(w) = \frac{1}{w}\int_0^w g(\zeta)\,d\zeta.$$

Proof. Since $T_f T_{\bar g} = T_{\bar g} T_f$, we have $T_g T_{\bar f} = T_{\bar f} T_g$ by taking adjoints. In particular, we have $T_g T_{\bar f}(\bar w) = T_{\bar f} T_g(\bar w)$. Now, repeating exactly the same argument as in the proof of Theorem 4 yields

$$P[f\overline{P(g\bar w)}] = P(fw\bar g),$$

so that

$$\langle f, P(g\bar w)w^{k+1}\rangle = \langle P[f\overline{P(g\bar w)}], w^{k+1}\rangle = \langle P(fw\bar g), w^{k+1}\rangle = \langle fw, gw^{k+1}\rangle$$

for all $k \geq 0$. Rearranging this expression by using Lemma 2, we have

$$\int_D f(w)\overline{g(w)w^k}(1-|w|^2)\,dA(w) = \int_D f(w)\overline{G(w)w^k}\,dA(w). \tag{8}$$

On the other hand, by Lemma 6,

$$\begin{aligned}\int_D & f(w)\overline{G(w)w^k}\,dA(w) \\ &= \int_D f(w)[\overline{G'(w)w + (k+2)G(w)}]\bar w^k(1-|w|^2)\,dA(w) \\ &= \int_D f(w)[\overline{g(w) + (k+1)G(w)}]\bar w^k(1-|w|^2)\,dA(w);\end{aligned}$$

thus, by (8),

$$\int_D f(w)\overline{G(w)w^k}(1-|w|^2)\,dA(w) = 0$$

for all $k \geq 0$. The proof is complete. □

We turn to characterizing the harmonic symbols of normal Toeplitz operators. This will show that only harmonic symbols of normal Toeplitz operators are obvious ones.

THEOREM 8. *Let $u \in b^2$. Then T_u is normal on b^2 if and only if $u(D)$ is contained in a straight line. In particular, for $f \in L^2_a$, T_f is normal on b^2 if and only if f is constant.*

Proof. Throughout the proof we assume $u(0) = 0$ without loss of generality. Suppose $u(D)$ is contained in a straight line. Then there exists a holomorphic function $f \in L^2$ such that $u = \alpha(f + \bar{f})$ for some constant α. Now one can easily check that T_u is normal.

Conversely, assume T_u is normal. First consider the case when u is holomorphic and use the temporary notation $u = f$. Since the adjoint operator of Toeplitz operator T_f is $T_{\bar{f}}$, we have $T_f T_{\bar{f}} = T_{\bar{f}} T_f$. Thus, setting

$$F(w) = \frac{1}{w} \int_0^w f(\zeta)\, d\zeta,$$

we see from Lemma 7 that

$$\begin{aligned} 0 &= \int_D f(w) \overline{F(w)} (1 - |w|^2)\, dA(w) \\ &= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \overline{\frac{F^{(n)}(0)}{n!}} \int_D |w|^{2n} (1 - |w|^2)\, dA(w). \end{aligned} \tag{9}$$

Note that

$$F^{(n)}(0) = \frac{f^{(n)}(0)}{n+1} \quad (n = 1, 2, \ldots).$$

Insert these into (9) to obtain

$$\sum_{n=1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \cdot \frac{1}{n+1} \int_D |w|^{2n} (1 - |w|^2)\, dA(w) = 0,$$

which means that $f^{(n)}(0) = 0$ for all $n \geq 1$. Thus f is identically 0, as desired.

Now, consider general $u = f + \bar{g}$, where $f, g \in L^2_a$ are nonconstant functions with $f(0) = g(0) = 0$. Then, by Theorem 4, there is a constant α such that $g = \alpha f$. Now, normality of T_u is equivalent to

$$(1 - |\alpha|^2)(T_f T_{\bar{f}} - T_{\bar{f}} T_f) = 0.$$

Since f is nonconstant, T_f is not normal by the previous case; thus, the preceding equality yields $|\alpha| = 1$. Consequently, $\bar{u} = \alpha u$ so that $\sqrt{\alpha} u$ is real-valued. This completes the proof. □

Before characterizing harmonic symbols of commuting Toeplitz operators (having one or more polynomial symbols), we first prove the following special case.

LEMMA 9. *Let $f, g \in L^2_a$ and suppose one of them is a polynomial. If T_f and $T_{\bar{g}}$ commute on b^2, then either f or g is constant.*

Proof. We assume that both f and g are nonconstant and proceed to derive a contradiction. Note that the adjoint operators $T_{\bar{f}}$ and $T_{\bar{g}}$ also commute. Thus, without loss of generality, we may assume that f is a polynomial of degree $n \geq 1$. We may further assume $f(0) = g(0) = 0$. Since g is nonconstant, there is a function $h \in L^2_a$, $h(0) \neq 0$, such that $g = w^k h$ for some positive integer k.

Consider the case $k \leq n$. Denote G for the function associated with g as in Lemma 7. It then follows from the lemma that

$$\begin{aligned} 0 &= \int_D f(w)\overline{G(w)w^{n-k}}(1-|w|^2)\,dA(w) \\ &= \frac{f^{(n)}(0)}{n!}\cdot\frac{\overline{G^{(k)}(0)}}{k!}\int_D |w|^{2n}(1-|w|^2)\,dA(w), \end{aligned}$$

so that

$$h(0) = \frac{g^{(k)}(0)}{k!} = (k+1)\cdot\frac{G^{(k)}(0)}{k!} = 0,$$

a contradiction.

Consider the case $k \geq n+1$. Since T_f and $T_{\bar{g}}$ commute, we have $T_f T_{\bar{g}}(\bar{w}^{n+1}) = T_{\bar{g}} T_f(\bar{w}^{n+1})$. Before calculating these, note that

$$P(\bar{w}^i w^j) = \begin{cases} 0 & \text{for } i > j, \\ \frac{j+1-i}{j+1} w^{j-i} & \text{for } i \leq j, \end{cases} \tag{10}$$

where i, j are nonnegative integers. This is easily verified by a straightforward calculation. Since f is a polynomial of degree n, it follows from (10) that

$$P(f\bar{g}\bar{w}^{n+1}) = P(f\bar{w}^{n+1}) = 0.$$

Now, as in the proof of Theorem 4, we have

$$\begin{aligned} T_f T_{\bar{g}}(\bar{w}^{n+1}) &= P(f\bar{g}\bar{w}^{n+1}) + \overline{P(\bar{f}gw^{n+1})} - P(f\bar{g}\bar{w}^{n+1})(0) \\ &= \overline{P(\bar{f}gw^{n+1})} \end{aligned}$$

and

$$\begin{aligned} T_{\bar{g}} T_f(\bar{w}^{n+1}) &= T_{\bar{g}} Q(f\bar{w}^{n+1}) \\ &= T_{\bar{g}}[P(f\bar{w}^{n+1}) + \overline{P(\bar{f}w^{n+1})} - P(f\bar{w}^{n+1})(0)] \\ &= T_{\bar{g}}[\overline{P(\bar{f}w^{n+1})}] \\ &= \overline{gP(\bar{f}w^{n+1})}, \end{aligned}$$

so that

$$P(\bar{f}gw^{n+1}) = gP(\bar{f}w^{n+1}).$$

In particular, we have

$$\langle P(\bar{f}gw^{n+1}), w^{k+1}\rangle = \langle gP(\bar{f}w^{n+1}), w^{k+1}\rangle.$$

On one hand, since fw^{k+1} is a polynomial of degree $n+k+1$, we see that

$$\begin{aligned}\langle P(\bar{f}gw^{n+1}), w^{k+1}\rangle &= \langle \bar{f}gw^{n+1}, w^{k+1}\rangle \\ &= \langle hw^{n+k+1}, fw^{k+1}\rangle \\ &= h(0)\cdot \frac{\overline{f^{(n)}(0)}}{n!}\int_D |w|^{2(n+k+1)}\,dA(w) \\ &= h(0)\cdot \frac{\overline{f^{(n)}(0)}}{n!(n+k+2)}.\end{aligned}$$

On the other hand, since

$$\begin{aligned}P(\bar{g}w^{k+1}) &= P(\bar{h}\bar{w}^k w^{k+1}) \\ &= \overline{h(0)}P(\bar{w}^k w^{k+1}) + \overline{h'(0)}P(|w|^{2(k+1)}) \\ &= \frac{1}{k+2}[2\overline{h(0)}w + \overline{h'(0)}],\end{aligned}$$

by (10) and

$$\langle \bar{f}w^{n+1}, 1\rangle = \langle w^{n+1}, f\rangle = 0,$$

$$\langle \bar{f}w^{n+1}, w\rangle = \langle w^{n+1}, fw\rangle = \frac{\overline{f^{(n)}(0)}}{n!(n+2)}$$

it follows that

$$\begin{aligned}\langle gP(\bar{f}w^{n+1}), w^{k+1}\rangle &= \langle P(\bar{f}w^{n+1}), \bar{g}w^{k+1}\rangle \\ &= \langle \bar{f}w^{n+1}, P(\bar{g}w^{k+1})\rangle \\ &= \frac{2h(0)}{k+2}\cdot\frac{\overline{f^{(n)}(0)}}{n!(n+2)}.\end{aligned}$$

In summary, we have

$$\frac{h(0)}{2}\cdot\frac{\overline{f^{(n)}(0)}}{n+k+2} = \frac{h(0)}{k+2}\cdot\frac{\overline{f^{(n)}(0)}}{n+2}.$$

One can check that this also is impossible, so the proof is complete. □

The following characterization shows that Toeplitz operators can commute only in the obvious cases if at least one of their symbols is a polynomial.

THEOREM 10. *Let $u, v \in b^2$ be nonconstant functions and suppose one of them is a polynomial. Then T_u and T_v commute on b^2 if and only if $v = \alpha u + \beta$ for some constants α, β.*

Proof. The sufficiency is trivial. For the proof of necessity, assume (without loss of generality) that $u(0) = v(0) = 0$. We split the proof into cases. Since the adjoint operators $T_{\bar{u}}$ and $T_{\bar{v}}$ also commute, we may assume that u is a polynomial and thus need only consider two cases as follows. (The remaining cases are contained in Theorem 5 and Lemma 9.)

Case 1 (u holomorphic, v not antiholomorphic). In this case, by Theorem 4 there is a constant α such that $\partial v = \alpha(\partial u)$. Accordingly, we may write $u = f$ and $v = \alpha f + \bar{g}$ for some $f, g \in L^2_a$. Thus, the assumption $T_u T_v = T_v T_u$ is equivalent to

$$T_f T_{\bar{g}} - T_{\bar{g}} T_f = 0;$$

that is, T_f and $T_{\bar{g}}$ commute. Since f is nonconstant, it follows from Lemma 9 that g must be constant (identically 0) and thus $v = \alpha u$.

Case 2 (u, v are neither holomorphic nor antiholomorphic). In this case, by Theorem 4 there are constants α, β such that $\partial v = \alpha(\partial u)$ and $\bar{\partial} v = \beta(\bar{\partial} u)$. Accordingly, we may write $u = f + \bar{g}$ and $v = \alpha f + \beta \bar{g}$ for some nonconstant $f, g \in L^2_a$ with $f(0) = g(0) = 0$. Thus, the assumption $T_u T_v = T_v T_u$ is equivalent to

$$(\alpha - \beta)(T_f T_{\bar{g}} - T_{\bar{g}} T_f) = 0.$$

Since f and g are nonconstant, it follows from Lemma 9 that $\alpha = \beta$ and hence $v = \alpha u$. The proof is complete. □

In view of theorems proved in this paper, one may naturally ask whether the same is true for general harmonic symbols. Hence, we close the paper with a question. To be more precise, let $u, v \in b^2$ be nonconstant functions. We do not know whether $T_u T_v = T_v T_u$ implies $v = \alpha u + \beta$ for some constants α, β. As we have seen in the proof of Theorem 10, this problem reduces to the following special case.

PROBLEM. Let $f, g \in L^2_a$ and suppose T_f and $T_{\bar{g}}$ commute on b^2. Does it follow that either f or g is constant?

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