Fibers over the Sphere of a Uniformly Convex Banach Space

Jeff D. Farmer

1. Introduction: Bounded Analytic Functions on the Unit Ball

Over the past years, a significant interest has developed in the study of holomorphic functions defined on a domain in an infinite-dimensional Banach space and of their constituents (via Taylor expansions), the homogeneous polynomials. Many of the questions that have been studied have arisen from considerations of infinite-dimensional topology and from standard function algebra questions [CCG]. Recently, there has been an interest in connecting the well-developed theory of the geometry of Banach spaces with the function theory questions that have been studied classically, and some progress has been made in this direction [ACG; D; F; CCG; CGJ]. In addition, connections between properties of polynomials and geometry of the unit ball has been of interest (see [GJL] for a survey of this topic).

The present work is an attempt to study some of the properties of bounded analytic functions on the unit ball of an infinite-dimensional Banach space. In particular, we are interested in understanding something of boundary behavior; we combine techniques from the several fields to investigate it, especially with regard to the interplay with convexity and smoothness.

Many of the results here apply to the classical "nice" reflexive spaces, such as l_p and L_p $(1 < p < \infty)$. It is almost certain that there is much more to be learned even about the Hilbert space case.

We consider the boundary behavior of H^∞ functions on B, the open unit ball of an infinite-dimensional complex Banach space that has the geometric properties of uniform convexity, uniform smoothness, or both. By uniform smoothness, we mean uniform (real) Frechet differentiability of the norm, with the space considered as a real Banach space. Uniform convexity will mean that the dual is uniformly smooth; since spaces with either property are reflexive, this definition is complete. To be specific however, we state the following (after [LT]).

Definition 1.1. A complex Banach space is said to be *uniformly convex* (u.c.) if

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;\middle|\; x, y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} > 0 \quad \forall \varepsilon > 0.$$

Received October 10, 1996. Revision received September 9, 1997.
Michigan Math. J. 45 (1998).

In geometric terms, uniform convexity means that, for each point on the sphere and for each direction one can move from that point, the norm increases strictly in a way that is uniform with respect to the point and the direction. This notion can be quantified, in a manner that we will use repeatedly.

COROLLARY 1.2. *If X is u.c., if $x^*(x) = \|x\| = \|x^*\| = 1$, and if* $\mathrm{Re}[x^*(y)] > 1 - 2\delta(\varepsilon)$, *then* $\|x - y\| \leq \varepsilon$.

Proof. The proof is immediate from applying the functional x^* to the vector $x + y$. □

An important consequence of uniform smoothness is that every norm-1 vector has a unique norming functional in the dual; uniform convexity implies that any norming functional will (by itself) generate the norm topology on the sphere at that point. Thus weak and norm topologies, each restricted to the ball, will coincide at points of the sphere.

The algebra $H^\infty(B)$ is a nice Banach function algebra having a spectrum that fibers over points of the closed ball (actually, the closed ball of the second dual [CCG; ACG]; here our spaces are reflexive). If we denote by ι the inclusion of X^* into $H^\infty(B)$, then the fibering is given by

$$\mathcal{M}_x = (\iota^*)^{-1}(x).$$

There is a standard metric on the spectrum:

$$d(\phi, \psi) = \sup_{f \in H^\infty(B),\, \|f\|=1} |\phi(f) - \psi(f)|.$$

(This is just the norm in the space $(H^\infty(B)^*$.) It can be used to define Gleason parts, one of which is the (evaluations at points of the) open ball. A natural question to consider (see [ACG] or [CGJ]) is what kinds of analytic structure exist in the spectrum; we obtain some new results here.

The paper is organized as follows. In Section 2 we consider certain subalgebras of $H^\infty(B)$, some of which have been previously studied, that may be useful to us. We also connect with previous work; in particular we recall the radius function of [ACG], which we will use later. In Section 3 we look at slices and single-functional subalgebras, and we begin to consider how the theory of boundary behavior in one dimension applies to our situation; we also explore a consequence of uniform convexity.

In Section 4 we identify a class of interpolating sequences for some subalgebras on the sphere as well as an associated class in the ball. Then we generalize the classical result that if a sequence of homomorphisms in fibers over the boundary converges (Gelfand) to another homomorphism, then all but a finite number already lie in the limit fiber.

In Section 5 we move to a consideration of analytic structure in fibers. We prove the most technical theorem of the paper (and one that is unique to the infinite-dimensional situation); namely, that one can find uniformly homeomorphic copies of balls of ultrapowers of l_p-spaces in fibers over boundary points of the sphere in

l_p. We note that our technique of proof gives us a way to find these in fibers over interior points as well; copies of nonseparable l_2 were first found in interior fibers by Cole, Gamelin, and Johnson [CGJ].

2. Subalgebras and Fiberings; the Radius Function

The Banach algebra $H^\infty(B)$ is extremely large. It therefore makes sense to begin by considering some smaller subalgebras contained within it that have accessible properties and are related to it. An example appears in [ACG], where $H^\infty(B)$ is studied by first considering the Frechet algebra $\mathcal{H}_b$ of entire functions that are bounded on bounded subsets of X with the topology of uniform convergence on bounded subsets. Aron, Cole, and Gamelin consider a radius function defined on the spectrum $\mathcal{M}_b$ of $\mathcal{H}_b$ in the following way: For $\phi \in \mathcal{M}_b$, $R(\phi)$ is the infimum of positive real numbers r with ϕ continuous with respect to uniform convergence on rB. Notice that, by considering functions in this algebra as functions on the ball, we can view $\mathcal{H}_b$ as a subset of $H^\infty(B)$. Thus there is a natural adjoint that maps $\mathcal{M}_{H^\infty(B)}$ to $\mathcal{M}_b$; this projection is 1–1 on elements whose images have radius less than 1 [ACG]. This map then induces a radius function on $\mathcal{M}_{H^\infty(B)}$ (i.e., $R(\phi) = \sup\{ r : \phi \text{ continuous on } rB \}$).

In our approach to the subject we will have occasion to consider the subalgebra of elements whose Taylor series have weakly continuous partial sums. Recalling that any $f \in H^\infty(B)$ can be written $f = \sum_{i=0}^\infty f_i$ with f_i an i-homogeneous analytic polynomial, we define

$$H_w^\infty(B) = \left\{ f \in H^\infty(B) \;\middle|\; f = \sum_{i=0}^{\infty} f_i \text{ with } f_i \text{ weakly continuous } \forall i \right\}$$

We can equivalently require that f be weakly continuous on any ball of radius less than 1; this algebra is simply the closure of the weakly continouous polynomials in the topology of uniform convergence on balls of radius strictly less than 1. This topology is a topology in which the Taylor series for f converges; we will refer to it from here on as the *ucb topology* (notice that this is the analog of the topology of uniform convergence on compact subsets in the classical case, for closed bounded subsets are weakly compact in a reflexive Banach space).

It has been previously shown (see [F]) that, in the context of reflexivity and the approximation property (AP), the requirement that $H^\infty(B) = H_w^\infty(B)$ is equivalent to so-called polynomial reflexivity, that is, the property that the spaces of n-homogeneous polynomials are reflexive for every n (which is the same as saying that all polynomials are weakly continuous modulo the AP). This is one context in which this subalgebra natually appears.

These algebras will not ever be the same in what follows here, however, for the following reason. Uniformly convex spaces are superreflexive and therefore have nontrivial type. By a geometric argument involving spreading models (see [FJ]), it can be shown that such spaces are "polynomially Schur" (i.e., they enjoy the "Λ-property" of [CCG]); as a consequence there always exists a homogeneous polynomial of some degree that is not weakly continuous.

The properties shown to be satisfied by $H^\infty(B)$ in [F] for polynomially reflexive spaces actually hold for the subalgebra $H^\infty_w(B)$ whenever X is reflexive. Owing to the inclusion $H^\infty_w(B) \subset H^\infty(B)$, there is a natural projection of the spectrum of the second into that of the first (it acts by restriction). We can now adapt one fact in particular from [F] to our situation here. Define

$$\mathcal{K}_x = \mathcal{M}_x \cap (\mathrm{cl}\, B),$$

where cl B is closure in the Gelfand topology.

PROPOSITION 2.1 (a consequence of [F, Thm. 4.5]). *Let* $\pi_w\colon \mathcal{M}_{H^\infty(B)} \to \mathcal{M}_{H^\infty_w(B)}$ *by* $\pi_w(\phi)(f) = \phi(f)$. *If* $R(\phi) < 1$ *and* $\phi \in \mathcal{K}_x$, *then* $\pi_w\phi = \delta_x$.

This simply says that the projection π_w maps all elements of the noncorona part of the fiber (i.e., the part that lies in the closure of the evaluations) that have radius function less than 1 to the evaluations. Thus the only other homomorphisms in fibers over interior points are elements whose radius function is 1; this says that, in order for a net of evaluations to converge to something in the fiber over an interior point, the net must move to the boundary.

We will find this subalgebra of $H^\infty(B)$ and its spectrum to be useful to us in the sequel—many of the functions we use will come from this algebra. In the next section we will also see that this algebra is closely related to the single-functional subalgebras.

3. Single-Functional Subalgebras; Restrictions to Slices

A (real) observer standing on the unit sphere of a complex infinite-dimensional Banach space (say at a point x) can gaze in many directions; one of these is distinguished as being the direction of complex rotation of the vector x. By restricting consideration to the (complex) 1-dimensional subspace generated by x, our observer finds an exact copy of the classical H^∞ space as follows. Let x^* be any norming functional for x. The map that takes $g \in H^\infty$ to $g \circ x^* \in H^\infty(B)$ is an isometric embedding of H^∞ into $H^\infty(B)$. We call the image of this map $H^\infty_{x^*}(B)$, its adjoint projection on the maximal ideal spaces π_{x^*}, with

$$\pi_{x^*}\colon \mathcal{M}_{H^\infty(B)} \to \mathcal{M} \quad \text{by} \quad \pi_{x^*}(\phi) = \phi|_{H^\infty_{x^*}(B)} \;\; \forall \phi \in \mathcal{M}_{H^\infty(B)}.$$

PROPOSITION 3.1. *The map* π_{x^*} *is Gelfand continuous, onto, and takes Gleason parts onto Gleason parts.*

The proof of the proposition is a straightforward verification: Since the norm of the map is 1, it is a contraction in the Gleason metric; it is also clearly weak* (Gelfand) continuous. We will see that it is onto in a moment.

PROPOSITION 3.2. *Let* $x \in S$, *the sphere of a uniformly convex Banach space, with* x^* *any norming functional. Let* $g \in H^\infty$, $y \in S$, $y \neq \lambda x$, *and* $\lambda \in \mathbf{C}$. *Then* $g \circ x^*$ *is weakly continuous at* y.

Proof. First, if $y \neq \lambda x$ then $|x^*(y)| < 1$ by uniform convexity, so $g \circ x^*$ is defined at y and is norm continuous there. But if $y_\alpha \to y$ weakly then this says exactly that $x^*(y_\alpha) \to x^*(y)$ and we are done. (In fact, this is true for any Banach space and any point x of strict convexity on its sphere.) The map π_{x^*} produces an additional fibering via its pullback, this time of $\mathcal{M}_{H^\infty(B)}$ over $\mathcal{M}$. We will call these fibers $(\pi_{x^*}^{-1}(\phi),\ \phi \in \mathcal{M}_{H^\infty(B)})$ "x^*-fibers."

On the other hand, given any $x \in S$, we can define a map

$$\rho_x \colon H^\infty(B) \to H^\infty \quad \text{by} \quad \rho_x(f)(\lambda) = f(\lambda x),\ |\lambda| < 1,$$

by restricting each function in $H^\infty(B)$ to the span of a single vector (the intersection of this span with the ball being a copy of the unit disk). We will call the adjoint of this map $\rho_x^* \colon \mathcal{M}_{H^\infty} \to \mathcal{M}_{H^\infty(B)}$. It is clear that, whenever x^* norms x, $\pi_{x^*} \circ \rho_x^*$ is simply the identity on the spectrum of H^∞. (Verifying this statement is simple, and proves the "onto" part of Proposition 3.1.) □

The single functional subalgebras (denoted $H_{x^*}^\infty(B)$) are related to the algebra $H_w^\infty(B)$ considered in the previous section. Each subalgebra is clearly contained in $H_w^\infty(B)$ and thus so is the closure of the algebra they generate in the ucb topology. On the other hand, weakly continuous polynomials on B will be in the uniform closure of the algebra generated by the single-functional subalgebras if the space has an appropriate approximation property. Denote by $A(B)$ the uniform closure of the polynomials weakly continuous on B and by $P(B)$ the uniform closure in $H^\infty(B)$ of the algebra generated by X^*.

Proposition 3.3. *The ucb closure of* $\operatorname{alg}\big(\bigcup\{H_{x^*}^\infty(B) \mid \|x^*\| = 1\}\big)$ *is contained in* $H_w^\infty(B)$. *If the space X has the property that every weakly continuous polynomial on B is a uniform limit of polynomials in linear functionals—that is, if $P(B) = A(B)$—then equality holds.*

Proof. We need merely observe that the stated property will guarantee that the partial sums of the Taylor series of elements in $H_w^\infty(B)$ are weakly continuous on B; since the Taylor series converges in the ucb topology, we are finished. □

We now turn to an illustration of how certain results on boundary behavior from the classical theory can be adapted to the infinite-dimensional situation. We use the notation from [H].

Let X be any complex Banach space; let $x \in S_X$, $x^*(x) = 1$, and $\|x^*\| = 1$. Define $W = \{\phi \in \mathcal{M}_y \mid y = \lambda x,\ |\lambda| = 1\}$. Let $W_+ = W \cap \pi_{x^*}^{-1}\{y = \lambda x \mid \operatorname{Im}(x^*(y)) > 0\}$ and likewise for W_-, while $W_0 = \mathcal{M}_x - \operatorname{cl} W_+ - \operatorname{cl} W_-$. Also, refer to the classical fiber in H^∞ over the point 1 as $\mathcal{M}_1$. We then have the following.

Theorem 3.4. *Let X be any uniformly convex Banach space; let $x \in S_X$, $x^*(x) = 1$, $\|x^*\| = 1$, and $\phi \in \mathcal{M}_x \subset \mathcal{M}_{H^\infty(B)}$. Then the sets $\mathcal{M}_x \cap \operatorname{cl} W_+$, $\mathcal{M}_x \cap \operatorname{cl} W_-$, and W_0 are disjoint and nonempty. In addition* (*if* $\dim X > 1$), *each x^*-fiber over a classical homomorphism in $\pi_{x^*}(W_0)$ contains many homomorphisms in the closure of*

$$(\mathcal{M}_{H^\infty(B)} - W) - \bigcup\{\mathcal{M}_z \mid z \in B\},$$

that is to say, the closure of the set of homomorphisms in fibers over points of the sphere that are not complex rotations of x.

Proof. Following [Ho, pp. 165–166], define $f \in H^\infty$ to be holomorphic on the disk and continuous at every point on the circle except at 1 and -1; f has the value e on the lower half and 1 on the upper half of the circle and has radial limit $\sqrt{e}$ at ± 1. Now, by Lemma 4.4 (to be proved in the next section), $f \circ x^* \in H^\infty(B)$ will be constant on fibers over points λx if $\lambda \neq \pm 1$; thus, it will separate cl W_+ and cl W_- since $W_+ \subset \widehat{f \circ x^*}^{-1}(1)$ and $W_- \subset \widehat{f \circ x^*}^{-1}(e)$. The sets are seen to be nonempty by lifting via ρ_x^* from the analogous sets provided for the classical case by [Ho, p. 165], bearing in mind that $(\pi_{x^*} \circ \rho_x^*)(\nu) = \nu$ $(\nu \in \mathcal{M})$.

We prove the additional fact. First, suppose that $\phi \in W_0$; it lies in the x^*-fiber $\pi_{x^*}^{-1}\pi_{x^*}\phi$. By Carleson's theorem, find $z_\alpha \in D$ converging to $\pi_{x^*}\phi$. Make any selection (using choice and the fact that dim $X > 1$) of points $w_\alpha \in S_X$ and homomorphisms satisfying

$$x^*(w_\alpha) = z_\alpha \quad \text{and} \quad \phi_\alpha \in \mathcal{M}_{w_\alpha}.$$

Now the net $\{\phi_\alpha\}$ must have a convergent subnet in $\mathcal{M}_{H^\infty(B)}$, say to ϕ'; this is the point we seek:

$$\pi_{x^*}\phi' = \pi_{x^*} \lim_\beta \phi_\beta = \lim_\beta \pi_{x^*}\phi_\beta = \lim_\beta w_\beta = \pi_{x^*}\phi,$$

by Gelfand continuity of π_{x^*}. □

We can actually say a bit more about this situation. The function f has radial limit $\sqrt{e}$ at 1, so it has nontangential limit $\sqrt{e}$ there. If the $\{\phi_\alpha\}$ are in fibers over points $w_\alpha \in S_X$ having the property that the $x^*(w_\alpha) = z_\alpha \in D$ approach 1 nontangentially, then we know that any limit point will be in W_0 because

$$\{\phi_\alpha\}(f \circ x^*) = \pi_{x^*}\phi_\alpha(f) = f(z_\alpha) \to \sqrt{e},$$

since $\pi_{x^*}\phi_\alpha$ is just a classical homomorphism in the fiber over $z_\alpha \in D$, a fiber that contains only the evaluation. Thus, in order to have a net approaching an element not in W_0, it must lie in fibers over points whose projection by x^* approaches the boundary quickly. We quantify this statement as follows.

THEOREM 3.5. *Let $\delta\colon (0,1] \to (0,1]$ be the modulus of uniform convexity for X. Let $x \in S_x$, $x^*(x) = \|x^*\| = 1$, $\phi_\alpha \in \mathcal{M}_{w_\alpha} \to \phi$, and $x^*(w_\alpha) = z_\alpha \to \pi_{x^*}\phi$, and suppose*

$$\liminf_\alpha \left\{ \frac{\delta(d(w_\alpha, \{e^{i\theta}x\}))}{\|w_\alpha - x\|} \right\} = c > 0.$$

Then $\phi \in W_0$.

Proof. The claim is that the condition of the proposition guarantees that the z_α converge nontangentially to 1, which will finish the proof by our previous remarks.

Let $c > c' > 0$. Then by hypothesis there exists a β such that for all $\alpha > \beta$ we have

$$\delta(d(w_\alpha, \{e^{i\theta}x\})) > c'\|w_\alpha - x\|$$

or

$$\delta(\|e^{i\theta}w_\alpha - x\|) > c'\|w_\alpha - x\| \quad \forall\theta,$$

which by uniform convexity (Corollary 1.2) implies that

$$\mathrm{Re}[x^*(e^{i\theta}w_\alpha)] < 1 - 2c'\|w_\alpha - x\| \quad \forall\theta$$

and hence

$$|z_\alpha| = |x^*(w_\alpha)| < 1 - 2c'\|w_\alpha - x\| < 1 - 2c'|z_\alpha - 1|$$

or

$$2c'|z_\alpha - 1| < 1 - |z_\alpha|.$$

This is exactly what we need for nontangential convergence in the unit disk. □

4. Fibers over the Boundary; Interpolating Sequences

An important consequence of uniform convexity for us is that at any point $x \in S$, the intersection of a weak neighborhood determined by x^* and S is contained in the intersection of S with a norm neighborhood of x; thus, the weak and norm topologies (on the ball) coincide at points of the sphere. This allows us to create some peaking functions.

PROPOSITION 4.1. *For any uniformly convex Banach space X, for all $\varepsilon > 0$, $1/2 > \delta > 0$, $\lambda \in \mathbf{C}$, and $x \in S$, there is a function in $f \in A(B)$ satisfying*

$$f(x) = \lambda, \quad \|f\| \le |\lambda|,$$

and

$$|f(y)| < \varepsilon \quad \textit{whenever} \;\; y \in \bar{B} \;\; \textit{and} \;\; \|x - y\| > \delta.$$

Proof. Let x^* be a norming functional for x. By uniform convexity there is a number $\eta > 0$ such that, for $y \in \bar{B}$, $\mathrm{Re}(x^*(y)) \ge 1 - \eta$ implies that $\|x - y\| \le \delta$. Now there is a number $\sigma > 0$ such that, if $\mathrm{Re}(x^*(y)) < 1 - \eta$ with $y \in \bar{B}$, then $\left|\left(\frac{1+x^*}{2}\right)(y)\right| < 1 - \sigma$. Simply choose m so that $|\lambda|(1-\sigma)^m < \varepsilon$ and let $f = \lambda\left(\frac{1+x^*}{2}\right)^m$. Then $f \in P(B) \subset A(B)$ has the desired properties. □

LEMMA 4.2. *Let X be u.c. Suppose $\{\Phi_\alpha\} \subset \mathcal{M}_{H^\infty(B)}$ with $\Phi_\alpha \to \Phi$ in the Gelfand topology. Let $\Phi_\alpha|_{X^*} = x_\alpha$ and $\Phi|_{X^*} = x \in S$ (i.e., each Φ_α lies in the fiber over x_α). Then $x_\alpha \to x$ in norm.*

Proof. Recalling that the Gelfand topology is the weak-star topology on $H^\infty(B)^*$ restricted to the spectrum, it is immediate that the convergence occurs weakly by restricting the action of the homomorphisms to $X^* \subset H^\infty(B)$. But the weak and norm topologies agree with regard to points on the sphere, so we are done. □

Since there is no known corona theorem even for two dimensions we do not expect one in our context soon. This means that on those occasions where we must approach points of the spectrum from the ball, we will look only at the closure of the ball in the spectrum and at fibers in this closure.

LEMMA 4.3. *Let B be the unit ball of any Banach space. Let $f \in H^\infty(B)$ be weakly continuous at $x \in \bar{B}$. Then f is constant on $\mathcal{K}_x$, the noncorona fiber over x (i.e., $\phi(f) = f(x)$ for all $\phi \in \mathcal{K}_x$).*

The proof is an exercise. If x is a point on the sphere of a uniformly convex Banach space where weak and norm continuity coincide, then f need only be norm continuously extendable to x, and we can say a bit more; in this case we can actually discuss the whole fiber.

LEMMA 4.4. *Let B be the ball of a u.c. Banach space. Then $f \in H^\infty(B)$ is continuously extendable to a point x on the sphere if and only if f is constant on the whole fiber $\mathcal{M}_x$ (i.e., $\phi(f) = f(x)$ for all $\phi \in \mathcal{M}_x$).*

Proof. The sufficiency is clear by taking cluster points of evaluations in the spectrum. The necessity is a consequence of the fact that each boundary point is a strong peak point for the ball algebra, as follows. Suppose that f is continuously extendable to x with value zero. Take a sequence of strongly peaking functions (p_n) in $P(B)$ that converge to zero uniformly on the ball off of norm neighborhoods of x (Proposition 4.1 shows how this may be done). Then $(1 - p_n)f$ converges uniformly to f on the ball, so that for any homomorphism $\phi \in \mathcal{M}_x$,

$$\phi(f) = \lim_n \phi((1 - p_n)f) = [\phi(1) - \phi(p_n)]\phi(f) = \lim 0 = 0$$

by choice of the p_n as strongly peaking elements of $P(B)$, the algebra generated by linear functionals. □

DEFINITION. We shall say that a set $\{x_n\}_{n\in I}$ in S is *well-separated* if, for all k, $\mathrm{d}(x_k, \{x_n\}_{n\in I} - \{x_k\}) = \delta_n > 0$ and if for all $i \neq j$ there is no complex number λ such that $x_i = \lambda x_j$. We call the numbers δ_n the *separation indices* of the set.

If the set is finite or countably infinite, then it is enough to check that it does not contain any of its accumulation points and that no element is a complex multiple of another.

THEOREM 4.5. *Let $\{x_n\}_{n\in\mathbf{N}}$ be well-separated in the unit sphere. Let $\lambda_n \in \mathbf{C}$, $|\lambda_n| = 1$, and $1 > \varepsilon_n > 0$, $\varepsilon_n \to 0$. Then there exists a bounded analytic function $f \in H^\infty(B)$, continuous everywhere on the closed unit ball except at accumulation points of $\{x_n\}$, satisfying*

(i) $|f(x_n) - \lambda_n| < \varepsilon_n$ *and*
(ii) $\|f\| \le \sup_n\{1 + \varepsilon_n\}$.

Proof. Let x_n^* be the norming functionals for the x_n, and let δ_n be the separation indices. Apply Proposition 4.1 with $x = x_1$, $\varepsilon = 2^{-1-2}\varepsilon_1$, $\delta = \frac{1}{2}\delta_1$, and $\lambda = \lambda_1$, obtaining f_1. To obtain f_n for $n \geq 2$, do the same thing except let $x = x_n$, $\delta = \frac{1}{2}\delta_n$, $\lambda = \lambda_n - \sum_{i=1}^{n-1} f_i(x_n)$, and $\varepsilon = \inf_{k \leq n}\{2^{-n-2}\varepsilon_k\}$. Note that we will always have $|\lambda| < 1 + \frac{1}{2}\sup_n \varepsilon_n$. Then define $f = \sum_{i=1}^{\infty} f_i$. One checks easily that the estimate is obtained:

$$f(x_n) = \sum_{j=1}^{\infty} f_j(x_n) = \sum_{j=1}^{n} f_j(x_n) + \sum_{j=n+1}^{\infty} f_j(x_n) = \lambda_n + \sum_{j=n+1}^{\infty} f_j(x_n).$$

Hence

$$|f(x_n) - \lambda_n| \leq \sum_{j=n+1}^{\infty} 2^{-j-2}\varepsilon_n < \varepsilon_n.$$

The same kind of calculation proves uniform convergence on bounded subsets of B, and in any neighborhood of a point of S that does not contain an accumulation point of the x_n, since $d(y, x_n) < \frac{1}{2}\delta_n$ for at most one value of n.

For example, if y happens to be close to x_p (i.e., if $\|y - x_p\| < \frac{1}{2}\delta_p$, which implies $\|y - x_j\| > \frac{1}{2}\delta_j$ for all $j \neq p$), then

$$\begin{aligned} |f(y)| &\leq \sum_{j \neq p} 2^{-j-2}\left(\inf_{k \leq j} \varepsilon_k\right) + f_p(y) \\ &< \frac{1}{2}\sup_i \varepsilon_i + \|f_p\| \\ &\leq \frac{1}{2}\sup_i \varepsilon_i + 1 + \frac{1}{2}\sup_i \varepsilon_i \\ &= 1 + \sup_i \varepsilon_i. \end{aligned}$$

□

Theorem 4.5 has some immediate corollaries; all apply to a uniformly convex space.

COROLLARY 4.6. *If X is uniformly convex, then any two points of $\mathcal{M}$ in different fibers over boundary points are in different Gleason parts.*

Proof. Let $\Phi_i \in \mathcal{M}_{x_i}$ $(i = 1, 2)$. If $\{x_1, x_2\}$ is well-separated, apply the theorem to obtain $1 = f(x_1) = -f(x_2)$; if not well-separated, the result is classical (by means of Möbius transformations). □

Now we have the following generalization of a classical result.

THEOREM 4.7. *Let $\Phi_i \in \mathcal{K} \subset \mathcal{M} - B$ lie in fibers over the boundary such that $\Phi_i \to \Phi$ in the Gelfand topology. Then all but a finite number of the Φ_i lie in the same fiber as Φ does.*

Proof. Suppose not. Now $\Phi_i \in \mathcal{M}_{x_i}$. By passing to a subsequence, we may assume that all of the x_i are distinct and that none of them are complex multiples of x. (For if at any point all of the remaining elements of the sequence *were* complex multiples of x then we could apply the classical result to $\rho_x^*\Phi_i$; see [Ho].) Since $x_i \to x$ in norm, we may also assume by passing to a subsequence that none of them are complex multiples of one another. But this means that the sequence is well-separated, so we may apply Theorem 4.5 to obtain a function that is close to both -1 and 1 on infinitely many of the x_i and also norm continuous (hence also weakly) at each one of them. Since this f will then be constant on the fibers over the x_i, we will have $\Phi(f)$ equal to 1 and -1, a contradiction. □

5. Analytic Structures in Fibers

The investigation of analytic structures that can be biholomorphically mapped into fibers was begun in the Banach space context by Cole, Gamelin, and Johnson [CGJ], who showed that the fiber over zero contains (at the least) a large number of analytic disks and, in the case of superreflexive spaces, a copy of the ball of a nonseparable Hilbert space. In this section, we wish to investigate structures in boundary fibers. As a consequence of our construction, we will see that some very large structures appear in interior fibers as well.

Our venue will be the l_p-spaces $(1 < p < \infty)$, and the goal will be to prove the following theorem.

THEOREM 5.1. *Let $x \in S_{l_p}$. Then there is a copy of $B_{\Pi_{\mathcal{U}} l_p} \times (\beta\mathbf{N} - \mathbf{N})$ in the fiber over x (in $\mathcal{M}_{H_w^\infty(B)}$), embedded via a uniform homeomorphism with uniformly continuous inverse, that is analytic for each point of $\beta\mathbf{N} - \mathbf{N}$.*

The proof will also give us a new structure in fibers over interior points.

COROLLARY 5.2. *Theorem 5.1 remains true if $x \in B_{l_p}$.*

By the Dvoretzky theorem, this also gives us a copy of the unit ball of a nonseparable Hilbert space in each fiber. However, this structure will be seen to be distinct from that embedded in the fiber over 0 by [CGJ].

The idea here will be to embed copies of the ball of the l_p^n in very small slices near elements of a convergent sequence x_i, and disjoint from them with respect to the basis. The fact that we can explicitly calculate the norm in l_p will be of great help to us. We begin with some lemmas from calculus. For the remainder of the proof, fix $1 < p < \infty$.

LEMMA 5.3. *For any $\lambda > 0$ and $n \in \mathbf{N}$, the function $g_n(a,b) = \lambda^{-1} a \left(\frac{1+b}{2}\right)^n$ subject to $a^p + b^p = 1$ has a single maximum value on the unit square at (a_0, b_0), where*

$$\frac{n-1}{n+1} < b_0^p < \frac{n}{n+2} \quad \text{and hence} \quad \frac{2}{n+2} < a_0^p < \frac{2}{n+1};$$

the maximum value $g_n(a_0, b_0) < \lambda^{-1}\left(\frac{2}{n}\right)^{1/p}$.

Proof. Since the function is zero at the endpoints and nonnegative, it should have a maximum. Lagrange multipliers tell us that at this point

$$\left(\frac{a}{b}\right)^{p-1} = \frac{1+b}{an} \quad \text{or} \quad \frac{a^p}{b^p} = \frac{1+b}{bn}.$$

Solving for a^p and plugging into the constraint gives

$$n = b^{p-1} + b^p(1+n),$$

from which the estimates are obtained by noting that $b^p < b^{p-1} < 1$. □

COROLLARY 5.4. *If* $\lambda^p = 1 - (2^{1-1/n} - 1)^p$ *then* $\lim_{n\to\infty} \lambda^p n = 2p\log 2$. *Hence, for any* $C > 1$ *there exists an* $N \in \mathbf{N}$ *such that, for* $n \geq N$ *and* λ *as before,* $g_n(a,b) < C(p\log 2)^{-1/p}$.

Proof. This is an application of L'Hopital's rule and Lemma 5.3. □

LEMMA 5.5. *For fixed* $0 < a, b < 1$, $a^p + b^p = 1$, *and* $\lambda = \lambda(n)$ *as before:*

(i) $\lim_{n\to\infty} g_n(a,b) = 0$; *and*
(ii) *for any* n, *if* $b < b_0(n)$ *(from Lemma 5.3) then* $g_n(a',b') < g_n(a,b)$ *whenever* $b' < b$.

Proof. (i) To check this, apply L'Hopital's rule once to the pth power of g; the new limit is easily seen to be zero.

(ii) Calculus and Lemma 5.3. □

We are now ready to begin proving Theorem 5.1. Choose $x_i \in S$ finitely supported on the basis and converging to x, say with $\operatorname{supp} x_i \subset \sigma_i$, where the supports form an increasing sequence of finite subsets of the integers ($\sigma_1 \subset \sigma_2 \subset \cdots$). We also choose the sequence so that none of the elements are multiples of x and so that, at every (kth) basis element where both are nonzero,

$$\arg(x^k) = \arg(x_i^k)$$

and $\arg(x_i^k) = 0$ otherwise. This is done slightly differently depending on whether x is finitely supported or not, but is nevertheless easily done in either case. All of this will result in the sequence being well-separated; we can thus find numbers δ_i to satisfy the following: $\operatorname{Re}(x_i^*(y)) > \delta_i$ implies $\operatorname{Re}(x_j^*(y)) < \delta_j$ whenever $j \neq i$. (Here $x_i^*(x_i) = \|x_i^*\| = 1$ and naturally it has the same support as x_i.) Hence each δ_i is an amount by which the hyperplane $[\operatorname{Re}(x_i^*) = 0]$ may be translated in order for the slices thus formed to be disjoint. (The δ_i may be explicitly calculated using Corollary 1.2; if μ_i are the separation indices defined in Section 4, then $\delta_i = 1 - 2\delta(\mu_i/2)$ will do.) Call the slices C_i' (i.e., $C_i' = B \cap [\operatorname{Re}(x_i^*) = \delta_i]$). Letting λ_i be defined by n_i as in Corollary 5.4, choose n_i large enough so that the maximum of g_i $(= g_{n_i})$ occurs at $b_{0,i}$ with $\delta_i < b_{0,i} < 1$ (by Lemma 5.3), $g_i((1-\delta_i^p)^{1/p}, \delta_i) < 2^{-i}$ (by Lemma 5.5(i)), and also so that $(2^{1-1/n_i} - 1)^p = 1 - \lambda_i^p > \delta_i^p$ (clear since the δ_i are defined). Let E_i be finite-dimensional l_p-spaces of increasing dimension. We will consider these spaces isometrically embedded on $\dim E_i$ elements of the standard basis in the following rather careful way.

Go out far enough on the basis to choose a block ρ_1 such that $\|x_i|_{\rho_1}\| < 2^{-7}\lambda_1$ for all $i > 1$ (where $\dim E_1 = |\rho_1|$) and such that ρ_1 is disjoint from the support of x_1. This can be accomplished because $x_i \to x \in l_p$. Continue similarly so that

$$\|x_i|_{\rho_j}\| < 2^{-6-j}\lambda_j \quad \forall i > j,$$

where ρ_j is disjoint from σ_j (the support of x_j). Map E_i isometrically to $l_p(\rho_i)$.

We will suppress any further reference to this embedding, and just think of the E_i as living on these blocks of the basis. Let $D_i = (2^{1-1/n_i} - 1)x_i + (\lambda_i/3)B_{E_i}$ be isometric affine copies of the ball of E_i scaled by $\lambda_i/3$; call the affine embedding $\psi_i : B_{E_i} \to D_i$, where

$$\psi_i(y) = (2^{1-1/n_i} - 1)x_i + \frac{\lambda_i}{3}y \quad \text{for } \; y \in B_{E_i}$$

and where $D_i \subset C_i = [\mathrm{Re}(x_i^*) = (2^{1-1/n} - 1)] \cap B$. Our previous choice of n_i, which determines λ_i, ensures that the slice C_i is closer to the boundary than C_i'. This map uses one third of the room available in the slice C_i in these directions.

Let $\mathcal{U}$ be a free ultrafilter on the positive integers; form the ultraproduct of the E_i along $\mathcal{U}$, which is isometrically an ultrapower of l_p. (This space contains both a nonseparable l_p and a nonseparable L_p; for details, see [He].) If $z_i \in B_{E_i}$, then $z = \{z_i\}_{i=1}^{\infty}$ is a representative of an element in $\Pi_{\mathcal{U}} E_i$ and we define

$$\Psi(z) = \lim_{\mathcal{U}} \psi_i(z_i),$$

where the points $\psi_i(z_i)$ are identified with evaluations and the limit is taken in the (compact) maximal ideal space. First of all, since $\lambda_i \to 0$, each of these limits lies in the fiber over x (this is true also in the case of Corollary 5.2, where the x_i converge only weakly). The map is naturally analytic, since the embeddings are affine. Our task is to show that the embedding is uniformly continuous in both directions (which, by the definition of the ultraproduct, will also show that it is well-defined).

Let $z = \{z_i\}_{i=1}^{\infty}$ and $w = \{w_i\}_{i=1}^{\infty}$ be representatives of distinct points in the ball of our big l_p-space. Then $\mathrm{dist}(z, w) = \lim_{\mathcal{U}} \|z_i - w_i\|$.

Distance in the maximal ideal space considered as a subset of the dual space of $H_w^{\infty}(B)$ is now given by

$$\begin{aligned}\|\Psi(z) - \Psi(w)\| &= \sup_{\|f\|_\infty = 1} \{|\Psi(z)(f) - \Psi(w)(f)|\} \\ &= \sup_{\|f\|_\infty = 1} \left\{ \lim_{\mathcal{U}} |f(\psi_i(z_i)) - f(\psi_i(w_i))| \right\},\end{aligned}$$

where each f is in $H_w^{\infty}(B)$. So this comes down to checking each coordinate map ψ_i. Here we use the Schwartz lemma. Let us omit the tedious (fixed) index i for a moment on w, z, E, λ, and ψ. Notice that no matter where z $(= z_i)$ is mapped, we can center a complex disk there in the direction of w with radius $2\lambda/3$ (since z is already in a disk of radius $\lambda/3$ centered in the slice C, which is of size λ in the relevant directions—namely, those of E, which has support on the basis disjoint from x_i). Call the map from the unit disk to this disk h; that is,

$$h(v) = \psi z + \frac{2v\lambda}{3\|\psi(w) - \psi(z)\|}(\psi(w) - \psi(z)).$$

The Schwartz estimate for the composition $\frac{1}{2}(f - f(\psi z)) \circ h$ (when applied to the point $h^{-1}(\psi(w)) = \frac{3}{2\lambda}\|\psi(w) - \psi(z)\|$; note $h^{-1}(\psi(z)) = 0$ here) gives

$$\frac{1}{2}|f(\psi(w)) - f(\psi(z))| = \frac{1}{2}(f - f(\psi(z))) \circ h\left(\frac{3}{2\lambda}\|\psi(w) - \psi(z)\|\right)$$
$$< \frac{3}{2\lambda}\|\psi(w) - \psi(z)\| = \frac{1}{2}\|w - z\|$$

(recall that the affine maps h and ψ scale by $2\lambda/3$ and $\lambda/3$, respectively). Hence Ψ is uniformly continuous.

To see that Ψ does not appreciably decrease distances as well, we need to construct a specific $H^\infty_w(B)$ function f that will separate two given points in the image of the ball of the ultraproduct. To this end, let z and w be representatives as before. (We return to the use of the index i). Take ϕ_i in $B_{E_i^*}$ so that $\phi_i(w_i - z_i) = \|w_i - z_i\|$. (We will not distinguish between ϕ_i in E_i^* and $\phi_i \in l_p^*$ supported on ρ_i.) Due to the disjoint supports of the E_i, we have $\phi_j(\psi(w_i)) = \phi_j(\psi(z_i)) = 0$ if $i \neq j$. By the definition of the maps ψ_i, we also have

$$x_i^*(\psi(w_i)) = x_i^*(\psi(z_i)) = 2^{1-1/n_i} - 1.$$

Now define

$$f = \sum_{i=1}^{\infty} \lambda_i^{-1}\phi_i\left(\frac{1 + x_i^*}{2}\right)^{n_i}.$$

Let us make some remarks about f. It is defined on the unit ball. The main difficulty is to show that it is bounded there. Let $y \in B_{l_p}$. Choose y' so that the modulus of each coordinate is the same as that of y, but rotated so that $x_i^*(y')$ is always positive. This can be done because of the way the x_i were chosen; the arguments of their coordinates agree for each coordinate in their common support. Now notice that

$$|f(y)| \leq \sum_{i=1}^{\infty} \tau_i\lambda_i^{-1}\phi_i(y)\left(\frac{1 + x_i^*(y)}{2}\right)^{n_i}$$
$$\leq \sum_{i=1}^{\infty} \tau_i'\lambda_i^{-1}\phi_i(y')\left(\frac{1 + x_i^*(y')}{2}\right)^{n_i},$$

where τ_i and τ_i' are constants of modulus 1 chosen in each case to make each term positive. The first inequality is due to simple properties of the complex numbers; the second is due to the fact that the size of each (now positive) term is only increased by the change from y to y', by the choice of y'. Because of the foregoing inequality, we need simply assume that we are working in real l_p (in fact, in the positive cone). Now the maximum of Lemma 5.3 also holds in the set $\{a^p + b^p \leq 1\}$, and $x_i^*(y) > \delta_i$ for at most one index, say k; we thus may write

$$f(y) \leq \sum_{i \neq k} g_i(\phi_i(y), x_i^*(y)) + g_k(\phi_k(y), x_k^*(y));$$

by Lemma 5.5(ii) and our choice of δ_i, this means that

$$f(y) \le \sum_{i\neq k} 2^{-i} + g_k(\phi_k(y), x_k^*(y)) \le 1 + 2 = 3$$

owing to Corollary 5.4. Thus $\|f\|_\infty < 3$ and $f \in H_w^\infty(B)$.

Let us now calculate

$$f(\Psi w) = \lim_{\mathcal{U}} f(\psi_i(w_i)) = \lim_{\mathcal{U}} f\left((2^{1-1/n} - 1)x_i + \frac{\lambda_i}{3} w_i\right).$$

Since $\phi_j(w_i) = 0$ whenever $i \neq j$ and $\phi_j(x_i) = 0$ if $i < j$, we have

$$\begin{aligned} f(\Psi w) &= \lambda_i^{-1}\phi_i\left(\frac{\lambda_i}{3} w_i\right)\left(\frac{1 + 2^{1-1/n_i} - 1}{2}\right)^{n_i} \\ &\quad + \sum_{j=1}^{i-1} \lambda_j^{-1}\phi_j((2^{1-1/n_i} - 1)x_i)\left(\frac{1 + x_j^*(\psi(w_i))}{2}\right)^{n_j} \\ &= \frac{1}{2}\lambda_i^{-1}\phi_i\left(\frac{\lambda_i}{3} w_i\right) + \sum_{j=1}^{i-1} \lambda_j^{-1}\phi_j((2^{1-1/n_i} - 1)x_i)\left(\frac{1 + x_j^*(\psi(w_i))}{2}\right)^{n_j}. \end{aligned}$$

Let's call the preceding summation term the "error". A similar calculation works for z_i, and since ϕ_i is linear we can say that

$$\begin{aligned} &|f(\Psi w) - f(\Psi z)| \\ &\quad = \left|\lim_{\mathcal{U}} f(\psi(w_i)) - \lim_{\mathcal{U}} f(\psi(z_i))\right| \\ &\quad \ge \lim_{\mathcal{U}}\left(\frac{1}{2}\lambda_i^{-1}\phi_i\left(\frac{\lambda_i}{3} w_i\right) - \frac{1}{2}\lambda_i^{-1}\phi_i\left(\frac{\lambda_i}{3} z_i\right) - |\text{error}_w - \text{error}_z|\right) \\ &\quad = \lim_{\mathcal{U}}\left(\tfrac{1}{6}\phi_i(w_i - z_i) - |\text{error}_w - \text{error}_z|\right) \\ &\quad = \lim_{\mathcal{U}}\left(\tfrac{1}{6}\|w_i - z_i\| - |\text{error}_w - \text{error}_z|\right), \end{aligned}$$

which will be greater than $\frac{1}{10}\|w - z\|$ provided the error terms are small enough. But the error terms may be bounded as a proportion of the distance between w and z by a Schwartz estimate as before in conjuntion with our choices of supports for the E_i. Note that

$$\begin{aligned} &|\text{error}_w - \text{error}_z| \\ &\quad = \sum_{j=1}^{i-1} \lambda_j^{-1}\phi_j((2^{1-1/n_i} - 1)x_i)\left\{\left(\frac{1 + x_j^*(\psi(w_i))}{2}\right)^{n_j} - \left(\frac{1 + x_j^*(\psi(z_i))}{2}\right)^{n_j}\right\}. \end{aligned}$$

Now by the choice of supports for the E_i we have $|\phi_j(x_i)| \le 2^{-6-j}\lambda_j$. By the Schwartz lemma, applied as before but now to the function $((1 + x_j^*)/2)^{n_j}$ (with a suitable constant added), which has norm 1 in B, we bound the error by $2^{-6}\|w - z\|$.

This shows that the map has uniformly continuous inverse and so completes the proof of Theorem 5.1. The same construction will work in the same way in $H^\infty(B)$. □

Some remarks are in order. First, let us note that nothing in the theorem changes if the sequence (x_i) converges weakly to a point in the interior of the ball. Thus we see that the same structure can be found in fibers over interior points. (This is Corollary 5.2.) We can check that this is an essentially different structure (in case $p = 2$; in other cases, it is trivially different) than that found in [CGJ] by noting that, in our construction (unlike theirs), the value of the radius function (see [ACG]) is equal to 1 at every point of the embedded ball.

Second, we note that the structure of the l_p-spaces is very important in this construction. Although something similar might be true for some wider class of spaces with very nice structure, it is not immediately clear how. But in spaces without unconditional structure of some sort, boundary slices tend to be shaped very oddly; although one could imagine finding some sort of topological nonstandard hull of bits of boundary slices in a fiber, it seems possible that nothing looking very much like an infinite-dimensional unit ball is there.

Third, this construction sheds some light on an interesting special case, namely that of the unit ball of Hilbert space. We now have identified several different kinds of Gleason parts, some in interior fibers and some over the boundary, which have copies of a ball of nonseparable Hilbert space in them. One wonders if the construction of [CGJ] could be combined with this one to find more, and if some sort of classification of parts is possible. Very little is known even in this “nicest possible” case.

ACKNOWLEDGMENTS. This project was initiated while the author was a participant in the Workshop in Linear Analysis and Probability at Texas A&M University in the summer of 1993 (supported by NSF DMS-9311902). The author would like to thank the referee for a number of helpful suggestions.

References

[ACG] R. M. Aron, B. J. Cole, and T. W. Gamelin, *Spectra of algebras of analytic functions on a Banach space,* J. Reine Angew. Math. 415 (1991), 51–93.

[CCG] T. K. Carne, B. J. Cole, and T. W. Gamelin, *A uniform algebra of analytic functions on a Banach space,* Trans. Amer. Math. Soc. 314 (1989), 639–659.

[CGJ] B. J. Cole, T. W. Gamelin, and W. B. Johnson, *Analytic disks in fibers over the unit ball of a Banach space,* Michigan Math. J. 39 (1992), 551–569.

[D] S. Dineen, *The Schwartz lemma,* Clarendon Press, Oxford, 1989.

[F] J. Farmer, *Polynomial reflexivity in Banach spaces,* Israel J. Math. 87 (1994), 257–273.

[FJ] J. Farmer and W. B. Johnson, *Polynomial Schur and polynomial Dunford–Pettis properties,* Contemp. Math., 144, Amer. Math. Soc., Providence, RI, 1993.

[GJL] J. Gutierrez, J. Jaramillo, and J. Llavona, *Polynomials and geometry of Banach spaces,* Extracta Math. 10 (1995), 79–114.

[He] S. Heinrich, *Ultraproducts in Banach space theory,* J. Reine Angew. Math. 313 (1980), 72–104.

[Ho] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, NJ, 1962 [reprinted Dover, New York, 1988].
[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function spaces,* Springer, New York, 1979.

Department of Mathematical Sciences
University of Northern Colorado
Greeley, CO 80639