Besov Spaces and Outer Functions

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1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}$ its boundary, and m the normalized Lebesgue measure on $\mathbb{T}$. For a function $f \in L^p$ $(:= L^p(\mathbb{T}, m))$, we define its L^p modulus of continuity by

$$\omega_p(t, f) := \sup_{-t \le h \le t} \left(\int_{\mathbb{T}} |f(e^{ih}\zeta) - f(\zeta)|^p \, dm(\zeta) \right)^{1/p}$$

for $0 \le t \le \pi$, and by

$$\omega_p(t, f) := \omega_p(\pi, f) \quad \text{for } \pi < t < \infty.$$

Further, given $0 < s < 1$, $0 < p < \infty$, and $0 < q < \infty$, the *Besov space* $B^s_{pq} = B^s_{pq}(\mathbb{T})$ is introduced as follows:

$$B^s_{pq} := \left\{ f \in L^p : \int_0^\infty \frac{\omega_p(t, f)^q}{t^{sq+1}} \, dt < \infty \right\}.$$

We shall mainly be concerned with the *analytic subspace*

$$AB^s_{pq} := B^s_{pq} \cap H^p,$$

where H^p is the classical Hardy space in the disk (see [9, Chap. II]). Alternatively, the class AB^s_{pq} can be described [15; 17] as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\mathbb{T}} |f'(r\zeta)|^p \, dm(\zeta) \right)^{q/p} dr < \infty. \tag{1.1}$$

We remark that there is also a natural way to define the spaces B^s_{pq} and AB^s_{pq} with $s \ge 1$, but these are not considered in the present paper.

The problem we treat here is to characterize (the boundary values of) the moduli of functions in AB^s_{pq}. Thus, we consider a nonnegative function $\varphi \in L^p$ with

$$\int_{\mathbb{T}} \log \varphi \, dm > -\infty \tag{1.2}$$

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(it is well known that (1.2) characterizes the moduli of nonzero H^p functions) and ask whether there exists an $f \in AB^s_{pq}$ such that $|f| = \varphi$ almost everywhere on $\mathbb{T}$. Equivalently, we have to ascertain when the *outer function* $\mathcal{O}_\varphi$, given by

$$\mathcal{O}_\varphi(z) := \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta)\, dm(\zeta) \right), \quad z \in \mathbb{D},$$

belongs to AB^s_{pq}. (The equivalence of the two settings is due to the fact that $|\mathcal{O}_\varphi| = \varphi$ a.e. on $\mathbb{T}$ and to the fact that the outer factor of a function in AB^s_{pq} must itself belong to AB^s_{pq}. The latter can be established along the lines of [10]; see also [11, Chap. III, Sec. 3.4].)

Yet another necessary condition, to be imposed on φ along with (1.2), is

$$\varphi \in B^s_{pq} \tag{1.3}$$

(just note that $\omega_p(t, |f|) \le \omega_p(t, f)$). However, (1.2) and (1.3) together are far from being sufficient to ensure that $\mathcal{O}_\varphi \in AB^s_{pq}$.

To make further discussion possible, we introduce some notation. Let μ_z stand for the harmonic measure representing a point $z \in \mathbb{D}$, so that

$$d\mu_z(\zeta) = \frac{1 - |z|^2}{|\zeta - z|^2}\, dm(\zeta), \quad \zeta \in \mathbb{T},$$

and let $\Psi(z, \varphi)$ be the function associated with a given $\varphi \in L^1$, $\varphi \ge 0$, via the formula

$$\Psi(z, \varphi) := \int_{\mathbb{T}} \varphi\, d\mu_z - \exp\left(\int_{\mathbb{T}} \log \varphi\, d\mu_z \right), \quad z \in \mathbb{D}$$

(in case (1.2) fails, it is understood that $\exp(-\infty) = 0$). Note that $\Psi(z, \varphi) \ge 0$ by Jensen's inequality. Finally, for a given $\varphi \in L^2$, $\varphi \ge 0$, we set

$$\Phi(z, \varphi) := \Psi(z, \varphi^2), \quad z \in \mathbb{D}.$$

In order to make the results of this paper look more natural, we now cite their prototypes that were previously obtained by the author for the Lipschitz spaces $\Lambda^\alpha := B^\alpha_{\infty\infty}$ with $0 < \alpha < 1$. More precisely, the Lipschitz space is defined by

$$\Lambda^\alpha = \{ f \in C(\mathbb{T}) : \omega_\infty(t, f) = O(t^\alpha) \},$$

where

$$\omega_\infty(t, f) := \sup\{ |f(e^{ih}\zeta) - f(\zeta)| : \zeta \in \mathbb{T},\ |h| \le t \}, \quad 0 \le t \le \pi.$$

The following Theorems A and B (see [6; 7] for the proofs) provide explicit characterizations of the outer functions in Λ^α in terms of their moduli.

THEOREM A. *Let $\varphi \in L^2$ be a nonnegative function satisfying* (1.2). *Then, for $0 < \alpha < \frac{1}{2}$, the following are equivalent*:

(i.A) $\mathcal{O}_\varphi \in \Lambda^\alpha$;
(ii.A) $\Phi(z, \varphi) = O((1 - |z|)^{2\alpha})$, $z \in \mathbb{D}$.

THEOREM B. *Let* $0 < \alpha < 1$, *and let* $\varphi \in \Lambda^\alpha$ *be a nonnegative function satisfying* (1.2). *The following are equivalent*:

(i.B) $\mathcal{O}_\varphi \in \Lambda^\alpha$.
(ii.B) $\Psi(z, \varphi) = O((1 - |z|)^\alpha)$, $z \in \mathbb{D}$.

When passing from Λ^α to general Besov spaces B^s_{pq}, it would be natural to expect that the desired membership criteria for $\mathcal{O}_\varphi$ might be obtained by replacing the uniform estimates (ii.A) and (ii.B) by suitable integral conditions on $\Phi(z, \varphi)$ and/or $\Psi(z, \varphi)$. Following the strategy of [6], we show that this is indeed the case and exhibit the appropriate integral conditions. However, the passage from "uniform smoothness" to the "smoothness in the mean" is no routine matter and requires a great deal of effort.

The rest of the paper is organized as follows. In Section 2, we provide a certain "BMO-type" characterization of AB^s_{pq} in terms of the mean oscillation $\left(\int |f - f(z)|^\sigma \, d\mu_z\right)^{1/\sigma}$, where σ is a new parameter.

In Section 3, we use this characterization, with $\sigma = 2$, to derive an integral version of Theorem A. It states that, in the case $0 < s < \frac{1}{2}$ and $p \geq 2$, the inclusion $\mathcal{O}_\varphi \in AB^s_{pq}$ is equivalent to the convergence of a certain integral involving $\Phi(z, \varphi)$. Further, we briefly discuss the inner–outer factorization of analytic functions in the Besov space. Also, combining our results with a lemma due to Aleman [1], we obtain, as a byproduct, the following fact: Given $p \geq 2$, $0 < s < \frac{1}{2}$, and $q > 0$, every function in AB^s_{pq} is the ratio of two bounded functions in the same class. (The case $p = q = 2$ was treated in [1].)

In Section 4, we cite (a special case of) a recent result of Shirokov [14] which gives an alternative description of outer functions in AB^s_{pq}, provided that

$$1 < p < \infty, \quad 1 \leq q < \infty, \quad \text{and} \quad 1/p < s < 1. \tag{1.4}$$

Shirokov's result enables us to prove the following auxiliary assertion: If p, q, and s satisfy (1.4), and if f is an outer function in $AB^{s/2}_{2p,2q}$ such that $|f|^2 \in B^s_{pq}$, then $f^2 \in AB^s_{pq}$. This last fact is in turn used, in conjunction with preceding results from Sections 2 and 3, to derive an integral analog of Theorem B. Namely, once p, q, and s are related by (1.4), the inclusion $\mathcal{O}_\varphi \in AB^s_{pq}$ is shown to be equivalent to an appropriate integrability condition on $\Psi(z, \varphi)$.

Finally, Section 5 contains a few concluding remarks and open questions.

2. BMO-Type Characterizations of Besov Spaces

THEOREM 2.1. *Let* $1 \leq \sigma \leq p < \infty$, $0 < q < \infty$, *and* $0 < s < 1/\sigma$. *Given a function* $f \in H^p$, *the following are equivalent*:

(i) $$f \in AB^s_{pq};$$

(ii) $$\int_0^1 \left\{ \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(\zeta) - f(r\eta)|^\sigma \, d\mu_{r\eta}(\zeta) \right)^{p/\sigma} dm(\eta) \right\}^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Proof. (ii) $\Rightarrow$ (i). For $z \in \mathbb{D}$, Cauchy's formula yields

$$\begin{aligned} |f'(z)| &= \frac{1}{1-|z|^2}\left|\frac{1}{2\pi i}\int_{\mathbb{T}} (f(\zeta)-f(z))\frac{1-|z|^2}{(\zeta-z)^2}\,d\zeta\right| \\ &\le \frac{1}{1-|z|}\int_{\mathbb{T}} |f(\zeta)-f(z)|\,d\mu_z(\zeta) \\ &\le \frac{1}{1-|z|}\left(\int_{\mathbb{T}} |f(\zeta)-f(z)|^\sigma\,d\mu_z(\zeta)\right)^{1/\sigma}. \end{aligned}$$

Consequently,

$$\int_{\mathbb{T}} |f'(r\eta)|^p\,dm(\eta) \le \frac{1}{(1-r)^p}\int_{\mathbb{T}}\left(\int_{\mathbb{T}} |f(\zeta)-f(r\eta)|^\sigma\,d\mu_{r\eta}(\zeta)\right)^{p/\sigma} dm(\eta).$$

Raising both sides to the power q/p, multiplying by $(1-r)^{(1-s)q-1}$, and integrating over r shows that condition (ii) implies (1.1) and hence also (i).

(i) $\Rightarrow$ (ii). For $0 < r < 1$, set

$$\Omega(r) := \left\{\int_{\mathbb{T}}\left(\int_{\mathbb{T}} |f(\zeta)-f(r\eta)|^\sigma\,d\mu_{r\eta}(\zeta)\right)^{p/\sigma} dm(\eta)\right\}^{1/p}.$$

Minkowski's inequality, applied twice, gives

$$\begin{aligned} \Omega(r) &\le \left\{\int_{\mathbb{T}}\left(\int_{\mathbb{T}} |f(\zeta)-f(\eta)|^\sigma\,d\mu_{r\eta}(\zeta)\right)^{p/\sigma} dm(\eta)\right\}^{1/p} \\ &\quad + \left\{\int_{\mathbb{T}} |f(\eta)-f(r\eta)|^p\,dm(\eta)\right\}^{1/p} \\ &=: \# + \flat. \end{aligned} \tag{2.1}$$

It is known [16] that

$$\flat = O(\omega_p(1-r, f)), \tag{2.2}$$

so we proceed by estimating the first term, #.

Once $r \in (0, 1)$ and $\eta \in \mathbb{T}$ are fixed, we have

$$\begin{aligned} \int_{\mathbb{T}} |f(\zeta)-f(\eta)|^\sigma\,d\mu_{r\eta}(\zeta) &= \int_{\mathbb{T}} |f(\xi\eta)-f(\eta)|^\sigma\,d\mu_r(\xi) \\ &= \sum_{k=0}^{N+1}\int_{I_k} |f(\xi\eta)-f(\eta)|^\sigma\,d\mu_r(\xi), \end{aligned} \tag{2.3}$$

where $N = N(r)$ is the integer such that

$$2^N < \frac{\pi}{1-r} \le 2^{N+1},$$

while the subsets $I_k \subset \mathbb{T}$ are defined by

$$\begin{gathered} I_0 := \{\, e^{ih} : |h| < 1-r \,\}, \\ I_k := \{\, e^{ih} : 2^{k-1}(1-r) \le |h| < 2^k(1-r) \,\} \quad (k = 1, \dots, N), \\ I_{N+1} := \{\, e^{ih} : 2^N(1-r) \le |h| \le \pi \,\}. \end{gathered}$$

Raising (2.3) to the power $1/\sigma$ and using the inequality $\|\cdot\|_{l^\sigma} \le \|\cdot\|_{l^1}$, we obtain

$$\left(\int_{\mathbb{T}} |f(\zeta) - f(\eta)|^\sigma \, d\mu_{r\eta}(\zeta)\right)^{1/\sigma} \le \sum_{k=0}^{N+1} \left(\int_{I_k} |f(\xi\eta) - f(\eta)|^\sigma \, d\mu_r(\xi)\right)^{1/\sigma}. \quad (2.4)$$

Hölder's inequality, together with the fact that

$$\mu_r(I_k) \le \text{const} \cdot 2^{-k}, \quad (2.5)$$

where the constant is independent of r and k, yields

$$\left(\int_{I_k} |f(\xi\eta) - f(\eta)|^\sigma \, d\mu_r(\xi)\right)^{1/\sigma} \le \text{const} \cdot 2^{-k(1/\sigma - 1/p)} \left(\int_{I_k} |f(\xi\eta) - f(\eta)|^p \, d\mu_r(\xi)\right)^{1/p}. \quad (2.6)$$

Substituting (2.6) into (2.4) gives

$$\left(\int_{\mathbb{T}} |f(\zeta) - f(\eta)|^\sigma \, d\mu_{r\eta}(\zeta)\right)^{1/\sigma} \le \text{const} \cdot \sum_{k=0}^{N+1} 2^{-k(1/\sigma - 1/p)} \left(\int_{I_k} |f(\xi\eta) - f(\eta)|^p \, d\mu_r(\xi)\right)^{1/p}.$$

Passing to L^p-norms with respect to $dm(\eta)$, we get

$$\# \le \text{const} \cdot \sum_{k=0}^{N+1} 2^{-k(1/\sigma - 1/p)} \left\{\int_{\mathbb{T}} dm(\eta) \int_{I_k} |f(\xi\eta) - f(\eta)|^p \, d\mu_r(\xi)\right\}^{1/p}. \quad (2.7)$$

We now look at the double integral $\{\dots\}$ on the right:

$$\{\dots\} = \int_{I_k} d\mu_r(\xi) \int_{\mathbb{T}} |f(\xi\eta) - f(\eta)|^p \, dm(\eta) \le \mu_r(I_k) \cdot \omega_p(2^k(1-r), f)^p \le \text{const} \cdot 2^{-k} \cdot \omega_p(2^k(1-r), f)^p$$

(we have once again used (2.5)). Thus

$$\{\dots\}^{1/p} \le \text{const} \cdot 2^{-k/p} \cdot \omega_p(2^k(1-r), f),$$

and so (2.7) implies

$$\# \le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-k/\sigma} \omega_p(2^k(1-r), f). \quad (2.8)$$

Comparing (2.1), (2.2), and (2.8) yields a similar estimate for $\Omega(r)$:

$$\Omega(r) \le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-k/\sigma} \omega_p(2^k(1-r), f). \quad (2.9)$$

Further, we must distinguish two cases.

Case 1: $0 < q < 1$. Using (2.9) and the elementary inequality

$$\left(\sum_j a_j\right)^q \le \sum_j a_j^q \quad (a_j \ge 0),$$

we obtain

$$\Omega(r)^q \le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-kq/\sigma} \omega_p(2^k(1-r),\, f)^q.$$

Hence

$$\int_0^1 \frac{\Omega(r)^q}{(1-r)^{sq+1}}\, dr \le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-kq/\sigma} \int_0^1 \frac{\omega_p(2^k t,\, f)^q}{t^{sq+1}}\, dt$$

$$\le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-kq/\sigma} J_k, \tag{2.10}$$

where

$$J_k := \int_0^{\infty} \frac{\omega_p(2^k t,\, f)^q}{t^{sq+1}}\, dt. \tag{2.11}$$

A change of variables gives

$$J_k = 2^{kqs} J_0, \tag{2.12}$$

whereas the integral J_0 converges by virtue of (i). In view of (2.12), (2.10) yields

$$\int_0^1 \frac{\Omega(r)^q}{(1-r)^{sq+1}}\, dr \le \text{const} \cdot J_0 \cdot \sum_{k=0}^{\infty} 2^{-kq(1/\sigma - s)} < \infty.$$

Case 2: $1 \le q < \infty$. Applying Minkowski's inequality, we deduce from (2.9) that

$$\left(\int_0^1 \frac{\Omega(r)^q}{(1-r)^{sq+1}}\, dr\right)^{1/q} \le \text{const} \cdot \sum_{k=0}^{\infty} 2^{-k/\sigma} J_k^{1/q}, \tag{2.13}$$

where J_k is again defined by (2.11). Now (2.12) shows that the right-hand side of (2.13) equals

$$\text{const} \cdot J_0^{1/q} \sum_{k=0}^{\infty} 2^{-k(1/\sigma - s)}$$

and is, therefore, finite (as long as (i) holds true).

Thus, in both cases we have

$$\int_0^1 \frac{\Omega(r)^q}{(1-r)^{sq+1}}\, dr < \infty,$$

which proves (ii). □

Having in mind some further applications, we point out two special cases of Theorem 2.1.

PROPOSITION 2.2. (a) *Let* $1 \le p < \infty$, $0 < q < \infty$, *and* $0 < s < 1$. *Given* $f \in H^p$, *one has* $f \in AB^s_{pq}$ *if and only if*

$$\int_0^1 \left\{ \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(\zeta) - f(r\eta)| \, d\mu_{r\eta}(\zeta) \right)^p dm(\eta) \right\}^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty. \quad (2.14)$$

(b) *Let* $2 \le p < \infty$, $0 < q < \infty$, *and* $0 < s < \frac{1}{2}$. *Given* $f \in H^p$, *one has* $f \in AB^s_{pq}$ *if and only if*

$$\int_0^1 \left\{ \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(\zeta)|^2 \, d\mu_{r\eta}(\zeta) - |f(r\eta)|^2 \right)^{p/2} dm(\eta) \right\}^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty. \quad (2.15)$$

Proof. To prove (a), apply Theorem 2.1 with $\sigma = 1$. To prove (b), set $\sigma = 2$ and note that

$$\int_{\mathbb{T}} |f - f(z)|^2 \, d\mu_z = \int_{\mathbb{T}} |f|^2 \, d\mu_z - |f(z)|^2, \quad z \in \mathbb{D}. \qquad \square$$

REMARKS. (1) In the case $p = q = 2$, Proposition 2.2(b) is implied by Proposition 2.4 in Aleman's paper [1]. However, the techniques of [1] are quite different from ours and do not appear suitable when dealing with the non-Hilbert case.

(2) Of course, Theorem 2.1 and Proposition 2.2 provide equivalent norms (or quasinorms, if $0 < q < 1$) on AB^s_{pq}, which are obtained by raising the left-hand sides of (ii), (2.14), and (2.15) to the power $1/q$.

3. On the Multiplicative Properties of Functions in AB^s_{pq}

In this section, we restrict ourselves to the case where

$$2 \le p < \infty, \quad 0 < q < \infty, \quad \text{and} \quad 0 < s < 1/2 \quad (3.1)$$

and derive several corollaries of Proposition 2.2(b). The first of these can be viewed as an integral version of Theorem A (see Section 1).

THEOREM 3.1. *Let* (3.1) *hold, and let* $\varphi \in L^p$ *be a nonnegative function satisfying* (1.2). *The following are equivalent*:

(i) $$\mathcal{O}_\varphi \in AB^s_{pq};$$

(ii) $$\int_0^1 \left(\int_{\mathbb{T}} \Phi(r\eta, \varphi)^{p/2} \, dm(\eta) \right)^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Proof. Apply Proposition 2.2(b) with $f = \mathcal{O}_\varphi$ and note that

$$\int_{\mathbb{T}} |\mathcal{O}_\varphi|^2 \, d\mu_z - |\mathcal{O}_\varphi(z)|^2 = \Phi(z, \varphi), \quad z \in \mathbb{D}. \qquad \square$$

Before stating our next result, we recall that an *inner function* is, by definition, an H^∞ function whose modulus equals 1 almost everywhere on $\mathbb{T}$.

THEOREM 3.2. *Assume that p, q, and s are as in* (3.1), *$f \in H^p$, and θ is an inner function. In order that $f\theta \in AB^s_{pq}$, it is necessary and sufficient that $f \in AB^s_{pq}$ and*

$$\int_0^1 \left\{ \int_{\mathbb{T}} |f(r\eta)|^p (1 - |\theta(r\eta)|)^{p/2} \, dm(\eta) \right\}^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty. \tag{3.2}$$

Proof. For $g \in H^2$, set

$$\Lambda_g(z) := \int_{\mathbb{T}} |g|^2 \, d\mu_z - |g(z)|^2, \quad z \in \mathbb{D}. \tag{3.3}$$

Further, let X^s_{pq} denote the set of all functions $h \in C(\mathbb{D})$, $h \ge 0$, for which

$$\int_0^1 \left(\int_{\mathbb{T}} h(r\eta)^{p/2} \, dm(\eta) \right)^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

By Proposition 2.2(b), one has $f\theta \in AB^s_{pq}$ if and only if

$$\Lambda_{f\theta} \in X^s_{pq}. \tag{3.4}$$

Writing

$$\Lambda_{f\theta}(z) = \Lambda_f(z) + |f(z)|^2 (1 - |\theta(z)|^2), \tag{3.5}$$

we see that (3.4) is equivalent to saying that both functions Λ_f and $|f|^2(1 - |\theta|^2)$ belong to X^s_{pq}. The first of these inclusions means, by Proposition 2.2(b), that $f \in AB^s_{pq}$, while the second one amounts to (3.2). □

Our next result generalizes a theorem of Aleman [1]. The method of proof is also borrowed from [1], except that the underlying Proposition 2.2(b) was proved differently from its counterpart in [1].

Given a nonzero $f \in H^p$, set

$$g_f(z) := \exp\left\{ -\int_{\{|f|>1\}} \frac{\zeta + z}{\zeta - z} \log|f(\zeta)| \, dm(\zeta) \right\}, \quad z \in \mathbb{D},$$

so that g_f is the outer function with modulus $\min(1, 1/|f|)$.

THEOREM 3.3. *Suppose* (3.1) *holds. If f is a nonzero function of class AB^s_{pq}, then so are the functions g_f, $1/g_f$, and fg_f. Moreover, their norms are bounded by a constant times the norm of f. (Here "the norm" means any reasonable norm, or quasinorm, on AB^s_{pq}.)*

We require the following fact (see Lemma 2.7 in [1]).

LEMMA A. *Let (X, μ) be a probability space, and let $\varphi \in L^1(X, \mu)$ be a nonnegative function with $\log \varphi \in L^1(X, \mu)$. Set*

$$E(\varphi) := \int_X \varphi \, d\mu - \exp\left(\int_X \log \varphi \, d\mu \right).$$

Then

$$E(\min(1, \varphi)) \le E(\varphi) \tag{3.6}$$

and

$$E(\max(1, \varphi)) \leq E(\varphi). \tag{3.7}$$

Proof of Theorem 3.3. Set $g = g_f$ and let $f = F\theta$ be the canonical factorization of f (here F is outer and θ is inner). Using the notation of (3.3), we have

$$\Lambda_{1/g}(z) = \Phi(z, \max(1, |f|)) \leq \Phi(z, |f|) = \Lambda_F(z) \leq \Lambda_f(z).$$

(We have used (3.7) with $\varphi = |f|^2$ and $(X, \mu) = (\mathbb{T}, \mu_z)$, and then (3.5) with f replaced by F.) The resulting inequality

$$\Lambda_{1/g}(z) \leq \Lambda_f(z), \quad z \in \mathbb{D}, \tag{3.8}$$

shows, in view of Proposition 2.2(b), that the hypothesis $f \in AB^s_{pq}$ yields $1/g \in AB^s_{pq}$.

This last inclusion implies, in turn, that $g \in AB^s_{pq}$. To see why, use the identity

$$g' = -g^2(1/g)', \tag{3.9}$$

the fact that $|g| \leq 1$ on $\mathbb{D}$, and the characterization (1.1) of AB^s_{pq}.

Finally, in order to check that $fg \in AB^s_{pq}$, we write

$$\Lambda_{fg}(z) = \Lambda_{Fg}(z) + |F(z)|^2|g(z)|^2(1 - |\theta(z)|^2). \tag{3.10}$$

Since $|g(z)| \leq 1$ and

$$\Lambda_{Fg}(z) = \Phi(z, \min(1, |f|)) \leq \Phi(z, |f|) = \Lambda_F(z)$$

(this time we have employed (3.6)), the relation (3.10) yields

$$\Lambda_{fg}(z) \leq \Lambda_F(z) + |F(z)|^2(1 - |\theta(z)|^2) = \Lambda_{F\theta}(z) = \Lambda_f(z).$$

Eventually, we obtain

$$\Lambda_{fg}(z) \leq \Lambda_f(z), \quad z \in \mathbb{D}, \tag{3.11}$$

and so, by Proposition 2.2(b), the hypothesis $f \in AB^s_{pq}$ is seen to imply that $fg \in AB^s_{pq}$.

The required inclusions are now verified, and the corresponding norm inequalities are, in fact, established as well. Actually, in light of Remark (2) at the end of Section 2, these inequalities are immediate from (3.8), (3.9), and (3.11). □

COROLLARY 3.4. *Under the assumption* (3.1), *every function in* AB^s_{pq} *is the ratio of two bounded functions in* AB^s_{pq}.

Proof. Given a nonzero $f \in AB^s_{pq}$, write $f = fg_f/g_f$; observe that $|fg_f| \leq 1$ and $|g_f| \leq 1$. □

REMARKS. (1) For $p > 2$, Corollary 3.4 gives a nontrivial result only if $0 < s \leq 1/p$; otherwise we have $B^s_{pq} \subset C(\mathbb{T})$.

(2) In the case $p = q = 2$, Theorem 3.3 and Corollary 3.4 are due to Aleman; see Theorem 2.6 and Corollary 2.8 in [1]. In fact, Aleman's results pertain to a somewhat more general situation (not encompassing, however, the non-Hilbert

spaces AB^s_{pq}). The Dirichlet space $AB^{1/2}_{2,2}$ was considered earlier by Richter and Shields [12].

(3) In connection with Theorem 3.2, we remark that "multiplication criteria" similar to (3.2) were obtained, by a different method, in [2, Sec. 3.3]. The topic involved (i.e., preservation of smoothness when multiplying or dividing by inner factors) is treated, among other places, in [3; 4; 5; 7]. See also the monograph [13] and the references therein.

4. Shirokov's Result and a Passage to Smaller *p* and Larger *s*

Let $p > 0$ and $q > 0$. In accordance with [14], we denote by $\mathbb{Q}^p_q$ the set of all functions $g(\zeta, h) \ge 0$, defined on $\mathbb{T} \times (0, 1)$, that have the following properties:

(a) For every $h \in (0, 1)$, one has $g(\cdot, h) \in L^p$ and, moreover,

$$\sup\left\{ \int_{\mathbb{T}} g(\zeta, h)^p \, dm(\zeta) : h \in (\varepsilon, 1) \right\} < \infty$$

whenever $0 < \varepsilon < 1$;

(b) $$\int_0^1 \left(\int_{\mathbb{T}} g(\zeta, h)^p \, dm(\zeta) \right)^{q/p} \frac{dh}{h} < \infty.$$

Further, given a function $\varphi \in C(\mathbb{T})$, $\varphi \ge 0$, and a point $z \in \mathbb{D}$, we set

$$M_\varphi(z) := \max\{ \varphi(\zeta) : \zeta \in \mathbb{T},\ |\zeta - z| \le 2(1 - |z|) \}.$$

The next result is due to Shirokov (in fact, a more general version is contained in Theorems 1, 2, and 3 of [14]).

THEOREM C. *Let*

$$1 < p < \infty, \quad 1 \le q < \infty, \quad \text{and} \quad 1/p < s < 1. \tag{4.1}$$

Given a nonnegative function $\varphi \in B^s_{pq}$ satisfying (1.2)*, the following are equivalent*:

(i.C) $$\mathcal{O}_\varphi \in AB^s_{pq}.$$

(ii.C) *There exist a function $F \in \mathbb{Q}^p_q$ and a constant $C > 0$ such that*

$$\int_{\mathbb{T}} \left| \log \frac{\varphi(\zeta)}{M_\varphi(z)} \right| d\mu_z(\zeta) \le C \tag{4.2}$$

whenever z is a point in $\mathbb{D} \setminus \{0\}$ for which

$$M_\varphi(z) \ge (1 - |z|)^s F\left(\frac{z}{|z|}, 1 - |z| \right).$$

From Theorem C we derive the following auxiliary proposition, to be employed later on.

COROLLARY 4.1. *Let p, q and s be as in* (4.1). *If f is an outer function in $AB^{s/2}_{2p,2q}$ such that $|f|^2 \in B^s_{pq}$, then $f^2 \in AB^s_{pq}$.*

Proof. Set $\varphi := |f|$, so that $f = \mathcal{O}_\varphi$. By Theorem C, the hypothesis $f \in AB^{s/2}_{2p,2q}$ can be restated by saying that (4.2) holds true for the points $z \in \mathbb{D} \setminus \{0\}$ satisfying

$$M_\varphi(z) \geq (1-|z|)^{s/2} F\left(\frac{z}{|z|}, 1-|z|\right),$$

where F is a suitable function in $\mathbb{Q}^{2p}_{2q}$. The latter condition is obviously equivalent to the requirement that

$$\int_{\mathbb{T}} \left| \log \frac{\varphi^2(\zeta)}{M_{\varphi^2}(z)} \right| d\mu_z(\zeta) \leq 2C$$

for those $z \in \mathbb{D} \setminus \{0\}$ for which

$$M_{\varphi^2}(z) \geq (1-|z|)^s F^2\left(\frac{z}{|z|}, 1-|z|\right).$$

Since $\varphi^2 \in B^s_{pq}$ (by assumption) and $F^2 \in \mathbb{Q}^p_q$ (because $F \in \mathbb{Q}^{2p}_{2q}$), another application of Theorem C yields $f^2 = \mathcal{O}_{\varphi^2} \in AB^s_{pq}$. □

We now combine Corollary 4.1 with preceding results from Sections 2 and 3 to obtain an integral version of Theorem B (see Section 1), stated in terms of the quantity

$$\Psi(z,\varphi) := \int_{\mathbb{T}} \varphi \, d\mu_z - \exp\left(\int_{\mathbb{T}} \log \varphi \, d\mu_z\right), \quad z \in \mathbb{D}.$$

THEOREM 4.2. *Assume that* (4.1) *holds and that $\varphi \in B^s_{pq}$ is a nonnegative function satisfying* (1.2). *Then the following are equivalent*:

(i) $$\mathcal{O}_\varphi \in AB^s_{pq};$$

(ii) $$\int_0^1 \left(\int_{\mathbb{T}} \Psi(r\eta, \varphi)^p \, dm(\eta) \right)^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Proof. (i) ⇒ (ii). Apply Proposition 2.2(a) with $f = \mathcal{O}_\varphi$ and observe that, for $z \in \mathbb{D}$,

$$\int_{\mathbb{T}} |\mathcal{O}_\varphi(\zeta) - \mathcal{O}_\varphi(z)| \, d\mu_z(\zeta) \geq \int_{\mathbb{T}} (|\mathcal{O}_\varphi(\zeta)| - |\mathcal{O}_\varphi(z)|) \, d\mu_z(\zeta) = \Psi(z,\varphi).$$

(ii) ⇒ (i). Set $\varphi_1 := \sqrt{\varphi}$ and rewrite (ii) in the form

$$\int_0^1 \left(\int_{\mathbb{T}} \Phi(r\eta, \varphi_1)^{2p/2} \, dm(\eta) \right)^{2q/2p} (1-r)^{-(s/2)\cdot 2q-1} \, dr < \infty. \tag{4.3}$$

Since $2p > 2$ and $s/2 < \frac{1}{2}$, Theorem 3.1 tells us that condition (4.3) is equivalent to the inclusion $\mathcal{O}_{\varphi_1} \in AB^{s/2}_{2p,2q}$. Recalling also that

$$|\mathcal{O}_{\varphi_1}|^2 = \varphi \in B^s_{pq}$$

and applying Corollary 4.1 with $f = \mathcal{O}_{\varphi_1}$, we obtain

$$\mathcal{O}^2_{\varphi_1} = \mathcal{O}_\varphi \in AB^s_{pq},$$

as desired. □

As before, the characterization we have just obtained enables us to derive certain information on the truncations of the functions in question.

COROLLARY 4.3. *If p, q, and s satisfy* (4.1), *and if f is an outer function of class AB^s_{pq}, then the three outer functions with moduli* $\max(1, |f|)$, $\min(1, |f|)$, *and* $\min(1, 1/|f|)$ *are also elements of AB^s_{pq}.*

The proof again relies on Lemma A and is completely similar to that of Theorem 3.3.

In order to state our final result, we introduce some more notation. Given p, q, and s as in (4.1) (so that $B^s_{pq} \subset C(\mathbb{T})$), we define the space $B^s_{pq}(\mathbb{D})$ to be the set of functions $f \in C(\operatorname{clos}\mathbb{D})$ having boundary values in $B^s_{pq} = B^s_{pq}(\mathbb{T})$ and satisfying

$$\int_0^1 \left(\int_{\mathbb{T}} |f(\eta) - f(r\eta)|^p \, dm(\eta) \right)^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Proceeding as in Section 2, it is not hard to show that the Poisson integral of a B^s_{pq} function always belongs to $B^s_{pq}(\mathbb{D})$. In other words, harmonic functions in $B^s_{pq}(\mathbb{D})$ are precisely the harmonic extensions of functions in B^s_{pq}, and that makes the notation reasonable.

This said, we are able to restate Theorem 4.2 in a very natural way.

THEOREM 4.4. *Suppose p, q, and s are related by* (4.1), *and let $\varphi \geq 0$ be a function in L^p satisfying* (1.2). *Consider the extension of φ into $\mathbb{D}$ given by*

$$\varphi(z) := \exp\left(\int_{\mathbb{T}} \log \varphi \, d\mu_z \right), \quad z \in \mathbb{D}.$$

In order that $\mathcal{O}_\varphi \in AB^s_{pq}$, it is necessary and sufficient that $\varphi \in B^s_{pq}(\mathbb{D})$.

Proof. Set $f = \mathcal{O}_\varphi$; observe that $\varphi = |f|$ everywhere on $\mathbb{D}$ and almost everywhere on $\mathbb{T}$. Now if $f \in AB^s_{pq}$, then f is the Poisson integral of $f|_{\mathbb{T}} \in B^s_{pq}$ and hence $f \in B^s_{pq}(\mathbb{D})$. The latter clearly implies that $\varphi \in B^s_{pq}(\mathbb{D})$.

Conversely, if $\varphi \in B^s_{pq}(\mathbb{D})$ then we write

$$\begin{aligned} \Psi(r\eta, \varphi) &= \int_{\mathbb{T}} \varphi \, d\mu_{r\eta} - \varphi(r\eta) \\ &\leq \left| \int_{\mathbb{T}} \varphi \, d\mu_{r\eta} - \varphi(\eta) \right| + |\varphi(\eta) - \varphi(r\eta)| \\ &=: h_1(r, \eta) + h_2(r, \eta), \end{aligned} \tag{4.4}$$

where $h_{1,2}(r, \eta)$ are just meant to denote the two terms, respectively. It remains to notice that

$$\int_0^1 \left(\int_{\mathbb{T}} h_j(r,\eta)^p \, dm(\eta) \right)^{q/p} \frac{dr}{(1-r)^{sq+1}} < \infty, \quad j = 1, 2. \tag{4.5}$$

Indeed, for $j = 1$, (4.5) actually means that the Poisson integral of the function $\varphi|_{\mathbb{T}} \in B^s_{pq}$ lies in $B^s_{pq}(\mathbb{D})$. For $j = 2$, (4.5) is due to the assumption that $\varphi \in B^s_{pq}(\mathbb{D})$.

Combining (4.4) and (4.5), we arrive at condition (ii) of Theorem 4.2, and the theorem tells us that $f \in AB^s_{pq}$. □

5. Concluding Remarks and Open Questions

(1) Throughout, we did not consider the endpoints $p = \infty$ and $q = \infty$. However, most of the above results remain true in these cases also, provided that we make some natural adjustments in the formulas involved.

Let us consider the case $q = \infty$ and describe the arising modifications, always assuming that p and s live in the same range as before (depending on the context). First of all, the space $B^s_{p\infty}$ is defined by

$$B^s_{p\infty} := \{ f \in L^p : \omega_p(t, f) = O(t^s) \},$$

while (1.1) should be changed to

$$\left(\int_{\mathbb{T}} |f'(r\zeta)|^p \, dm(\zeta) \right)^{1/p} = O((1-r)^{s-1}).$$

In Theorem 2.1, one replaces condition (ii) by

$$\int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(\zeta) - f(r\eta)|^\sigma \, d\mu_{r\eta}(\zeta) \right)^{p/\sigma} dm(\eta) = O((1-r)^{sp}) \tag{5.1}$$

(when $\sigma = 1$ or $\sigma = 2$, (5.1) gives the required version of condition (2.14) or (2.15), respectively, in Proposition 2.2). In Theorem 3.1, condition (ii) should be written in the form

$$\int_{\mathbb{T}} \Phi(r\eta, \varphi)^{p/2} \, dm(\eta) = O((1-r)^{sp}),$$

while Theorem 3.2 becomes valid for $q = \infty$ if one replaces (3.2) by

$$\int_{\mathbb{T}} |f(r\eta)|^p (1 - |\theta(r\eta)|)^{p/2} \, dm(\eta) = O((1-r)^{sp}).$$

Further, Theorem 3.3 and Corollary 3.4 hold true for $q = \infty$, no special changes being required, and so do Theorem C (with the appropriate interpretation of $\mathbb{Q}^p_\infty$) and Corollary 4.1. Finally, the $q = \infty$ version of condition (ii) in Theorem 4.2 reads

$$\int_{\mathbb{T}} \Psi(r\eta, \varphi)^p \, dm(\eta) = O((1-r)^{sp}),$$

while Corollary 4.3 and Theorem 4.4 remain intact.

In the case $p = \infty$, the results of Sections 2 and 3 become true if one similarly replaces the L^p-norm by the sup-norm. The Lipschitz case $p = q = \infty$ was studied in [6] and [7].

(2) Although Theorems 3.1 and 4.2 look quite similar, we note the following point of distinction between them: In Theorem 4.2 we assumed that $\varphi \in B^s_{pq}$ (which is obviously necessary to ensure that $\mathcal{C}_\varphi \in AB^s_{pq}$), whereas in Theorem 3.1 there were no a priori hypotheses on the smoothness of φ.

(3) The author suspects that Theorems 4.2 and 4.4 would remain valid if (4.1) were replaced by the wider range of indices

$$1 \le p \le \infty, \quad 1 \le q \le \infty, \quad \text{and} \quad 0 < s < 1. \tag{5.2}$$

The question would be settled if we could verify, under the assumptions (5.2), the statement of Corollary 4.1. This last task might probably be accomplished by means of the $\bar\partial$-techniques of Dyn'kin [8] (and without recourse to Shirokov's results). However, the author knows how to do it only in the Lipschitz case, where $p = q = \infty$ (see [6] and [7]).

(4) Assuming that (4.1) holds, it would be interesting to find a direct proof of the equivalence between condition (ii.C) in Theorem C and condition (ii) in Theorem 4.2.

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