Wandering Property in the Hardy Space

D. Khavinson, T. L. Lance, & M. I. Stessin

1. Introduction

Let X be a Hilbert space and let $V: X \to X$ be a bounded linear operator. If V is an isometry, then the well-known Wold decomposition theorem states that

$$X = X_0 \bigoplus_{n=0}^{\infty} V^n X_1, \tag{1}$$

where $X_1 = X \ominus VX$ is a wandering subspace and $X_0 = \bigcap_{n=0}^{\infty} V^n X$ [4]. If $X = H^2$ and V is the operator of multiplication by an inner function g, then the intersection $\bigcap_{n=0}^{\infty} V^n H^2 = \{0\}$ and the decomposition (1) implies that an orthonormal basis of $H^2 \ominus gH^2$, $\{s_1, \dots, s_n, \dots\}$, is a g-basis of H^2; that is, any function $f \in H^2$ can be written as

$$f(z) = \sum_{n=0}^{\infty} s_n(z) f_n(g(z)). \tag{2}$$

Any closed subspace $M \subset H^2$ that is invariant under multiplication by g could be considered as X, and therefore a relation similiar to (2) holds. We write this relation in the following form. Given a subset $A \subset H^2$, we denote by $[A]_g$ the minimal closed subspace of H^2 containing A that is invariant under multiplication by g. In our case the relation (1) could be written in these terms as follows. If M is invariant under multiplication by g, then

$$[M \ominus gM]_g = M. \tag{3}$$

It was shown in [5] that the relation (3) yields in this case some nice properties of functions from $M \ominus gM$ and leads to a generalization of classical canonical factorization. In general, (3) leads to description of multiplication invariant subspaces. In the case of the Bergman space in the unit disk, the validity of (3) when $g(z) = z$ was proved in [1].

In this paper we investigate the question when (3) holds if g is not inner. More precisely, let g be a bounded analytic function in the unit disk. We ask when (3) holds for any subspace $M \subset H^2$ that is invariant under multiplication by g. Our main result is the following.

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THEOREM. (i) *If* $g \in H^\infty$, $g(0) = 0$, *and* (3) *holds for any subspace* $M \subset H^2$ *that is invariant under multiplication by* g, *then there is a simply connected domain* $\Omega \subset \mathbf{C}$ *and an inner function* h *such that* $g = \varphi_\Omega \circ h$, *where* φ_Ω *is a Riemann mapping*

$$\varphi_\Omega: \Delta \to \Omega.$$

(ii) *If, in addition,* Ω *satisfies the condition that* φ_Ω *is a weak-$*$ generator of* H^∞ [7], *then condition* (i) *is also sufficient: for any inner function* h *and any* $M \subset H^2$ *that is invariant under multiplication by* $g = \varphi_\Omega \circ h$, *the relation* (3) *holds.*

COROLLARY 1. *Let* g *map the unit disc* Δ *onto itself. Then* g *is inner if and only if any* g*-invariant subspace* M *satisfies* (3).

An H^2-function f is said to be g-2-inner [5] if $\|f\|_{H^2} = 1$ and

$$\int_0^{2\pi} |f(e^{i\theta})|^2 (g(e^{i\theta}))^k \, d\theta = 0$$

for all $k = 1, 2, \dots$.

COROLLARY 2. *If* g *maps the unit disc* Δ *onto a bounded domain* Ω *such that the Riemann mapping* φ_Ω *is a weak-$*$ generator of* H^∞ *and any* g*-invariant subspace* M *satisfies* (3), *then the following equality holds for any* g*-2-inner function* f *and any polynomial* P:

$$\|f(z)P(g(z))\|_{H^2} = \|P(g(z))\|_{H^2}.$$

This paper is organized as follows. Section 2 contains results that deal with perturbations of reproducing kernels. These results are used in Section 3 to establish the theorem and corollaries just stated.

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2. Perturbation of Reproducing Kernels

PROPOSITION 1. *Let* $f(z)$ *be a holomorphic function in* Δ_r, *the closed disk of radius* r *centered at the origin. Let* $f(0) = 0$ *and* $|f'(z)| \ge A$ *in* Δ_r. *Then the range* $f(\Delta_r)$ *contains the disk of radius* Ar *centered at the origin.*

Proof. The result is straightforward. Indeed, let w_0 be a point in $\partial(f(\Delta_r))$ closest to the origin. Obviously, $|w_0| > 0$ and the whole straight interval from 0 to w_0 is in the range of f. Let γ be a curve that originates at the origin, terminates on the boundary of Δ_r, and satisfies $f(\gamma) = [0, w_0]$. Then

$$|w_0| = \int_\gamma |f'(z)| \, ds(z) \ge A\ell(\gamma) \ge Ar,$$

where ds is the arc-length differential on γ. □

Recall that if Ω is a bounded domain in $\mathbf{C}$ and if μ is a finite, nonnegative measure on $\bar{\Omega}$, then a point w is in bpe(μ) if the point evaluation at w is an $L^2(\mu)$-bounded linear functional on the set of analytic polynomials. In this case the Riesz theorem implies that there is an element $K_w(z)$ such that

$$f(w) = \int_{\bar{\Omega}} f(z)\overline{K_w(z)}\,d\mu(z)$$

for any polynomial f. If $w \in$ interior(bpe(μ)) and if point evaluations $K_w(z)$ depend antianalytically on w (and analytically on z) in a neighborhood of w, then $w \in$ abpe(μ). The following proposition can be found in [3, p. 63].

Proposition A. $\Omega \subset$ abpe(μ) *if and only if point evaluations are locally uniformly bounded in* Ω. *That is, for any* $w \in \Omega$ *there is a number* $r > 0$ *such that*

$$\sup\{\, \|K_z(\cdot)\|_{L^2(\Omega,\mu)} : |z - w| < r \,\} < \infty.$$

In the following two lemmas we assume that Ω and μ satisfy the following condition:

(a) $\Omega \subset$ abpe(μ).

Note that if Ω and μ satisfy the condition (a) and δ_w is a point mass at w, where $w \in \Omega$, then Ω and $\mu' = \mu + \lambda\delta_w$ satisfy (a) for any positive constant λ. Similarly, if Ω and μ satisfy (a), then Ω and $d\mu'(z) = \rho(z)d\mu(z)$ satisfy (a) for any positive step function $\rho(z)$ in $\bar{\Omega}$ that takes only a finite number of values.

We denote by $L^2_a(\Omega, \mu)$ the closed subspace of $P^2(\mu) =$ the closure of analytic polynomials in $L^2(\mu)$ (for a detailed description of $P^2(\mu)$ see [6; 8]), which consists of analytic functions in Ω. Thus the natural imbedding of $L^2_a(\Omega, \mu)$ into the space of all holomorphic functions in Ω is injective. Let $K_\mu(z, w)$ stand for the reproducing kernel of $L^2_a(\Omega, \mu)$. The kernel K_μ is analytic in z and antianalytic in w.

Lemma 1. *Let* $0 \in \Omega$, *and let* Ω *satisfy condition* (a) *with* μ *continuous* (*i.e., the measure of a point is zero*). *If there is a point* $\tau \in$ supp(μ) *in the interior of* Ω, *then there exist* $\tau_1, \tau_2 \in$ supp$(\mu) \cap \Omega$, $a \geq 1$, *and* $\alpha_1, \alpha_2 > 0$ $(\alpha_1 + \alpha_2 = 1)$ *such that the kernel* $K_{\mu'}(z, 0)$ *has zeros in* Ω, *where* μ' *is the measure*

$$\mu' = \frac{1}{a}\mu + \left(1 - \frac{1}{a}\right)(\alpha_1\delta_{\tau_1} + \alpha_2\delta_{\tau_2})$$

and δ_w *is the unit point mass at* w.

Proof. If $K_\mu(z, 0)$ has zeros in Ω, then set $a = 1$, $\alpha_1 = \alpha_2 = 1/2$, and $\tau_1 = \tau_2 = \tau$. Otherwise, denote by $\hat{\mu}$ the following measure:

$$\hat{\mu} = \frac{1}{a}\mu + \left(1 - \frac{1}{a}\right)\delta_\tau.$$

If f is a polynomial then

$$f(z) = \int_{\bar{\Omega}} f(w)\overline{K_{\hat{\mu}}(w,z)}\,d\hat{\mu}(w)$$
$$= \frac{1}{a}\int_{\bar{\Omega}} f(w)\overline{K_{\hat{\mu}}(w,z)}\,d\mu(w) + \left(1-\frac{1}{a}\right)f(\tau)\overline{K_{\hat{\mu}}(\tau,z)}.$$

Thus, if $f(\tau)=0$ we obtain

$$f(z) = \frac{1}{a}\int_{\bar{\Omega}} f(w)\overline{K_{\hat{\mu}}(w,z)}\,d\mu(w)$$
$$= \int_{\bar{\Omega}} f(w)\overline{K_{\mu}(w,z)}\,d\mu(w). \tag{4}$$

The relation (4) yields

$$K_{\hat{\mu}}(z,w) = aK_{\mu}(z,w) + \lambda(w)K_{\mu}(z,\tau).$$

Since the kernel is antisymmetric, we can easily obtain

$$K_{\hat{\mu}}(z,w) = aK_{\mu}(z,w) + \lambda K_{\mu}(z,\tau)K_{\mu}(\tau,w), \tag{5}$$

where λ is a constant. An application of (5) to $f(z)=1$ gives

$$\lambda = -\frac{a(a-1)}{1+(a-1)K_{\mu}(\tau,\tau)},$$

and therefore

$$K_{\hat{\mu}}(z,w) = \frac{a}{1+(a-1)K_{\mu}(\tau,\tau)} \times \big[K_{\mu}(z,w) + (a-1)\big(K_{\mu}(z,w)K_{\mu}(\tau,\tau) - K_{\mu}(z,\tau)K_{\mu}(\tau,w)\big)\big]. \tag{6}$$

Thus,

$$K_{\hat{\mu}}(z,0) = \frac{a}{1+(a-1)K_{\mu}(\tau,\tau)} \times \big[K_{\mu}(z,0) + (a-1)\big(K_{\mu}(z,0)K_{\mu}(\tau,\tau) - K_{\mu}(z,\tau)K_{\mu}(\tau,0)\big)\big]$$

and

$$K_{\hat{\mu}}(\tau,0) = \frac{aK_{\mu}(\tau,0)}{1+(a-1)K_{\mu}(\tau,\tau)}. \tag{7}$$

Further,

$$\frac{d}{dz}K_{\hat{\mu}}(z,0) = \frac{a}{1+(a-1)K_{\mu}(\tau,\tau)} \times \left[\frac{d}{dz}K_{\mu}(z,0) + (a-1)\left(\frac{d}{dz}K_{\mu}(z,0)K_{\mu}(\tau,\tau) - \frac{d}{dz}K_{\mu}(z,\tau)K_{\mu}(\tau,0)\right)\right]. \tag{8}$$

If

$$\frac{d}{dz}K_\mu(z,0)\Big|_{z=\tau}K_\mu(\tau,\tau)-\frac{d}{dz}K_\mu(z,\tau)\Big|_{z=\tau}K_\mu(\tau,0)\neq 0, \tag{9}$$

then it follows from (8) that there is a neighborhood $\mathcal{O}_\varepsilon(\tau)$ where

$$\left|\frac{d}{dz}K_{\hat\mu}(z,0)\right|\to\infty \quad \text{as } a\to\infty \tag{10}$$

uniformly in $\mathcal{O}_\varepsilon(\tau)$. Note that, by (7), we have

$$K_{\hat\mu}(\tau,0)\to\frac{K_\mu(\tau,0)}{K_\mu(\tau,\tau)} \quad \text{as } a\to\infty. \tag{11}$$

Now Proposition 1, (10), and (11) together imply that, if there is a $\tau\in\operatorname{supp}(\mu)\cap\Omega$ such that (9) holds, then $K_{\hat\mu}(z,0)$ has zeros in Ω if a is sufficiently large. In this case we set $\lambda_1=1$, $\lambda_2=0$, and $\mu'=\hat\mu$. Finally, suppose that

$$\frac{d}{dz}K_\mu(z,0)\Big|_{z=\tau}K_\mu(\tau,\tau)-\frac{d}{dz}K_\mu(z,\tau)\Big|_{z=\tau}K_\mu(\tau,0)=0 \tag{12}$$

for all $\tau\in\operatorname{supp}(\mu)\cap\Omega$. Let us show that there are $\nu,\tau\in\operatorname{supp}(\mu)\cap\Omega$ such that the measure $\tilde\mu=\frac{1}{b}\mu+\frac{b-1}{b}\delta_\nu$ satisfies (9) for some $b>1$; that is,

$$\frac{d}{dz}K_{\tilde\mu}(z,0)\Big|_{z=\tau}K_{\tilde\mu}(\tau,\tau)-\frac{d}{dz}K_{\tilde\mu}(z,\tau)\Big|_{z=\tau}K_{\tilde\mu}(\tau,0)\neq 0. \tag{13}$$

Suppose that (13) does not hold. To simplify the notation let us write $K(z,w)$ for $K_\mu(z,w)$ and $K_z(u,w)$ for $\frac{d}{dz}K_\mu(z,w)\big|_{z=u}$. We also denote by D the constant

$$D=\frac{b}{1+(b-1)K(\nu,\nu)}.$$

By (6) we have

$$K_{\tilde\mu}(z,w)=D\big[K(z,w)+(b-1)\big(K(z,w)K(\nu,\nu)-K(z,\nu)K(\nu,w)\big)\big].$$

A direct computation shows that (12) yields

$$\begin{aligned}0&=\frac{d}{dz}K_{\tilde\mu}(z,0)\Big|_{z=\tau}K_{\tilde\mu}(\tau,\tau)-\frac{d}{dz}K_{\tilde\mu}(z,\tau)\Big|_{z=\tau}K_{\tilde\mu}(\tau,0)\\&=D^2\big\{(b-1)\big[K(\nu,0)\big(K_z(\tau,\tau)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,\tau)\big)\\&\qquad-K(\nu,\tau)\big(K_z(\tau,0)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,0)\big)\big]\\&\quad+(b-1)^2K(\nu,\nu)\big[K(\nu,0)\big(K_z(\tau,\tau)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,\tau)\big)\\&\qquad-K(\nu,\tau)\big(K_z(\tau,0)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,0)\big)\big]\big\}.\end{aligned}$$

Since $b>1$ is arbitrary, the last relation implies that

$$\begin{aligned}&K(\nu,0)\big(K_z(\tau,\tau)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,\tau)\big)\\&\quad-K(\nu,\tau)\big(K_z(\tau,0)K(\tau,\nu)-K_z(\tau,\nu)K(\tau,0)\big)=0,\end{aligned} \tag{14}$$

and (14) holds for all $\tau,\nu \in \operatorname{supp}(\mu)\cap\Omega$. By (12) we have

$$K_z(\tau,\tau) = \frac{K_z(\tau,0)K(\tau,\tau)}{K(\tau,0)} \tag{15}$$

(recall that we assumed $K(z,0)$ does not vanish in Ω). Now (14) and (15) yield

$$\big(K_z(\tau,0)K(\tau,\nu) - K_z(\tau,\nu)K(\tau,0)\big) \times \big(K(\tau,\tau)K(\nu,0) - K(\nu,\tau)K(\tau,0)\big) = 0. \tag{16}$$

If

$$K(\tau,\tau)K(\nu,0) - K(\nu,\tau)K(\tau,0) = 0 \tag{17}$$

then

$$K(\nu,\tau) = \lambda K(\nu,0), \tag{18}$$

where λ is independent of ν. Fix τ. If (17) holds for infinitely many $\nu \in \mathcal{O}_\varepsilon(\tau)$ then, by the uniqueness theorem, (18) implies

$$K(z,\tau) = \lambda K(z,0), \quad z \in \Omega;$$

for any polynomial P we would then have

$$P(\tau) - \bar{\lambda}P(0) = \int_{\bar{\Omega}} P(z)\big(\overline{K(z,\tau)} - \bar{\lambda}\overline{K(z,0)}\big)\,d\mu(z) = 0, \tag{19}$$

a contradiction. Therefore, we conclude by (16) and the last argument that

$$K_z(\tau,0)K(\tau,\nu) - K_z(\tau,\nu)K(\tau,0) = 0$$

for all $\tau,\nu \in \operatorname{supp}(\mu)\cap\Omega$. This implies

$$\frac{d}{dz}\left(\frac{K(z,\nu)}{K(z,0)}\right)\bigg|_{z=\tau} = 0 \quad \text{for all } \tau,\nu \in \operatorname{supp}(\mu)\cap\Omega.$$

Fix ν. Because $K(z,\nu)/K(z,0)$ is analytic, we obtain $K(z,\nu) = \lambda K(z,0)$ for some constant λ. The same argument with the integral (19) leads to a contradiction. □

LEMMA 2. *Let μ be a continuous probability measure such that Ω and μ satisfy condition* (a). *Suppose that there is a point $\tau \in \operatorname{supp}(\mu)\cap\Omega$. Then there is a positive step function $\rho(z)$ in $\bar{\Omega}$ that takes a finite number of values and satisfies the following condition: the reproducing kernel $K_{\mu'}(z,0)$ has zeros in Ω, where $d\mu'(z) = \rho(z)\,d\mu(z)$.*

Proof. By Lemma 1, there exist $\tau_1,\tau_2 \in \operatorname{supp}(\mu)\cap\Omega$ and a pair of nonnegative numbers λ_1,λ_2 such that $K_{\tilde{\mu}}(z,0)$ has zeros in Ω, where

$$\tilde{\mu} = \mu + \lambda_1\delta_{\tau_1} + \lambda_2\delta_{\tau_2}.$$

Let $\varepsilon_n \to 0$ as $n \to \infty$. Write

$$\rho_n(z) = 1 + \frac{\lambda_1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_1))}\chi_{\mathcal{O}_{\varepsilon_n}(\tau_1)}(z) + \frac{\lambda_2}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_2))}\chi_{\mathcal{O}_{\varepsilon_n}(\tau_2)}(z),$$
$$d\mu_n(z) = \rho_n(z)d\mu(z),$$

where χ_A stands for the characteristic function of a set A. Note that Ω is in both $\operatorname{abpe}(\tilde{\mu})$ and $\operatorname{abpe}(\mu_n)$. This follows from the following simple fact: $\Omega \subset \operatorname{abpe}(\mu)$ and $\mu_1 \geq 0$ imply that $\mu_2 = \mu + \mu_1$ satisfies $\Omega \subset \operatorname{abpe}(\mu_2)$. Moreover, for every $w \in \Omega$ we have

$$\|K_{\mu_2}(z, w)\|_{L^2(\Omega,\mu_2)} \leq \|K_\mu(z, w)\|_{L^2(\Omega,\mu)}, \tag{20}$$

where the kernels in (20) are considered as functions of z. Further, the direct application of (20) yields

$$\begin{aligned}
&\left|K_{\mu_n}(\tau_i, w) - \frac{1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_i))} \int_{\bar{\Omega}} K_{\mu_n}(z, w) \chi_{\mathcal{O}_{\varepsilon_n}(\tau_i)}(z)\, d\mu(z)\right| \\
&\quad \leq \frac{1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_i))} \int_{\bar{\Omega}} \left|K_{\mu_n}(\tau_i, w) - K_{\mu_n}(z, w)\right| \chi_{\mathcal{O}_{\varepsilon_n}(\tau_i)}(z)\, d\mu(z) \\
&\quad \leq \sup_{z \in \mathcal{O}_{\varepsilon_n}(\tau_i)} \left|K_{\mu_n}(\tau_i, w) - K_{\mu_n}(z, w)\right| \leq \varepsilon_n \sup_{z \in \mathcal{O}_{\varepsilon_n}(\tau_i)} \left|\frac{\partial}{\partial z} K_{\mu_n}(z, w)\right| \\
&\quad \leq \varepsilon_n C(w) \|K_{\mu_n}(z, w)\|_{L^2(\Omega,\mu_n)} \\
&\quad \leq \varepsilon_n C(w) \|K_\mu(z, w)\|_{L^2(\Omega,\mu)} = \varepsilon_n C_1(w),
\end{aligned} \tag{21}$$

where $i = 1, 2$ and $C(w)$, $C_1(w)$ are constants that depend only on the distance from w to $\partial\Omega$. For any function f that is analytic in $\bar{\Omega}$, we have

$$\begin{aligned}
&\left|\frac{1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_i)} \int_{\bar{\Omega}} \overline{K_{\mu_n}(z, w)} \chi_{\mathcal{O}_{\varepsilon_n}(\tau_i)}(z) f(z)\, d\mu(z)\right. \\
&\quad \left. - \frac{f(\tau_i)}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_i))} \int_{\bar{\Omega}} \overline{K_{\mu_n}(z, w)} \chi_{\mathcal{O}_{\varepsilon_n}(\tau_i)}(z)\, d\mu(z)\right| \leq \varepsilon_n C_2(w) \|f\|_{L^2(\Omega,\mu)},
\end{aligned} \tag{22}$$

where, as in (21), the constant $C_2(w)$ depends only on the distance from w to $\partial\Omega$. Finally, (20), (21), and (22) yield

$$\begin{aligned}
&\left|\int_{\bar{\Omega}} \big(K_{\tilde{\mu}}(z, w) - K_{\mu_n}(z, w)\big) \overline{f(z)}\, d\tilde{\mu}(z)\right| \\
&\quad = \left|\int_{\bar{\Omega}} K_{\mu_n}(z, w) \overline{f(z)} (d\mu_n(z) - d\tilde{\mu}(z))\right| \\
&\quad = \left|\frac{\lambda_1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_1))} \int_{\bar{\Omega}} K_{\mu_n}(z, w) \chi_{\mathcal{O}_{\varepsilon_n}(\tau_1)}(z) \overline{f(z)}\, d\mu(z)\right. \\
&\qquad + \frac{\lambda_2}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_2))} \int_{\bar{\Omega}} K_{\mu_n}(z, w) \chi_{\mathcal{O}_{\varepsilon_n}(\tau_2)}(z) \overline{f(z)}\, d\mu(z) \\
&\qquad \left. - \lambda_1 K_{\mu_n}(\tau_1, w) \overline{f(\tau_1)} - \lambda_2 K_{\mu_n}(\tau_2, w) \overline{f(\tau_2)}\right| \\
&\quad \leq \lambda_1 |f(\tau_1)| \left|K_{\mu_n}(\tau_1, w) - \frac{1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_1))} \int_{\bar{\Omega}} K_{\mu_n}(z, w) \chi_{\mathcal{O}_{\varepsilon_n}(\tau_1)}(z)\, d\mu(z)\right| \\
&\qquad + \lambda_2 |f(\tau_2)| \left|K_{\mu_n}(\tau_2, w) - \frac{1}{\mu(\mathcal{O}_{\varepsilon_n}(\tau_2))} \int_{\bar{\Omega}} K_{\mu_n}(z, w) \chi_{\mathcal{O}_{\varepsilon_n}(\tau_2)}(z)\, d\mu(z)\right|
\end{aligned}$$

$$+ \varepsilon_n(\lambda_1 + \lambda_2)C_2(w)\|f\|_{L^2(\Omega,\tilde{\mu})}$$
$$\leq \varepsilon_n\big((\lambda_1|f(\tau_1)| + \lambda_2|f(\tau_2)|)C_1(w) + (\lambda_1 + \lambda_2)C_2(w)\|f\|_{L^2(\Omega,\tilde{\mu})}\big)$$
$$\leq \varepsilon_n C_3(w)\|f\|_{L^2(\Omega,\tilde{\mu})}. \tag{23}$$

Now, (23) implies that

$$\|K_{\mu_n}(z,w) - K_{\tilde{\mu}}(z,w)\|_{L^2(\Omega,\tilde{\mu})} \to 0 \quad \text{as } n \to \infty \tag{24}$$

for all w. Therefore, $K_{\mu_n}(z,w) \to K_{\tilde{\mu}}(z,w)$ uniformly on compacta. Since $K_{\tilde{\mu}}(z,0)$ has zeros in Ω, the same is true for $K_{\mu_n}(z,0)$ if n is sufficiently large.

□

3. Proof of the Main Theorem

To prove statement (i) of the theorem, it is enough to show that even the weaker property

$$\big[[f]_g \ominus g[f]_g\big]_g = [f]_g \quad \text{for all } f \in H^2 \tag{25}$$

implies $g = \varphi_\Omega \circ h$, where h is inner and φ_Ω is a Riemann mapping of the unit disc onto a simply connected domain Ω. (Recall that $[f]_g$ is the minimal closed subspace of H^2 that contains f and is invariant under multiplication by g.)

Suppose that (25) holds for all $f \in H^2$. Let g map the unit disc Δ onto a domain Ω. Denote by μ the pullback measure on $\bar{\Omega}$ that is generated by g in the following way. Since g is bounded, it has boundary values almost everywhere with respect to the normalized Lebesgue measure m on the unit circle. We denote the boundary values by the same letter g. If A is a Borel subset of $\bar{\Omega}$, define $\mu(A)$ by

$$\mu(A) = m\big(\{z \in \mathbf{T} : g(z) \in A\}\big), \tag{26}$$

where $\mathbf{T}$ stands for the unit circle. It is easy to see that Ω and μ satisfy condition (a) of the previous section. Indeed, let f be a polynomial and $w \in \Omega$. If $\tau \in \Delta$ is a preimage of w under g, then we have

$$\begin{aligned} |f(w)| = |f(g(\tau))| &\leq \frac{1}{\sqrt{1-|\tau|^2}}\|f \circ g\|_{H^2} \\ &= \frac{1}{\sqrt{1-|\tau|^2}}\left(\int_{\mathbf{T}} |f(g(z))|^2\,dm(z)\right)^{1/2} \\ &= \frac{1}{\sqrt{1-|\tau|^2}}\left(\int_{\bar{\Omega}} |f(z)|^2\,d\mu(z)\right)^{1/2} \\ &= \frac{1}{\sqrt{1-|\tau|^2}}\|f\|_{L^2(\Omega,\mu)}. \end{aligned}$$

Since g is an open mapping, the last estimate is locally uniform. This shows that $\Omega \subset \operatorname{abpe}(\mu)$.

Suppose that g maps some subset $A \subset \mathbf{T}$ of positive linear measure into Ω. Then there is a point $\tau \in \operatorname{supp}(\mu) \cap \Omega$. By Lemma 2, there is a positive step function $\rho(z)$ such that $K_{\mu'}(z,0)$ has zeros in Ω. Define the function $f(z)$ in the unit disc by

$$f(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\sqrt{\rho(g(e^{i\theta}))}\,d\theta\right\}. \tag{27}$$

Then $f \in H^\infty$ and, for boundary values of f, we have

$$|f(e^{i\theta})| = \sqrt{\rho(g(e^{i\theta}))}.$$

Now, if $\varphi_i = fP_i(g)$, where P_i are polynomials ($i = 1, 2$), then

$$\begin{aligned}\langle\varphi_1, \varphi_2\rangle_{H^2} &= \frac{1}{2\pi}\int_0^{2\pi} P_1(g(e^{i\theta}))\overline{P_2(g(e^{i\theta}))}|f(e^{i\theta})|^2\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_1(g(e^{i\theta}))\overline{P_2(g(e^{i\theta}))}\rho(g(e^{i\theta}))\,d\theta \\ &= \frac{1}{2\pi}\int_{\bar\Omega} P_1(z)\overline{P_2(z)}\,d\mu'(z).\end{aligned}$$

Thus, $[f]_g \ominus g[f]_g$ is spanned by the function $\psi = f \cdot R(g)$, where R is in $L^2_a(\Omega, \mu')$ and orthogonal to all functions that vanish at the origin. This implies that $R(z) = K_{\mu'}(z, 0)$. Since this kernel has zeros in Ω, this yields that the closed subspace of H^2 spanned by $\{f(z)K_{\mu'}(g(z), 0)g(z)^k,\ k = 0, 1, \dots\}$ consists of functions that have zeros in Δ and hence cannot contain f, a contradiction. Therefore, the (linear Lebesgue) measure of the set of points of the unit circle that are mapped by g into the interior of Ω is equal to zero. Moreover, the preceding argument shows that there are no points in $\operatorname{supp}(\mu)$ that are in the interior of the component of $\operatorname{abpe}(\mu)$ containing Ω (we denote this component by $\hat\Omega$). Indeed, any function from $L^2_a(\Omega, \mu)$ can be analytically extended to $\hat\Omega$. Apply the same argument as before to $L^2_a(\hat\Omega, \mu)$. We obtain that there is an invertible function $f \in H^2$ given by (27) such that the reproducing kernel for the measure μ' on $\hat\Omega$ associated with f has zeros in $\hat\Omega$. This implies that $K_{\mu'}(z, 0)$ is not a cyclic element of $L^2_a(\hat\Omega, \mu)$. In particular, a constant function is not in the z-invariant subspace of $L^2_a(\hat\Omega, \mu)$ generated by $K_{\mu'}(z, 0)$. This is equivalent to the fact that f is not in the closed subspace of H^2 spanned by $\{f(z)K_{\mu'}(g(z), 0)g(z)^k,\ k = 0, 1, \dots\}$.

Let $\varphi_{\hat\Omega}^{-1}$ be the inverse Riemann mapping that maps $\hat\Omega$ onto the unit disc Δ. The foregoing argument shows that the composition mapping

$$h = \varphi_{\hat\Omega}^{-1} \circ g \colon \Delta \to \Delta$$

is bounded and that $|h| = 1$ a.e. on the unit circle. Since g is not a constant, h is an inner function and $g = \varphi_\Omega \circ h$. The statement (i) is proved.

To prove (ii) we suppose that $g = \varphi_\Omega \circ h$, where h is inner, $h(0) = 0$, and $\varphi_\Omega \colon \Delta \to \Omega$ is the Riemann mapping, $\varphi_\Omega(0) = 0$. If φ_Ω is a weak-$*$ generator of H^∞ then it is easily seen that, for any subset $M \subset H^2$, $[M]_g = [M]_h$. In particular, any g-invariant subspace of H^2 is h-invariant. Moreover, $gL = hL$ for any g-invariant subspace L of H^2. Thus, for any g-invariant subspace L, we have

$$[L \ominus gL]_g = [L \ominus gL]_h = [L \ominus hL]_h = L,$$

where the last equality follows from the validity of (3) for inner functions. □

REMARK 1. Bercovici [2] asked whether the above weak-$*$ generator of H^∞ condition is also necessary: if the wandering property (3) holds, and $g = \varphi_\Omega \circ h$, then φ_Ω is a weak-$*$ generator of H^∞. We know of no counterexamples to this conjecture.

REMARK 2. It follows from the proof just given that the condition $g(0) = 0$ in the statement of the main theorem could be replaced with "zero is in the range of g". This remark makes Corollary 1 straightforward.

To prove Corollary 2 we note that, if φ_Ω is a weak-$*$ generator of H^∞, then any g-2-inner function is h-2-inner. Further, for any polynomial in g there is a bounded analytic function F in the unit disk such that $P(g(z)) = F(h(z))$, $z \in \Delta$. Now Corollary 2 follows from the similar property for inner functions [5, Prop. 8].

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D. Khavinson
Department of Mathematical Sciences
University of Arkansas at Fayetteville
Fayetteville, AR 72701

dmitry@comp.uark.edu

T. L. Lance
Department of Mathematics and Statistics
State University of New York at Albany
Albany, NY 12222

lance@math.albany.edu

M. I. Stessin
Department of Mathematics and Statistics
State University of New York at Albany
Albany, NY 12222

stessin@math.albany.edu