Singular Factors are Rare

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Let ϕ be an analytic function in the Hardy space H^p on the open unit disc $\Delta = \{z : |z| < 1\}$ for $0 < p \leq \infty$. It is classical that ϕ has a factorization $\phi = BSF$, where B is a Blaschke product, S is a singular function, and F is an outer function. Specifically, these factors are

$$B(z) = z^m \Pi \mu_k \frac{z_k - z}{1 - z\bar{z}_k}, \quad \mu_k = \frac{|z_k|}{z_k},$$

where m is the order of the zero of ϕ at the origin and $z_1, z_2, \dots$ are the zeros of ϕ in $\Delta \setminus \{0\}$;

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\nu(t)\right\},$$

where ν is a nonnegative measure singular with respect to Lebesgue measure; and

$$F(z) = \lambda \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|\phi(e^{it})|\, dt\right\},$$

where λ is a unimodular constant. See [2] for a full description of these functions and their properties.

It is a well-known theorem of Frostman that if ϕ is an inner function, then $[w - \phi(z)]/[1 - \bar{w}\phi(z)]$ is a Blaschke product for all $w \in \Delta$ with the exception at most of a set of capacity zero.

Caughran and Shields [1] raised this question: How big is the set of complex numbers w such that $\phi(z) - w$ has a nontrivial singular function as a factor? They showed that if ϕ' is in the Hardy space H^1 then the set of such w is countable. Fisher [3] showed, with no assumption on ϕ', that the set of w for which the singular function has an atom in its associated measure is countable.

A theorem due to Rudin [8, Thm. 4] can be used to give an answer to the above question. The theorem deals with functions of n complex variables, but we will use the restriction to $n = 1$. The class N_* is defined (in [8]) to be the set of all analytic functions f on the unit disc such that the functions $\log^+|f_r|$ have uniformly absolutely continuous integrals. (Here f_r is defined by $f_r(z) = f(rz)$, $z \in \Delta$, and $r \in (0, 1)$.) What this means explicitly is that for each $\varepsilon > 0$ there should exist a $\delta > 0$ such that

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$$\int_A \log^+|f(rw)|\,dm(w) < \varepsilon$$

for all $A \subset T$ (the unit circle) with $m(A) < \delta$, and for all $r \in (0, 1)$. The class N is the usual Nevanlinna class, which can be viewed as the space of all functions on the unit disc that are quotients of bounded analytic functions. N^+ is the class of all functions on the unit disc that can be written as the quotient of a bounded analytic function with an outer function. See [2] for details about these classes. It is left as an exercise for the reader to see that $N_* \subset N$, and, in fact, $N_* = N^+$. The following is then a corollary to Rudin's theorem.

THEOREM 1. *Let $\phi \in N_*$. Then the set of points w for which $\phi(z) - w$ has a nontrivial singular inner factor has logarithmic capacity zero. Conversely, given any (compact) set E of logarithmic capacity zero, there is a bounded analytic function ϕ such that $\phi(z) - w$ has a nontrivial singular inner factor if and only if $w \in E$.*

The converse statement is well known; see [3]. Let E be a compact set of capacity zero in Δ. The covering map F of the domain $\Delta \setminus E$ is an inner function since E has capacity zero. For each $w \in E$, $[F(z) - w]/[1 - \bar{w}F(z)]$ is a nonvanishing inner function and so is singular. Thus, since $1 - \bar{w}F(z)$ is an outer function, $F(z) - w$ is a function with nontrivial singular inner factor for all w in E.

Sarason produced a different sort of extension of Frostman's result, which appears in a paper by Mortini [7] as part of a constructive proof of the Beurling–Rudin theorem. He proved that, for mutually prime inner functions u and v (by which we mean that u and v have no zero in common and that there is no singular inner function S with $u = Su_1$ and $v = Sv_1$ for inner functions u_1 and v_1) and for $\rho > 0$, the function $u(z) + \rho e^{it}v(z)$ has a trivial singular inner factor for almost all (with respect to Lebesgue measure) real t.

Here we provide a generalization to the theorems of Frostman, Rudin, and Sarason that will further answer the general question of when singular inner factors disappear.

THEOREM 2. *Let $f, g \in H^p$, $0 < p \leq \infty$, have mutually prime singular inner factors. Then the set of points w for which $f(z) - wg(z)$ has a nontrivial singular inner factor has logarithmic capacity zero.*

REMARK. In our Theorem 2, if g is an outer function, then we see that the lack of a singular factor in $f(z) - wg(z)$ is equivalent to the lack of a singular factor in the decomposition of the function $f(z)/g(z) - w$ (in N_*), and is thus covered in Theorem 1.

Proof. We write

$$f(z) - wg(z) = F_w(z)B_w(z)S_w(z), \tag{1}$$

where the product on the right-hand side of (1) is of the outer, Blaschke product and of singular inner factors, respectively. We first comment on the dependence of the factors on w. Since

$$|F_w(z)| = \exp\left\{\frac{1}{2\pi}\int P_z(e^{i\theta})\log|f(e^{i\theta}) - wg(e^{i\theta})|\,d\theta\right\},$$

where P_z is the Poisson kernel for $z \in \Delta$, it follows that $|F_w(z)|$ is a continuous function of w on Δ. That is, if $w_n \to \zeta$ and we write $f(z) - w_n g(z) = F_n(z)B_n(z)S_n(z)$ and $f(z) - \zeta g(z) = F(z)B(z)S(z)$, then $|F_n| \to |F|$ uniformly on compact subsets of Δ. Hence, $|B_n S_n| \to |BS|$ uniformly on compact subsets of Δ.

Suppose now that E is a compact set of positive capacity. By adjusting by a scale factor, we may assume that $\max_E |w| < \frac{1}{2}$. There is a positive measure μ on E with integral 1 such that

$$v(z) = \int_E \log|w - z|\,d\mu(w) \tag{2}$$

is continuous in the whole plane [6, Thm. 1.8 and Sec. I.3]. Since μ has compact support, $v(z)$ is also bounded below.

The function u defined by

$$u(z) = \int_E \log|B_w(z)S_w(z)|\,d\mu(w)$$

is nonpositive in Δ. Moreover,

$$u(z) = \int_E \log|f(z) - wg(z)|\,d\mu(w) - \int_E \log|F_w(z)|\,d\mu(w). \tag{3}$$

Let $d\sigma$ be the normalized Lebesgue measure on the unit circle T. We shall show that

$$\lim_{r\to 1}\int_T u(re^{i\theta})\,d\sigma(\theta) = 0. \tag{4}$$

This will give us the first equation in the following:

$$\begin{aligned} 0 &= \lim_{r\to 1}\int_T\int_E \log|B_w(re^{i\theta})S_w(re^{i\theta})|\,d\mu(w)\,d\sigma(\theta) \\ &= \lim_{r\to 1}\int_E\left\{\int_T \log|B_w(re^{i\theta})S_w(re^{i\theta})|\,d\sigma(\theta)\right\}d\mu(w) \\ &\leq \int_E\left\{\limsup_{r\to 1}\int_T \log|B_w(re^{i\theta})S_w(re^{i\theta})|\,d\sigma(\theta)\right\}d\mu(w). \end{aligned}$$

This last inequality holds by Fatou's lemma, since the term inside the bracket is nonpositive. Hence

$$\lim_{r\to 1}\int_T \log|B_w(re^{i\theta})S_w(re^{i\theta})|\,d\sigma(\theta) = 0 \quad \text{a.e. } [\mu],$$

and thus $B_w(z)S_w(z)$ is a Blaschke product [5, p. 75, Prob. 6]. Thus, $S_w(z)$ is constant for w outside a set of μ-measure zero.

We proceed to prove (4) by considering two terms:

$$I(r) = \iint \log|f(re^{i\theta}) - wg(re^{i\theta})|\, d\mu(w)\, d\sigma(\theta) \quad \text{and}$$
$$II(r) = \iint \log|F_w(re^{i\theta})|\, d\mu(w)\, d\sigma(\theta).$$

We wish to show that, for

$$G(z) = \int_E \log|f(z) - wg(z)|\, d\mu(w),$$

we have

$$\lim_{r\to 1} I(r) = \lim_{r\to 1} \int G(re^{i\theta})\, d\sigma(\theta) = \int G(e^{i\theta})\, d\sigma(\theta)$$
$$= \iint \log|f(e^{i\theta}) - wg(e^{i\theta})|\, d\mu(w)\, d\sigma(\theta). \tag{5}$$

Note here that

$$G(z) = \int_E \log|g(z)|\, d\mu(w) + \int_E \log\left|\frac{f(z)}{g(z)} - w\right| d\mu(w)$$
$$= \log|g(z)| + v\left(\frac{f(z)}{g(z)}\right)$$

and that μ was chosen so that $v(z)$ is continuous; hence we easily see that, for almost every θ,

$$\lim_{r\to 1} G(re^{i\theta}) = G(e^{i\theta}).$$

Equation (5) will then be true by a variant of the dominated convergence theorem, provided there is a family of nonnegative integrable functions $V_r(\theta)$ and an integrable function V such that $V_r(\theta) \to V(\theta)$,

$$\lim_{r\to 1} \int V_r(\theta)\, d\sigma(\theta) = \int V(\theta)\, d\sigma,$$

and

$$|G(re^{i\theta})| \le V_r(\theta).$$

To demonstrate the existence of such a family $V_r(\theta)$, we will show separately that $G(re^{i\theta})$ has such an upper bound and a lower bound. We will need to use the factorization of f and g into outer and inner factors: $f(z) = O_f(z)I_f(z)$ and $g(z) = O_g(z)I_g(z)$.

To find the upper bound, note that for every $w \in E$,

$$|f(re^{i\theta}) - wg(re^{i\theta})| \le |O_f(re^{i\theta})| + |O_g(re^{i\theta})|. \tag{6}$$

This leads us to choose

$$V_r(\theta) = \log^+|O_f(re^{i\theta})| + \log^+|O_g(re^{i\theta})| + 1 \quad \text{and} \tag{7}$$
$$V(\theta) = \log^+|O_f(e^{i\theta})| + \log^+|O_g(e^{i\theta})| + 1.$$

We can easily see, by combining (6) and (7) (and using the inequality $\log(a+b) \leq \log^+ a + \log^+ b + 1$ for all real $a, b > 0$), that

$$\begin{aligned} G(re^{i\theta}) &= \int_E \log|f(re^{i\theta}) - wg(re^{i\theta})|\, d\mu(w) \\ &\leq \log(|O_f(re^{i\theta})| + |O_g(re^{i\theta})|) \\ &\leq V_r(\theta), \end{aligned}$$

and it is clear that $V_r(\theta) \to V(\theta)$ pointwise. Since O_f and O_g are outer functions, we obtain $\int V_r(\theta)\, d\sigma \to \int V(\theta)\, d\sigma$ [2, Thm. 2.10].

To find the lower bound, we need the following lemma.

LEMMA 1. *There is a constant K such that, for all $z \in \Delta$,*

$$G(z) \geq \log(\max\{|f(z)|, |g(z)|\}) + K. \tag{8}$$

For the proof of the lemma, we break the unit disc up into two pieces, A and B, where A consists of those points z where $|g(z)| \geq |f(z)|$, and B those points where $|g(z)| < |f(z)|$. We will prove the lemma separately for points in A and points in B.

If $z \in A$, then $\log(\max\{|f(z)|, |g(z)|\}) = \log|g(z)|$. Recall that

$$G(z) = \log|g(z)| + v\left(\frac{f(z)}{g(z)}\right)$$

and that $v(z)$, defined in equation (2), is bounded below, so the term on the right above can be written as in (8).

If $z \in B$, then we note that $|f(z) - wg(z)| > \frac{1}{2}|f(z)|$, so

$$G(z) = \int_E \log|f(z) - wg(z)|\, d\mu(w) > \log\left(\frac{1}{2}|f(z)|\right).$$

Also, for $z \in B$, $\log(\max\{|f(z)|, |g(z)|\}) = \log|f(z)|$, and again (8) can be satisfied. This completes the proof of the lemma.

Now we will find the family $V_r(\theta)$ just as before. We note that

$$\begin{aligned} \log(\max\{|f(re^{i\theta})|, |g(re^{i\theta})|\}) &\geq \log^-|O_f(re^{i\theta})| + \log^-|O_g(re^{i\theta})| \\ &\quad + \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\}). \end{aligned} \tag{9}$$

Take

$$\begin{aligned} V_r(\theta) &= \log^-|O_f(re^{i\theta})| + \log^-|O_g(re^{i\theta})| \\ &\quad + \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\}) \end{aligned} \tag{10}$$

and

$$V(\theta) = \log^-|O_f(e^{i\theta})| + \log^-|O_g(e^{i\theta})|.$$

Putting together (8), (9), and (10) gives us

$$G(re^{i\theta}) \geq V_r(\theta) + K.$$

The sum of the first two terms in the definition of $V_r(\theta)$ in (10) clearly approaches $V(\theta)$ pointwise, and the third term approaches zero pointwise. Furthermore,

$$\int_T V_r(\theta)\, d\sigma = \int_T \log^-|O_f(re^{i\theta})| + \log^-|O_g(re^{i\theta})|\, d\sigma(\theta) \\ + \int_T \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\})\, d\sigma(\theta),$$

and the first term on the right approaches $\int V(\theta)\, d\sigma(\theta)$, just as in the upper-bound case. The second term approaches zero, which we can see from the following lemma, due to Sarason.

LEMMA 2. *If u_1 and u_2 are inner functions without a common factor, then*

$$\lim_{r\to 1} \int_T \log(\max\{|u_1(re^{i\theta})|, |u_2(re^{i\theta})|\})\, d\sigma(\theta) = 0.$$

Sarason's proof of this lemma can be found in [7], but we include it here, with his permission, for completeness. The limit on the left side is the value at the origin of the least harmonic majorant in Δ of the subharmonic function $\max\{\log|u_1|, \log|u_2|\}$. It thus remains to show that this least harmonic majorant is the constant function 0. Let h denote this least harmonic majorant. Then $\log|u_1| \le h \le 0$. This implies that h has radial limits 0 almost everywhere on T. So, if h is not identically zero, then h is the Poisson integral of a negative singular measure on T. Hence $\varphi = e^{h+i\tilde{h}}$ is a singular inner function (here $\tilde{h}$ denotes the harmonic conjugate of h in Δ). Since $|u_1| \le |e^{h+i\tilde{h}}|$, the inner function φ divides u_1. But $|u_2| \le |e^{h+i\tilde{h}}|$ implies that φ also divides u_2, contradicting our assumption about u_1 and u_2. Thus $h \equiv 0$. This completes the proof of the lemma.

Our conditions for convergence of the integrals in (5) are satisfied, so we have

$$\lim_{r\to 1} I(r) = \int\int \log|f(e^{i\theta}) - wg(e^{i\theta})|\, d\sigma(\theta)\, d\mu(w). \tag{11}$$

Next,

$$\begin{aligned} II(r) &= \int\int \log|F_w(re^{i\theta})|\, d\mu(w)\, d\sigma(\theta) \\ &= \int\int\int \log|f(e^{it}) - wg(e^{it})| P_r(\theta - t)\, d\sigma(t)\, d\sigma(\theta)\, d\mu(w) \\ &= \int\int \log|f(e^{it}) - wg(e^{it})|\, d\sigma(t)\, d\mu(w). \end{aligned} \tag{12}$$

We have $\int u(re^{i\theta})\, d\sigma(\theta) = I(r) - II(r)$. When (11) and (12) are used in this, we find $\lim_{r\to 1} \int u(re^{i\theta})\, d\sigma(\theta) = 0$. As explained earlier, this finishes the proof that the set of w for which $f(z) - wg(z)$ has a nontrivial singular factor has capacity zero. □

References

[1] J. Caughran and A. Shields, *Singular inner factors of analytic functions,* Michigan Math. J. 16 (1969), 409–410.

[2] P. Duren, *Theory of H^p spaces,* Academic Press, New York, 1970.

[3] S. Fisher, *The singular set of a bounded analytic function,* Michigan Math. J. 20 (1973), 257–261.
[4] J. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.
[5] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, NJ, 1962.
[6] N. S. Landkof, *Foundations of modern potential theory,* Springer, New York, 1972.
[7] R. Mortini, *A constructive proof of the Beurling–Rudin theorem,* preprint.
[8] W. Rudin, *A generalization of a theorem of Frostman,* Math. Scand. 21 (1967), 136–143.

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