Sampling Sequences for $A^{-\infty}$

C. HOROWITZ, B. KORENBLUM, & B. PINCHUK

I. Introduction

For every $n > 0$, we define A^{-n} to be the Banach space of all functions f analytic in the unit disc U such that

$$\|f\|_{A^{-n}} \equiv \sup_{z\in U} |f(z)|(1-|z|^2)^n < \infty.$$

If $f \in A^{-n}$ and if $\Gamma \subset U$ is any subset then we can define

$$\|f|_\Gamma\|_{A^{-n}} = \sup_{z\in\Gamma} |f(z)|(1-|z|^2)^n.$$

Thus we always have

$$\|f|_\Gamma\|_{A^{-n}} \le \|f\|_{A^{-n}}.$$

Γ is called an A^{-n} *sampling set* if there exists a constant L such that, for every $f \in A^{-n}$,

$$\|f\|_{A^{-n}} \le L\|f|_\Gamma\|_{A^{-n}}.$$

The smallest such L, designated $L(\Gamma, n)$ is called the *sampling constant* of Γ. In an important paper, Seip [4] gave a complete characterization of A^{-n} sampling sets in terms of a certain density that he defined.

The space $A^{-\infty}$ is defined by

$$A^{-\infty} = \bigcup_{n>0} A^{-n};$$

that is, it is the algebra of functions analytic in U satisfying

$$|f(z)| \le \frac{M}{(1-|z|)^n} \quad \text{for some constants } M \text{ and } n.$$

Equipped with the inductive limit topology, $A^{-\infty}$ becomes a topological algebra. The zero sets and closed ideals of $A^{-\infty}$ were completely characterized in [2] and [3].

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For $f \in A^{-\infty}$, we define the type of f by

$$T(f) = \inf\{n : f \in A^{-n}\} = \overline{\lim_{|z|\to 1}} \frac{\log|f(z)|}{|\log(1-|z|)|}. \tag{1.1}$$

For every subset $E \subset U$ such that $\sup_{z\in E}|z| = 1$ and for every $f \in A^{-\infty}$, we can define

$$T_E(f) = \inf\{n : \sup_{z\in E}|f(z)|(1-|z|^2)^n < \infty\}$$

$$= \overline{\lim_{\substack{|z|\to 1\\ z\in E}}} \frac{\log|f(z)|}{|\log(1-|z|)|}. \tag{1.2}$$

Thus we always have $T_E(f) \leq T(f)$.

(1.3) DEFINITION. $E \subset U$ is called an $A^{-\infty}$ sampling set if, for all $f \in A^{-\infty}$, $T_E(f) = T(f)$. If E is also a discrete sequence in U then it is called an $A^{-\infty}$ sampling sequence.

At first glance one might conjecture that E is an $A^{-\infty}$ sampling sequence if and only if it is a sampling sequence for all A^{-n}. In fact, one of our main results is the following.

(1.4) THEOREM. *Let E be a sampling sequence for all A^{-n}. Then E is an $A^{-\infty}$ sampling sequence. However, there exists an $A^{-\infty}$ sampling sequence that is not a sampling set for any space A^{-n}.*

The structure of the paper is as follows. In Section 2 we prove Theorem 1.4. Section 3 deals with the characterization of certain circularly symmetric sampling sequences. In Section 4 we prove a general necessary condition for sampling sequences and show that it is sufficient (in a certain sense) for symmetric sequences of the type discussed in Section 3.

II. Comparison with $A^{-\infty}$ Sampling Sequences

As stated previously, the purpose of this section is to prove Theorem (1.4). We first treat its positive assertion—namely, that any sequence which is sampling for all A^{-n} is also sampling for $A^{-\infty}$. We wish to thank the referee of this paper who suggested the very short proof which follows, replacing a much longer argument constructed by the authors.

(2.1) LEMMA. *Let $n > 0$ be given, and let Γ be a sampling sequence for A^{-n}. If $f \in A^{-\infty}$ and $T(f) = n$, then $T_\Gamma(f) = n$.*

Proof. Since $T_\Gamma(f) \leq T(f) = n$ for any Γ, we need only prove that $T_\Gamma(f) \geq n$. To that end, it suffices to show that for all $m < n$,

$$\sup_{z\in\Gamma}|f(z)|(1-|z|^2)^m < \infty \implies \sup_{z\in U}|f(z)|(1-|z|^2)^m < \infty. \tag{2.2}$$

To prove (2.2) we make use of a formula of Seip [4, eq. (30)], which for our purposes can be stated as follows.

Let $\Gamma = \{z_k\}_{k=1}^{\infty}$ be an A^{-n} sampling sequence. Then, for $\varepsilon > 0$ sufficiently small, there exist functions $h_k(\zeta)$ satisfying $\sum_k |h_k(\zeta)| \le C$ where C is independent of ζ, such that for every $f \in A^{-\infty}$ with $T(f) = n$, for every $s > 0$, and for all $\zeta \in U$,

$$(1-|\zeta|^2)^{n+\varepsilon} f(\zeta) = \sum_k (1-|z_k|^2)^{n+\varepsilon} f(z_k) \left(\frac{1-|\zeta|^2}{1-\bar{\zeta} z_k}\right)^s h_k(\zeta). \qquad (2.3)$$

Now, for $m < n$ we set $s = n - m + \varepsilon$ and rewrite (2.3) in the form

$$\begin{aligned}(1-|\zeta|^2)^{n+\varepsilon} f(\zeta) \\ = \sum_k (1-|z_k|^2)^m f(z_k) \frac{(1-|z_k|^2)^{n-m+\varepsilon}}{(1-\bar{\zeta} z_k)^{n-m+\varepsilon}} (1-|\zeta|^2)^{n-m+\varepsilon} h_k(\zeta).\end{aligned}$$

We conclude from this that if $\sup_{z_k \in \Gamma} |f(z_k)|(1-|z_k|^2)^m < \infty$ then, for all $\zeta \in U$,

$$(1-|\zeta|^2)^m |f(\zeta)| \le M \sum_k |h_k(\zeta)|,$$

and thus $\sup_{\zeta \in U} (1-|\zeta|^2)^m |f(\zeta)| < \infty$. This proves (2.2) and hence Lemma (2.1). □

An immediate corollary of the lemma is the first part of Theorem 1.4.

We turn to the negative converse assertion of Theorem (1.4). Specifically, we shall construct a sampling set for $A^{-\infty}$ (as in Definition (1.3)) that is not a set of sampling for any A^{-n}. This example can easily be modified to a parallel result concerning sampling *sequences*; the procedure for passing from arbitrary sampling sets to sampling sequences will be described in detail in Section 3.

(2.4) PROPOSITION. *Let $E = \bigcup_n \{z : |z| = r_n\}$ where $0 < r_1 < r_2 \cdots$ and $\lim_{n\to\infty} |\log(1-r_{n+1})|/|\log(1-r_n)| = 1$. Then E is a sampling set for $A^{-\infty}$.*

Proof. Let $f \in A^{-\infty}$, and let $\varepsilon > 0$ be given. Then we can find n_0 such that, for all $n > n_0$,

$$\frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} < 1+\varepsilon \quad \text{and} \quad \sup_{|\xi|=r_n} \frac{\log|f(\xi)|}{|\log(1-r_n)|} < T_E(f) + \varepsilon.$$

Now, if $n > n_0$ and if $r_n \le |z| \le r_{n+1}$ then we have

$$\begin{aligned}\frac{\log|f(z)|}{|\log(1-|z|)|} &\le \sup_{|\xi|=r_{n+1}} \frac{\log|f(\xi)|}{|\log(1-r_n)|} = \sup_{|\xi|=r_{n+1}} \frac{\log|f(\xi)|}{|\log(1-r_{n+1})|} \frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} \\ &\le (1+\varepsilon)[T_E(f)+\varepsilon].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that $T(f) \le T_E(f)$ for every $f \in A^{-\infty}$; therefore, E is an $A^{-\infty}$ sampling set. □

(2.5) EXAMPLE. *Let $E = \bigcup_n \{z : |z| = r_n\}$ where*

$$r_n = 1 - \exp(-e^{\sqrt{n}}).$$

Then E is a sampling set for $A^{-\infty}$, but not for any space A^{-n}.

Proof. It follows immediately from Proposition (2.4) that E is a sampling set for $A^{-\infty}$. On the other hand, if E is a sampling set for some A^{-n} then Seip's characterization in [4] shows that E must have a uniformly discrete subsequence of positive lower density (as defined there). However, if $F \subset E$ is uniformly discrete then it is easy to see that, for each n,

$$\sum_{\substack{z\in F\\ |z|=r_n}} \log\frac{1}{|z|} = O(1).$$

Thus,

$$\sum_{\substack{z\in F\\ |z|\le r_n}} \frac{\log\frac{1}{|z|}}{\log\frac{1}{1-r_n}} = \frac{O(n)}{e^{\sqrt{n}}} \to 0.$$

This implies that the set F has lower density zero, so it cannot be a sampling set for any A^{-n}. □

III. Symmetric Sampling Sets and Sequences

We begin this section with a converse to Proposition (2.4). The proof will depend on the following theorem, which is an immediate consequence of [1, Thm. 2].

(3.1) THEOREM. *For $r \in [0,1]$, let $k(r)$ be an unbounded increasing function such that* $\sup_{0\le r<1} k(r) - k(r^2) < \infty$. *Then there exists a function $f(z)$ analytic in U such that, for* $0 < r < 1$,

$$\max_{|z|=r} \log|f(z)| = k(r) + O(1). \tag{3.2}$$

Proof. In [1, Thm. 2] it was shown that, under our hypotheses, there exists f analytic in U satisfying

$$\log|f(z)| \le k(|z|) + O(1)$$

and whose zeros $\{z_k\}$ satisfy

$$\sum_{|z_k|<r} \log\frac{r}{|z_k|} = k(r) + O(1), \quad 0 < r < 1.$$

Thus, (3.2) follows directly from Jensen's formula. □

We note that in [1] there was a standing assumption that k should be a convex function of $\log r$; however, this was not used in the proof of the result cited here.

(3.3) PROPOSITION. *If* $0 < r_1 < r_2 < \cdots \to 1$ *and if*

$$\overline{\lim_{n\to\infty}} \left|\frac{\log(1-r_{n+1})}{\log(1-r_n)}\right| > 1$$

then $E = \bigcup_n \{z : |z| = r_n\}$ *is not a sampling set for* $A^{-\infty}$.

Proof. Under our hypothesis we can define an increasing function $k(r)$, $0 < r < 1$, such that

$$k(r_n) = \log \frac{1}{1 - r_n} \tag{3.4}$$

for each n and

$$1 < \varlimsup_{r \to 1} \frac{k(r)}{|\log(1-r)|} \le 2; \tag{3.5}$$

for example, for $r_n < r < r_{n+1}$, let

$$k(r) = \min\bigl(|\log(1 - r_{n+1})|, 2|\log(1-r)| - |\log(1-r_n)|\bigr).$$

In particular, $k(r^2) - k(r)$ is bounded and so by Theorem (3.1) there exists a function f analytic in U such that, for $0 < r < 1$,

$$\sup_{|z|=r} \log|f(z)| = k(r) + O(1).$$

By (3.4) $T_E(f) = 1$ while by (3.5) $T(f) > 1$. Thus E is not an $A^{-\infty}$ sampling set. □

The following lemma will help us to pass from arbitrary sampling sets to sampling sequences.

(3.6) LEMMA. *Let $f \in A^{-\infty}$ and let $\{z_k\}$ be a discrete sequence in U such that $|z_k| \to 1$ and*

$$\lim_{k \to \infty} \frac{\log|f(z_k)|}{|\log(1 - |z_k|)|} = T(f).$$

(Clearly such sequences must exist.) Then if $q > 0$, $\varepsilon > 0$, and $\{w_k\}$ is another sequence in U satisfying

$$|z_k - w_k| < q(1 - |z_k|)^{1+\varepsilon}, \quad k = 1, 2, \ldots,$$

we have also

$$\lim_{k \to \infty} \frac{\log|f(w_k)|}{|\log(1 - |w_k|)|} = T(f).$$

Proof. Define $N = T(f)$. Thus, if $N < N_1 < N + \varepsilon/2$ then $f \in A^{-N_1}$ and, by [4, Lemma 2.1] for all k we have

$$|f(z_k)|(1 - |z_k|^2)^{N_1} - |f(w_k)|(1 - |w_k|^2)^{N_1} \le M\|f\|_{A^{-N_1}} \left| \frac{z_k - w_k}{1 - \bar{z}_k w_k} \right|, \tag{3.7}$$

where M is a constant depending only on N. For each k and m define

$$p_{k,m} = |f(z_k)|(1 - |z_k|^2)^m \quad \text{and} \quad q_{k,m} = |f(w_k)|(1 - |w_k|^2)^m.$$

Then, for all k,

$$\begin{aligned} p_{k,N} - q_{k,N} = {} & (p_{k,N_1} - q_{k,N_1})(1 - |z_k|^2)^{N-N_1} \\ & + q_{k,N_1}[(1 - |z_k|^2)^{N-N_1} - (1 - |w_k|^2)^{N-N_1}]. \end{aligned} \tag{3.8}$$

By our hypothesis, it follows that for all k we have

$$1 - |w_k| = 1 - |z_k| + O(1 - |z_k|)^{1+\varepsilon}.$$

Thus

$$c_1(1 - |z_k|) \le 1 - |w_k| \le c_2(1 - |z_k|)$$

and so, by the mean value theorem, for each real $\alpha \neq 0$ there exists a constant c_α such that

$$\left|(1 - |w_k|^2)^\alpha - (1 - |z_k|^2)^\alpha\right| \le c_\alpha(1 - |z_k|)^{\alpha+\varepsilon}, \quad k = 1, 2, \dots .$$

Returning to (3.8), since $f \in A^{-N_1}$, the numbers q_{k,N_1} are bounded and

$$\left|(1 - |z_k|^2)^{N-N_1} - (1 - |w_k|^2)^{N-N_1}\right| \le c(1 - |z_k|)^{N-N_1+\varepsilon} \tag{3.9}$$

Since $|1 - \bar{z}_k w_k| \ge 1 - |z_k|^2 - |\bar{z}_k(w_k - z_k)| \ge c(1 - |z_k|^2)$, we can conclude from (3.7), (3.8), and (3.9) that, for all k,

$$|p_{k,N} - q_{k,N}| \le c(1 - |z_k|^2)^{\varepsilon/2}. \tag{3.10}$$

Now our hypothesis in this lemma is just that

$$\lim_{k\to\infty} \frac{|\log p_{k,N}|}{|\log(1 - |z_k|)|} = 0.$$

Thus, if $0 < \delta < \varepsilon/2$ then we have

$$\frac{\log \frac{1}{p_{k,N}}}{\log \frac{1}{1-|z_k|}} < \delta$$

for all large k, which means that $p_{k,N} > (1 - |z_k|)^\delta$. By (3.10) we obtain $q_{k,N} > c(1 - |z_k|)^\delta$, and it follows easily that

$$\overline{\lim_{k\to\infty}} \frac{|\log q_{k,N}|}{|\log(1 - |z_k|)|} < \delta.$$

Since $\delta > 0$ is arbitrary, we have in fact

$$\lim_{k\to\infty} \frac{|\log q_{k,N}|}{|\log(1 - |z_k|)|} = 0,$$

which proves the lemma. □

(3.11) PROPOSITION. *Let $\{z_k\}$ be an $A^{-\infty}$ sampling sequence, and let $\{w_k\}$ be a neighboring sequence as in Lemma* (3.6). *Then $\{w_k\}$ is an $A^{-\infty}$ sampling sequence.*

Proof. By hypothesis, if $f \in A^{-\infty}$ then there is a subsequence $\{z_{k_n}\}$ such that $\log|f(z_{k_n})|/|\log(1 - |z_{k_n}|)| \to T(f)$. By the lemma, the same holds for the subsequence $\{w_{k_n}\}$. □

CONJECTURE. *Proposition* (3.11) *is actually true under the weaker hypothesis that*

$$|z_k - w_k| = o\left(\frac{1 - |z_k|}{|\log(1 - |z_k|)|}\right). \tag{3.12}$$

We note that the analog of Lemma (3.6) is *not* true under the hypothesis (3.12). Indeed consider the function

$$f(z) = \frac{1}{1-z}\prod_{k=1}^{\infty}\frac{w_k - z}{1 - w_k z}$$

where, for all k, $w_k = 1 - 1/2^k$ and z_k are real numbers satisfying

$$|z_k - w_k| = \frac{1 - w_k}{\log_2^2(1 - w_k)} = \frac{1}{k^2 2^k}.$$

Then $\{z_k\}$ "samples" f whereas $\{w_k\}$ clearly does not.

(3.13) Definition. Let $0 < r_n \nearrow 1$, let $\sup[(1 - r_{n+1})/(1 - r_n)] < 1$, and let $\varepsilon > 0$. A *symmetric sequence* based on $\{r_n\}$ with exponent ε is a sequence consisting precisely of $[(1 - r_n)^{-1-\varepsilon}]$ points symmetrically placed on the circle $|z| = r_n$.

(3.14) Proposition. *A symmetric sequence based on* $\{r_n\}$ *is an* $A^{-\infty}$ *sampling sequence if and only if*

$$\lim_{n\to\infty}\frac{|\log(1 - r_{n+1})|}{|\log(1 - r_n)|} = 1.$$

Proof. If the sequence is sampling then clearly $E = \bigcup_n \{|z| = r_n\}$ is sampling, which by Proposition (3.3) implies our condition. Conversely, if the condition holds then Proposition (2.4) shows that E is a sampling set. This means that, for every $f \in A^{-\infty}$, there is a sequence $\{z_k\}$ contained in E such that $|z_k| \to 1$ and

$$\lim_{k\to\infty}\frac{\log|f(z_k)|}{|\log(1 - |z_k|)|} = T(f).$$

Our given sequence (call it $\{w_l\}$) is symmetric, so we can pair each z_k with an appropriate w_{l_k} such that $|z_k - w_{l_k}| < q(1 - |z_k|)^{1+\varepsilon}$ for all k. Thus, by Lemma (3.6),

$$\lim_{k\to\infty}\frac{\log|f(w_{l_k})|}{|\log(1 - |w_{l_k}|)} = T(f).$$

It follows that $\{w_l\}$ is an $A^{-\infty}$ sampling sequence. □

IV. A General Necessary Condition

In this section we develop a general necessary condition for $A^{-\infty}$ sampling sets and prove that it is sufficient—in a certain sense—in the case of symmetric sets.

(4.1) Definition. $\xi = e^{i\alpha} \in \partial U$ is a *point of fast decline* for a function $f \in H^\infty$ if $\lim_{r\to 1}(1 - r)^{-n}|f(re^{i\alpha})| = 0$ for all $n > 0$.

(4.2) Theorem. *If* Γ *is an* $A^{-\infty}$ *sampling sequence, then for all* $\xi \in \partial U$, *for all* $\beta > 1$, *and for all* $q < 1/4$, *either* $\Gamma \cap G_{\xi,\beta,q}$ *is not a Blaschke sequence or* ξ *is a* *point of fast decline for the Blaschke product* $B_{\xi,\beta,q}$ *vanishing on the subsequence* $\Gamma \cap G_{\xi,\beta,q}$, *where*

$$G_{\xi,\beta,q} = \{z \in U : |z| < 1 - q|\operatorname{Arg}(z/\xi)|^{\beta}\}.$$

Proof. Suppose that some $\xi_0 \in \partial U$ is not a point of fast decline for $B = B_{\xi_0,\beta,q}$. Then there is a sequence $r_n \nearrow 1$ and a number $m > 0$ such that

$$|B(r_n\xi_0)| > (1 - r_n)^m \quad (n = 1, 2, \dots). \tag{4.3}$$

Now define

$$F_N(z) = B(z)(\xi_0 - z)^{-N}, \quad N \in \mathbb{N}.$$

Since $F_N|_{\Gamma\cap G_{\xi,\beta,q}} = 0$,

$$T_\Gamma(F_N) \le N/\beta.$$

However, (4.3) implies that

$$T(F_N) \ge N - m.$$

For N sufficiently large this implies

$$T(F_N) > T_\Gamma(F_N),$$

and so Γ is not an $A^{-\infty}$ sampling sequence. □

(4.4) DEFINITION. We call the condition in Theorem (4.2) "condition C".

(4.5) THEOREM. *Let* $0 < r_n \nearrow 1$ *in such a way that*

$$\sup_n \frac{1 - r_{n+1}}{1 - r_n} < 1 \quad \text{and} \quad \varlimsup_{n\to\infty} \frac{|\log(1 - r_{n+1})|}{|\log(1 - r_n)|} \ge 1 + 2\varepsilon$$

for some ε, $0 < \varepsilon < \frac{1}{2}$. *If* $\beta > 1$, $\delta > 0$, *and* $1 - 1/\beta + \delta < \varepsilon/3$, *then a symmetric sequence* S *of exponent* δ *based on* $\{r_n\}$ *(see Definition* (4.11)) *violates condition* C *with exponent* β.

Proof. We shall show that condition C fails on the real positive radius. Let $G = G_{1,\beta,\frac{1}{5}}$, and let B be the Blaschke product vanishing at the points $\{z_k\} = S \cap G$. It will suffice to show that there exists a constant $\gamma > 0$ such that, for every n satisfying

$$\frac{|\log(1 - r_{n+1})|}{|\log(1 - r_n)|} > 1 + \varepsilon$$

(i.e., for a full subsequence of $\{r_n\}$), we can find a point r'_n such that

$$r_n < r'_n < r_{n+1} \quad \text{and} \quad |B(r'_n)| > \gamma.$$

Now, since

$$B(z) = \prod_{z_k \in G} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z},$$

if $z > 0$ is a bit distant from all of the $\{r_n\}$, say

$$\inf_n \left|\frac{r_n - z}{1 - r_n z}\right|^2 \geq d > 0,$$

then

$$\log\frac{1}{|B(z)|} = \frac{1}{2}\sum_{z_k \in G} \log\left|\frac{z_k - z}{1 - \bar{z}_k z}\right|^{-2} \leq \frac{1}{2d}\sum_{z_k \in G} 1 - \left|\frac{z_k - z}{1 - \bar{z}_k z}\right|^2$$
$$= \frac{1}{2d}\sum_n \cdot \sum_{\substack{z_k \in G \\ |z_k| = r_n}} \frac{(1 - z^2)(1 - r_n^2)}{|1 - \bar{z}_k z|^2}.$$

For a given n, we have approximately $(1 - r_n)^{-1-\delta}$ points $\{z_k\}$ on $\{|z| = r_n\}$, so that the fraction $c(1 - r_n)^{1/\beta}$ of them belong to G. Hence we can conclude that

$$\log\frac{1}{|B(z)|} \leq c\sum_n \frac{(1 - z^2)(1 - r_n)^{1/\beta - \delta}}{(1 - z r_n)^2}$$
$$\leq c\sum_{r_n \leq z} \frac{(1 - z)(1 - r_n)^{1/\beta - \delta}}{(1 - r_n)^2} + c\sum_{r_n > z} \frac{(1 - r_n)^{1/\beta - \delta}}{(1 - z)}.$$

Since, by assumption, the numbers $(1 - r_n)$ decrease geometrically (at least) with n, we can estimate each of the foregoing sums by its largest term. Thus, if we choose N such that $r_N < z < r_{N+1}$, then

$$\log\frac{1}{|B(z)|} \leq c\left[\frac{(1 - z)}{(1 - r_N)^{2 - 1/\beta + \delta}} + \frac{(1 - r_{N+1})^{1/\beta - \delta}}{1 - z}\right].$$

Now, if $(1 - r_{N+1}) < (1 - r_N)^{1+\varepsilon}$ then we can choose z, $r_N < z < r_{N+1}$, such that $(1 - r_{N+1})^{1-\varepsilon/3} \leq (1 - z) \leq (1 - r_N)^{1+\varepsilon/3}$ and so obtain

$$\log\frac{1}{|B(z)|} \leq c\left[(1 - r_N)^{\varepsilon/3 - (1 - 1/\beta + \delta)} + (1 - r_{N+1})^{\varepsilon/3 - (1 - 1/\beta + \delta)}\right].$$

By hypothesis, $1 - 1/\beta + \delta < \varepsilon/3$, so the last expression actually tends to zero as $N \to \infty$, proving our theorem. □

The following theorem summarizes several of our results.

(4.6) THEOREM. *Assume that*

$$0 < r_n \nearrow 1 \quad \textit{and} \quad \sup_n \frac{1 - r_{n+1}}{1 - r_n} < 1.$$

Then the following are equivalent.

(a) *Every symmetric sequence based on $\{r_n\}$ is an $A^{-\infty}$ sampling set.*
(b) *Every symmetric sequence based on $\{r_n\}$ satisfies condition C.*
(c) $\lim_{n \to \infty} \dfrac{|\log(1 - r_{n+1})|}{|\log(1 - r_n)|} = 1.$

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C. Horowitz
Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900
Israel

B. Korenblum
Department of Mathematics
SUNY at Albany
Albany, NY 12222

B. Pinchuk
Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900
Israel