Geodesics in Hyperbolic 3-Folds

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Dedicated to Clifford Earle on the occasion of his sixtieth birthday

1. Introduction

Hyperbolic 3-space is the set

$$\mathbf{H}^3 = \{ (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 > 0 \}$$

endowed with the complete Riemannian metric $ds = |dx|/x_3$ of constant curvature equal to -1. A *Kleinian group* G is a discrete nonelementary subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$, where $\mathrm{Isom}^+(\mathbf{H}^3)$ is the group of orientation preserving isometries. In this setting, *nonelementary* means that the group G is not virtually abelian. Finally a *hyperbolic 3-orbifold* $\mathcal{Q}$ is the orbit space of a Kleinian group G,

$$\mathcal{Q} = \mathbf{H}^3/G. \tag{1.1}$$

The orbit space is a *hyperbolic 3-manifold* if the group G is torsion-free. For the general facts about hyperbolic geometry and Kleinian groups we refer the reader to the monographs [2], [16], [21], and [23].

This paper is concerned with the length of intersecting and nonsimple closed geodesics in hyperbolic 3-manifolds and 3-orbifolds. By a *geodesic* α we mean a complete geodesic in the induced metric of constant curvature -1; we denote by $\ell(\alpha)$ the hyperbolic length of α. A closed geodesic α is *simple* if it is embedded or a power of a closed embedded geodesic; otherwise, α is *nonsimple*. If α is nonsimple, then any points of self intersection are transverse. The same is true at the points of intersection of two distinct geodesics that are not both powers of some primitive geodesic (see e.g. [21]).

Here are our two main results. They are motivated by a beautiful result due to Beardon, Theorem 11.6.8 in [2], which gives sharp estimates for the case of closed geodesics on Riemann surfaces. For related results see also [11], [19], and [20] as well as [1] and [15].

THEOREM 1.2. *If α_1 and α_2 are closed geodesics in a hyperbolic 3-fold that intersect at an angle ϕ where $0 < \phi < \pi$, then*

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$$\sinh(\ell(\alpha_1))\sinh(\ell(\alpha_2))\sin^{4/3}(\phi) \geq \ell_1^2, \tag{1.3}$$

where $0.121 \leq \ell_1$. *The exponent of* $\sin(\phi)$ *in* (1.3) *cannot be replaced by any constant greater than* $4/3$.

THEOREM 1.4. *If* α *is a closed geodesic in a hyperbolic 3-fold with a self-intersection of angle* ϕ *where* $0 < \phi < \pi$, *then*

$$\sinh(\ell(\alpha))\sin(\phi) \geq \ell_2, \tag{1.5}$$

where $0.122 \leq \ell_2$. *The exponent of* $\sin(\phi)$ *in* (1.5) *cannot be replaced by any constant greater than* 1.

Theorems 1.2 and 1.4 are 3-dimensional analogs of the following theorem of Beardon's mentioned before.

THEOREM 1.6. *If* α_1 *and* α_2 *are closed geodesics in a Riemann surface* $\mathcal{S}$ *that intersect at an angle* ϕ *where* $0 < \phi < \pi$, *then*

$$\sinh(\ell(\alpha_1)/2)\sinh(\ell(\alpha_2)/2)\sin(\phi) \geq \ell^2, \tag{1.7}$$

where $\ell \geq 0.471\ldots$. *The exponent of* $\sin(\phi)$ *in* (1.7) *cannot be replaced by any constant greater than* 1.

We shall now reformulate and prove more precise versions of these results in the language of Kleinian groups.

2. A Reformulation of the Problem

Let $\mathbf{M}$ denote the group of all Möbius transformations of the extended complex plane $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We associate with each Möbius transformation

$$f = \frac{az+b}{cz+d} \in \mathbf{M}, \quad ad - bc = 1,$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$$

and set $\mathrm{tr}(f) = \mathrm{tr}(A)$, where $\mathrm{tr}(A) = a + d$ denotes the trace of the matrix A. Next, for each f and g in $\mathbf{M}$ we let $[f, g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\gamma(f,g) = \mathrm{tr}([f,g]) - 2, \quad \beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4 \tag{2.1}$$

the *parameters* of the two-generator group $\langle f, g\rangle$. These parameters are independent of the choice of matrix representations for f and g in $\mathrm{SL}(2, \mathbf{C})$, and they determine $\langle f, g\rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$.

The elements of f of $\mathbf{M}$, other than the identity, fall into three types.

(1) *Elliptic:* $\beta(f) \in [-4, 0)$ and f is conjugate to $z \mapsto \mu z$ where $|\mu| = 1$.
(2) *Loxodromic:* $\beta(f) \notin [-4, 0]$ and f is conjugate to $z \mapsto \mu z$ where $|\mu| > 1$; f is *hyperbolic* if, in addition, $\mu > 0$.
(3) *Parabolic:* $\beta(f) = 0$ and f is conjugate to $z \mapsto z + 1$.

Each Möbius transformation of $\bar{\mathbf{C}} = \partial\mathbf{H}^3$ extends uniquely via the Poincaré extension [2] to an orientation-preserving isometry of hyperbolic 3-space $\mathbf{H}^3$. In this way we identify Kleinian groups with discrete Möbius groups.

If $f \in \mathbf{M}$ is nonparabolic, then f fixes two points of $\bar{\mathbf{C}}$ and the closed hyperbolic line joining these two fixed points is called the *axis* of f, denoted by $\mathrm{ax}(f)$. In this case f translates along $\mathrm{ax}(f)$ by an amount $\tau(f) \geq 0$, the *translation length* of f, f rotates about $\mathrm{ax}(f)$ by an angle $\theta(f) \in (-\pi, \pi]$, and

$$\beta(f) = 4\sinh^2\left(\frac{\tau(f) + i\theta(f)}{2}\right). \tag{2.2}$$

It then follows from (2.2) that

$$\cosh(\tau(f)) = \frac{|\beta(f) + 4| + |\beta(f)|}{4} \tag{2.3}$$

and

$$\cos(\theta(f)) = \frac{|\beta(f) + 4| - |\beta(f)|}{4} \tag{2.4}$$

(cf. (15), (17), and (18) in [9]).

If $f, g \in \mathbf{M}$ are nonparabolic and if α is the hyperbolic line in $\mathbf{H}^3$ that is orthogonal to $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$, then

$$\frac{4\gamma(f, g)}{\beta(f)\beta(g)} = \sinh^2(\delta + i\phi), \tag{2.5}$$

where δ is the hyperbolic distance between $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$ and where $\phi \in [0, \pi]$ is the angle between the hyperplanes in $\mathbf{H}^3$ that contain $\mathrm{ax}(f) \cup \alpha$ and $\mathrm{ax}(g) \cup \alpha$, respectively (see [8, Lemma 4.2]). In particular, if $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$ intersect at an angle $\phi \in (0, \pi)$ then

$$\frac{4\gamma(f, g)}{\beta(f)\beta(g)} = -\sin^2(\phi). \tag{2.6}$$

The following two numbers, associated with the (2, 3, 7) triangle group, will occur frequently in what follows:

$$c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.048\ldots,$$
$$d = 2(1 - \cos(\pi/7)) = 0.198\ldots.$$

We show how one may establish the Kleinian group form of Theorem 1.6 from (2.2), (2.6), and the following sharp inequality (see Cor. in [22]).

LEMMA 2.7. *If $\langle f, g\rangle$ is a Fuchsian group, then*

$$|\gamma(f, g)| \geq d = 0.198\ldots. \tag{2.8}$$

THEOREM 2.9. *If $\langle f, g\rangle$ is a Kleinian group, if f and g are hyperbolics, and if $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$ intersect at an angle $0 < \phi < \pi$, then*

$$\sinh(\tau(f)/2)\sinh(\tau(g)/2)\sin(\phi) \geq \lambda^2, \tag{2.10}$$

where $\lambda = 0.471\ldots$. The constant λ is sharp and the exponent of $\sin(\phi)$ cannot be replaced by a constant greater than 1.

Proof. Let S be the hyperplane in $\mathbf{H}^3$ determined by $\operatorname{ax}(f)$ and $\operatorname{ax}(g)$. Then S is invariant under G, $F = G|S$ is conjugate to a Fuchsian group and

$$|\gamma(f, g)| \geq d = 0.198\ldots \tag{2.11}$$

by Lemma 2.7. Next, since f and g are hyperbolic, $\theta(f) = \theta(g) = 0$ and

$$|\beta(f)| = 4\sinh^2(\tau(f)/2), \qquad |\beta(g)| = 4\sinh^2(\tau(g)/2) \tag{2.12}$$

by (2.2). Thus,

$$\begin{aligned} 16\sinh^2(\tau(f)/2)\sinh^2(\tau(g)/2)\sin^2(\phi) &= |\beta(f)||\beta(g)|\sin^2(\phi) \\ &= 4|\gamma(f, g)| \\ &\geq 4d = 16\lambda^4 \end{aligned}$$

by (2.6), (2.11), and (2.12), and we obtain (2.10).

Inequality (2.10) holds with equality if f and g are hyperbolic generators for the $(2, 3, 7)$ triangle group with

$$\operatorname{par}(\langle f, g\rangle) = (-d, c, c) = (-0.198\ldots, 1.048\ldots, 1.048\ldots).$$

We give an example in Section 4 to show that the exponent 1 of $\sin(\phi)$ cannot be increased. □

We want now to modify the above argument in order to establish the two Kleinian group or 3-dimensional forms of Theorem 2.9 that correspond to Theorems 1.2 and 1.4. There are two difficulties to circumvent.

(1) We need an inequality similar to (2.11) for the commutator parameter $\gamma(f, g)$ when $\langle f, g\rangle$ is a Kleinian group.
(2) We need a relation similar to (2.12) involving only the trace parameter $\beta(f)$ and the translation length $\tau(f)$ when f is a loxodromic element.

An example due to Jørgensen [13] shows that there exists no absolute lower bound for $|\gamma(f, g)|$ when $\langle f, g\rangle$ is a Kleinian group. However, the following result, established in [14] and [7] and with the sharp constant by Cao in [3], will serve as a substitute for inequality (2.11).

LEMMA 2.13. *If $\langle f, g\rangle$ is a Kleinian group and if either*

$$|\beta(f)| \leq c = 1.048\ldots \quad \text{or} \quad \beta(f) = \beta(g), \tag{2.14}$$

then

$$|\gamma(f, g)| \geq d = 0.198\ldots. \tag{2.15}$$

This result is sharp under either assumption in (2.14).

The following extension of a lemma due to Zagier [18] will serve as a substitute for (2.12).

LEMMA 2.16. *For each loxodromic Möbius transformation f there exists an integer $m \geq 1$ such that*

$$|\beta(f^m)| \le \frac{4\pi}{\sqrt{3}} \sinh(\tau(f)). \tag{2.17}$$

The coefficient of $\sinh(\tau(f))$ *in* (2.17) *cannot be replaced by smaller constant.*

Proof. This is an immediate consequence of [4, Cor. 3.25] and the fact that

$$\beta(f^n) = 4\sinh^2\left(\frac{n(\tau(f) + i\theta(f))}{2}\right). \qquad \square$$

We will also need the following variant of [8, Thm. 3.4] in what follows.

LEMMA 2.18. *If* $\langle f, g\rangle$ *is a Kleinian group, if* f *is elliptic of order* $n \ge 3$, *and if* g *is not of order* 2, *then*

$$|\gamma(f, g)| \ge a(n), \tag{2.19}$$

where

$$a(n) = \begin{cases} 2\cos(2\pi/7) - 1 = 0.246\ldots & \text{if } n = 3, \\ 2\cos(2\pi/5) = 0.618\ldots & \text{if } n = 4, 5, \\ 2\cos(2\pi/6) = 1 & \text{if } n = 6, \\ 2\cos(2\pi/n) - 1 \ge 0.246\ldots & \text{if } n \ge 7. \end{cases} \tag{2.20}$$

Proof. Since $\langle f, g\rangle$ is nonelementary,

$$\text{fix}(f) \cap \text{fix}(g) = \emptyset, \quad \gamma(f, g) \ne 0.$$

If $\gamma(f, g) \ne \beta(f)$, then (2.19) follows from [8, Thm. 3.4]. Otherwise,

$$\gamma(f, gfg^{-1}) = \gamma(f, g)(\gamma(f, g) - \beta(f)) = 0, \quad \text{fix}(f) \cap g(\text{fix}(f)) \ne \emptyset,$$

g maps one fixed point of f onto the other and f is of order 3, 4, or 6 by [17, Prop. 1]. Hence

$$|\gamma(f, g)| = |\beta(f)| \ge 1 \ge a(n)$$

and we again obtain (2.19). $\square$

3. Main Results

We establish here the Kleinian group analogs of Theorem 2.9 that correspond to the estimates for hyperbolic geodesics in Theorems 1.2 and 1.4. We begin with the following key inequality.

LEMMA 3.1. *If* $\langle f, g\rangle$ *is discrete and if* f *and* g *are loxodromics with axes that intersect at an angle* ϕ *where* $0 < \phi < \pi$, *then*

$$|\beta(f)\beta(g)| \sin^{4/3}(\phi) \ge b, \tag{3.2}$$

where $0.777 \le b \le 0.884$.

Proof. By (2.5),

$$|\beta(f)\beta(g)|\sin^2(\phi) = |4\gamma(f,g)|. \tag{3.3}$$

We want to find a lower bound for

$$u = |\beta(f)\beta(g)|\sin^{4/3}(\phi).$$

By relabeling, we may assume that $|\beta(f)| \le |\beta(g)|$.

If $|\gamma(f,g)| \ge d = 0.198\ldots$ then, by (3.3),

$$u \ge |\beta(f)\beta(g)|\sin^2(\phi) = |4\gamma(f,g)| \ge 4d = 0.792\ldots.$$

Next, if $|\beta(f)| \le c = 1.048\ldots$ then $\gamma(f,g) \ne 0$, by (3.3), and $\langle f, g\rangle$ is Kleinian. Thus $|\gamma(f,g)| \ge d$ by Lemma 2.13, and

$$u \ge 4d = 0.792\ldots$$

as before. Finally, if $|\beta(f)| \ge c$ and $|\gamma(f,g)| \le d$, then

$$\gamma(f, gfg^{-1}) = \gamma(f,g)(\gamma(f,g) - \beta(f)) \ne 0 \tag{3.4}$$

and $\langle f, gfg^{-1}\rangle$ is Kleinian. Hence

$$\begin{aligned} c^{-2}u^3 + 4u^{3/2} &= |\beta(f)\beta(g)|^3 c^{-2}\sin^4(\phi) + 4|\beta(f)\beta(g)|^{3/2}\sin^2(\phi) \\ &\ge |\beta(f)\beta(g)|^2\sin^4(\phi) + 4|\beta(f)\beta(g)|\sin^2(\phi)|\beta(f)| \\ &\ge 16|\gamma(f,g)|^2 + 16|\gamma(f,g)||\beta(f)| \\ &\ge 16|\gamma(f,gfg^{-1})| \ge 16d = 3.168\ldots \end{aligned}$$

by (3.4) and Lemma 2.13, and we obtain

$$u \ge 0.777.$$

Thus (3.2) follows with $b \ge 0.777$.

If $\langle f, g\rangle$ is the (2, 3, 7) triangle group with

$$\operatorname{par}(\langle f,g\rangle) = (-d, c, c) = (-0.198\ldots, 1.048\ldots, 1.048\ldots),$$

then

$$u = \sin^{4/3}(\phi)|\beta(f)\beta(g)| = (4dc)^{2/3} \le 0.884$$

and this implies that $b \le 0.884$. □

If we combine Lemma 2.16 and Lemma 3.1, we obtain an inequality for Kleinian groups that corresponds to Theorem 1.2.

THEOREM 3.5. *If $\langle f, g\rangle$ is discrete, if f and g are loxodromics, and if* $\operatorname{ax}(f)$ *and* $\operatorname{ax}(g)$ *intersect at an angle ϕ where $0 < \phi < \pi$, then*

$$\sinh(\tau(f))\sinh(\tau(g))\sin^{4/3}(\phi) \ge \lambda_1^2, \tag{3.6}$$

where $0.121 \le \lambda_1 \le 0.265$. *In particular,*

$$\max(\tau(f), \tau(g)) \ge \mu_1, \tag{3.7}$$

where $0.121 \le \mu_1 \le 0.308$. *The exponent of* $\sin(\phi)$ *in inequality* (3.6) *cannot be replaced by a constant greater than* $4/3$.

Proof. By Lemma 2.16 we can choose integers $m, n \geq 1$ such that

$$|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}} \sinh(\tau(f)), \quad |\beta(g^n)| \leq \frac{4\pi}{\sqrt{3}} \sinh(\tau(g)). \tag{3.8}$$

Then $\langle f^m, g^n \rangle$ is Kleinian and we obtain

$$\left(\frac{4\pi}{\sqrt{3}}\right)^2 \sinh(\tau(f)) \sinh(\tau(g)) \sin^{4/3}(\phi) \geq |\beta(f^m)\beta(g^n)| \sin^{4/3}(\phi) \geq b$$

by (3.8) and (3.2). This implies (3.6) and (3.7) with

$$\lambda_1 \geq \frac{\sqrt{3b}}{4\pi} \geq 0.1215, \quad \mu_1 \geq \operatorname{arcsinh}(\lambda_1) \geq 0.121.$$

Next, the polynomial $p(z) = z^4 + 6z^3 + 12z^2 + 9z + 1$ is monic with a pair of complex conjugate roots $\gamma, \bar{\gamma}$, where

$$\gamma = -1.5 + i.6066\ldots$$

and with two real roots in $(-3, 0)$. Then, by either [5, Thm. 5.14] or [8, Sec. 8], there exist elliptics f and g of orders 3 and 2 such that $\langle f, g \rangle$ is Kleinian with

$$\operatorname{par}(\langle f, g \rangle) = (\gamma, -3, -4).$$

Hence $\langle fg, g \rangle$ is Kleinian with

$$\operatorname{par}(\langle fg, g \rangle) = (\gamma, \gamma - 1, -4).$$

Let h denote the Lie product of the loxodromic fg and g, that is, the Möbius transformation of order 2 that interchanges the endpoints of fg and interchanges the endpoints of g (see [12] or [8]). Then $\langle fg, fh \rangle$ is Kleinian with

$$\operatorname{par}(\langle fg, fh \rangle) = (-1, \gamma - 1, -\gamma - 4) = (-1, \gamma - 1, \bar{\gamma} - 1)$$

by [12, Sec. 4]. Next, fg and fh are loxodromic with

$$\tau(fg) = \operatorname{arccosh}\left(\frac{|\gamma + 3| + |\gamma - 1|}{4}\right) = 0.3074\ldots \tag{3.9}$$

and $\tau(fh) = \tau(fg)$ by (2.3), and $\operatorname{ax}(fg)$ and $\operatorname{ax}(fh)$ intersect at an angle ϕ where

$$\phi = \arcsin\left(\frac{2}{|\gamma - 1|}\right) = 0.8905\ldots$$

by (2.6). Thus

$$\sinh(\tau(fg)) \sinh(\tau(fh)) \sin^{4/3}(\phi) = 0.06975\ldots \tag{3.10}$$

and we have

$$\lambda_1 \leq 0.265, \quad \mu_1 \leq 0.308$$

by (3.10) and (3.9), respectively.

We will give in Section 4 an example to show that the exponent $4/3$ of $\sin(\phi)$ in (3.6) cannot be increased. □

Next, Lemmas 2.13 and 2.16 yield the following inequality for Kleinian groups that corresponds to Theorem 1.4.

THEOREM 3.11. *If* $\langle f, g\rangle$ *is discrete, if* f *and* g *are loxodromics with* $\beta(f) = \beta(g)$, *and if* $\mathrm{ax}(f)$ *and* $\mathrm{ax}(g)$ *intersect at an angle* ϕ *where* $0 < \phi < \pi$, *then*

$$\sinh(\tau(f))\sin(\phi) \geq \lambda_2, \tag{3.12}$$

where $0.122 \leq \lambda_2 \leq 0.435$. *In particular,*

$$\tau(f) \geq \mu_2, \tag{3.13}$$

where $0.122 \leq \mu_2 \leq 0.492$. *The exponent of* $\sin(\phi)$ *in* (3.12) *cannot be replaced by a constant greater than* 1.

Proof. By Lemma 2.16 we can choose an integer $m \geq 1$ such that

$$|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}}\sinh(\tau(f)). \tag{3.14}$$

Then $\langle f^m, g^m\rangle$ is Kleinian with $\beta(f^m) = \beta(g^m)$ and we obtain

$$\begin{aligned}\frac{4\pi}{\sqrt{3}}\sinh(\tau(f))\sin(\phi)) &\geq (|\beta(f^m)\beta(g^m)|\sin^2(\phi))^{1/2}\\ &= (4|\gamma(f^m, g^m)|)^{1/2} \geq 2\sqrt{d}\end{aligned}$$

from (2.17), (2.6), and (2.15). This implies (3.12) and (3.13) with

$$\lambda_2 \geq \frac{\sqrt{3d}}{2\pi} \geq 0.1226, \qquad \mu_2 \geq \operatorname{arcsinh}(\lambda_2) \geq 0.122.$$

If f and g are elliptic of order 3 and 2 with

$$\gamma = \gamma(f, g) = -4\cos^2(\pi/7) = -3.2469\ldots,$$

then $\langle f, g\rangle$ is a $\mathbf{Z}_2$-extension of the (2, 3, 7)-triangle group obtained by adjoining the Lie product of the generators of orders 2 and 3 (see [12] or [8]). Next,

$$\beta(gf) = \beta(fg) = \gamma - 1$$

and

$$\tau(gf) = \tau(fg) = \operatorname{arccosh}\left(\frac{|\gamma+3| + |\gamma-1|}{4}\right) = 0.4919\ldots$$

by (2.3). Then

$$\gamma(fg, gf) = \gamma(f^2, gf) = \gamma(f, gf)(\beta(f)+4) = \gamma(f, g) = \gamma,$$

and we see from (2.5) and (2.6) that $\mathrm{ax}(fg)$ and $\mathrm{ax}(gf)$ meet at the angle $\phi \in (0, \pi)$ where

$$\sin^2(\phi) = -\frac{4\gamma}{(\gamma-1)^2} = 0.7200\ldots.$$

Hence

$$\lambda_2 \leq 0.435\ldots, \qquad \mu_2 \leq 0.491\ldots.$$

The example in Section 4 shows that the exponent of $\sin(\phi)$ in (3.12) cannot exceed 1. □

Finally, we consider what can be said about the translation length of a loxodromic f whose axis intersects the axis of an elliptic g.

THEOREM 3.15. *If $\langle f, g\rangle$ is discrete, if f is loxodromic and g is elliptic of order $n \geq 3$, and if* $\operatorname{ax}(f)$ *and* $\operatorname{ax}(g)$ *meet at an angle ϕ where $0 < \phi < \pi$, then*

$$\sinh(\tau(f))\sin^2(\pi/n)\sin^2(\phi) \geq \lambda_3^2, \tag{3.16}$$

where $0.184 \leq \lambda_3 \leq 0.568$. *Moreover,*

$$\tau(f) \geq \mu_3, \tag{3.17}$$

where $0.141 \leq \mu_3 \leq 0.832$.

Proof. We shall prove a stronger result by exhibiting, for each $n \geq 3$, constants depending on n for the left-hand sides of (3.16) and (3.17).

We may assume without loss of generality that g is a primitive elliptic. Next, by Lemma 2.16 we can choose an integer $m \geq 1$ such that

$$|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}}\sinh(\tau(f)).$$

Then $\langle f^m, g\rangle$ is Kleinian and f^m is not of order 2, so we obtain

$$\frac{4\pi}{\sqrt{3}}\sinh(\tau(f))\sin^2(\pi/n)\sin^2(\phi) \geq |\gamma(f^m, g)| \geq a(n) \tag{3.18}$$

from (2.6) and Lemma 2.18, where $a(n)$ is as in (2.20). Inequality (3.18) then yields (3.16) with

$$\lambda_3 \geq \sqrt{\sqrt{3}a(3)/4\pi} \geq 0.184\ldots.$$

Next, (3.18) implies that

$$\tau(f) \geq \mu_{3,n} \tag{3.19}$$

for $n \geq 4$, where

$$\mu_{3,n} = \begin{cases} 0.141\ldots & \text{if } n = 3,\\ 0.169\ldots & \text{if } n = 4,\\ 0.244\ldots & \text{if } n = 5,\\ 0.526\ldots & \text{if } n = 6,\end{cases} \tag{3.20}$$

and

$$\mu_{3,n} = \operatorname{arcsinh}\left(\frac{\sqrt{3}}{4\pi}\frac{1 - 4\sin^2(\pi/n)}{\sin^2(\pi/n)}\right) \geq 0.179\ldots \tag{3.21}$$

for $n \geq 7$.

To obtain (3.19) when $n = 3$, we apply Theorem 3.11 to the two loxodromic elements f and $h = gfg^{-1}$ whose axes intersect. The dihedral angle between the hyperbolic plane containing $\operatorname{ax}(g)$ and $\operatorname{ax}(f)$ and that containing $\operatorname{ax}(g)$ and $\operatorname{ax}(h)$ is $\pi/3$. Then spherical trigonometry implies that the angle θ between the axes of f and h is given by

$$\sin(\theta/2) = \sin(\pi/6)\sin(\phi) \leq \sin(\pi/6).$$

Thus $\theta \leq \pi/3$ and Theorem 3.11 implies (3.19) for $n = 3$ and (3.17) with $\mu_3 = \mu_{3,3}$.

The following result yields upper bounds for λ_3 and μ_3 (cf. [5]).

LEMMA 3.22. *If f and g are elliptics of orders 4 and 2 with $\gamma(f, g) \neq 0$, then $f^2 g$ is nonparabolic,* $\operatorname{ax}(f^2g)$ *meets* $\operatorname{ax}(f)$ *at angle $\pi/2$, and*

$$\tau(f^2 g) = \operatorname{arccosh}\left(\frac{|\gamma(f, g) + 2| + |\gamma(f, g)|}{2}\right). \tag{3.23}$$

Proof. Since g is of order 2,

$$\begin{aligned}
\beta(f^2 g) &= \gamma(f^2, g) - \beta(f^2) - 4 \\
&= (\gamma(f, g) - \beta(f))(\beta(f) + 4) - 4 \\
&= 2(\gamma(f, g) + 2) - 4 \\
&= 2\gamma(f, g) = 2\gamma(f^2 g, f) \neq 0.
\end{aligned}$$

Hence $f^2 g$ is nonparabolic,

$$\frac{4\gamma(f^2 g, f)}{\beta(f^2 g)\beta(f)} = -1,$$

and $\operatorname{ax}(f^2 g)$ meets $\operatorname{ax}(f)$ at angle $\pi/2$ by (2.5). □

If f and g are primitive elliptics of orders 8 and 2 with

$$\gamma(f, g) = \sqrt{2} - 1, \qquad \gamma(f^2, g) = \sqrt{2},$$

then $\langle f, g\rangle$ is the (2, 3, 8)-triangle group. Hence $\operatorname{ax}(f^4 g)$ meets $\operatorname{ax}(f)$ at angle $\phi = \pi/2$,

$$\tau(f^4 g) = \operatorname{arccosh}(\sqrt{2} + 1) = 1.528\ldots,$$

and

$$\lambda_3^2 \leq \sinh(\tau(f^4 g)) \sin^2(\pi/8) \sin^2(\phi) = 0.322\ldots$$

by Lemma 3.22.

Similarly, if f and g are primitive elliptics of orders 4 and 2 with

$$\gamma(f, g) = \frac{-1 + i\sqrt{3}}{2},$$

then $\langle f, g\rangle$ is Kleinian by [8, Thm. 8.23], $\operatorname{ax}(f^2 g)$ meets $\operatorname{ax}(f)$ at angle $\phi = \pi/2$, and

$$\mu_3 \leq \tau(f^2 g) = 0.831\ldots,$$

again by Lemma 3.22. □

Theorem 3.15 holds only when g is an elliptic of order $n \geq 3$. Theorem 3.11 yields the following counterpart of Theorem 3.15 for the case when g is of order 2.

THEOREM 3.24. *If $\langle f, g\rangle$ is discrete, if f is loxodromic and g elliptic of order 2, and if* $\operatorname{ax}(f)$ *and* $\operatorname{ax}(g)$ *intersect at an angle ϕ where $0 < \phi < \pi/2$, then*

$$\sinh(\tau(f))\sin(2\phi) \geq \lambda_2 \tag{3.25}$$

and

$$\tau(f) \geq \mu_2, \tag{3.26}$$

where λ_2 and μ_2 are as in Theorem 3.11.

Proof. Let $h = gfg^{-1}$. Then $\langle f, h\rangle$ is discrete, $\beta(f) = \beta(h)$, and the axes of f and h meet at the angle 2ϕ. Hence (3.26) and (3.27) follow from (3.12) and (3.13). The example exhibited in the proof of Theorem 3.11 yields the same upper bounds for λ_2 and μ_2. □

4. An Example

We present here an example due to Jørgensen to show that the exponents of $\sin(\phi)$ in Theorems 2.9, 3.5, and 3.11 cannot be increased.

For $a > 2$, let

$$f(z) = -a^2 z, \qquad g(z) = \frac{(a^2 + a^{-2})z - 2}{2z - (a^2 + a^{-2})}.$$

Then, by [13], $\langle f, g\rangle$ is Kleinian with

$$\gamma(f, g) = 4(a - a^{-1})^{-2}, \quad \beta(f) = -(a + a^{-1})^2, \quad \beta(g) = -4.$$

Hence $\langle fg, f\rangle$ and $\langle fg, gf\rangle$ are Kleinian with

$$\gamma(fg, f) = \gamma(f, g) = 4(a - a^{-1})^{-2},$$
$$\gamma(fg, gf) = \gamma(fg, f^2) = \gamma(fg, f)(\beta(f) + 4) = -4,$$
$$\beta(fg) = \gamma(f, g) - \beta(f) - 4 = 4(a - a^{-1})^{-2} + (a - a^{-1})^2.$$

Because

$$-1 < \frac{4\gamma(fg, f)}{\beta(fg)\beta(f)} < 0,$$

we see from (2.5) and (2.6) that $\mathrm{ax}(fg)$ and $\mathrm{ax}(f)$ intersect at an angle $\phi_1 \in (0, \pi)$ where

$$\sin^2(\phi_1) = -\frac{4\gamma(fg, f)}{\beta(fg)\beta(f)} \sim \frac{16a^{-2}}{a^4} = \frac{16}{a^6}$$

as $a \to \infty$. Next, $\beta(fg) = \beta(gf) > 4$,

$$-1 < \frac{4\gamma(fg, gf)}{\beta(fg)\beta(gf)} < 0,$$

and $\mathrm{ax}(fg)$ and $\mathrm{ax}(gf)$ intersect at an angle $\phi_2 \in (0, \pi)$, where

$$\sin^2(\phi_2) = -\frac{4\gamma(fg, gf)}{\beta(fg)\beta(gf)} \sim \frac{16}{a^4}$$

as $a \to \infty$. By (2.3),

$$\cosh(\tau(f)) = \frac{|\beta(f) + 4| + |\beta(f)|}{4} \sim \frac{a^2}{2}$$

and

$$\cosh(\tau(fg)) = \frac{|\beta(fg)+4| + |\beta(fg)|}{4} \sim \frac{a^2}{2},$$

whence

$$\sinh(\tau(fg))\sinh(\tau(f)) \sim \cosh(\tau(fg))\cosh(\tau(f)) \sim \frac{a^4}{4}$$

and

$$\sinh(\tau(fg)) \sim \cosh(\tau(fg)) \sim \frac{a^2}{2}, \qquad \sinh(\tau(fg)/2) \sim \frac{a}{2}$$

as $a \to \infty$. Thus

$$\sinh(\tau(fg))\sinh(\tau(f))\sin^s(\phi_1) \sim 4^{s-1}a^{4-3s} \tag{4.1}$$

as $a \to \infty$, and the left-hand side of (4.1) is bounded away from 0 only if $s \le 4/3$. Thus the exponent of $\sin(\phi)$ in Theorem 3.5 cannot be greater than $4/3$. Next, fg and gf are hyperbolic with $\tau(fg) = \tau(gf)$ and with

$$\sinh(\tau(fg))\sin^s(\phi_2) \sim 2^{2s-1}a^{2-2s} \tag{4.2}$$

and

$$\sinh(\tau(fg)/2)\sinh(\tau(gf)/2)\sin^s(\phi_2) \sim 4^{s-1}a^{2-2s} \tag{4.3}$$

as $a \to \infty$, and the left-hand sides of (4.2) and (4.3) are bounded away from 0 only if $s \le 1$. Hence the exponents of $\sin(\phi)$ in Theorems 2.9 and 3.11 cannot exceed 1.

5. Final Remarks

The theorems in Section 1 concerning geodesics in hyperbolic 3-folds now follow from the corresponding theorems in Sections 2 and 3. For if $\mathcal{Q} = \mathbf{H}^3/G$ is a hyperbolic 3-orbifold, then a closed geodesic in $\mathcal{Q}$ lifts to a hyperbolic line stabilized by a loxodromic transformation $f \in G$ with $\ell(\alpha) = \tau(f)$. If α is not a simple geodesic and has an angle of self-intersection ϕ, then there is a conjugate g of f in G such that the axes of f and g meet at the angle ϕ; the angle is preserved because the projection map from $\mathbf{H}^3$ to the quotient $\mathcal{Q}$ is locally conformal. A similar situation arises for two different geodesics meeting at an angle ϕ. In particular, Theorem 1.2 is implied by Theorem 3.5, Theorem 1.4 is implied by Theorem 3.12, and Theorem 1.6 follows from Theorem 2.9.

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