The Moduli of Holomorphic Functions in Lipschitz Spaces

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1. Introduction and Results

Given $0 < \alpha < 1$, let Λ^α denote the classical *Lipschitz space* of the real line $\mathbb{R}$, that is, the set of all complex-valued functions $f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$|f(t_1) - f(t_2)| \le \text{const}\,|t_1 - t_2|^\alpha, \quad t_1, t_2 \in \mathbb{R}$$

(the constant on the right may depend only on f). Further, let $\Lambda^\alpha_{\mathrm{A}}$ stand for the corresponding *analytic subspace* consisting of those functions in Λ^α whose harmonic extensions (Poisson integrals) are holomorphic on

$$\mathbb{C}_+ \stackrel{\text{def}}{=} \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}.$$

In other words, elements of $\Lambda^\alpha_{\mathrm{A}}$ are just H^∞ functions with boundary values in Λ^α (as usual, H^∞ denotes the algebra of bounded holomorphic functions on $\mathbb{C}_+$).

The problem we treat here is to characterize the absolute values of $\Lambda^\alpha_{\mathrm{A}}$ functions. More precisely, given a nonnegative function φ on $\mathbb{R}$, we are concerned with explicit conditions under which φ agrees with (the boundary values of) the modulus $|f|$ of some function $f \in \Lambda^\alpha_{\mathrm{A}}$.

The two immediate necessary conditions are

$$\varphi \in \Lambda^\alpha \tag{1.1}$$

and, if we exclude the trivial function $\varphi \equiv 0$ from consideration,

$$\int_{-\infty}^{\infty} \frac{\log \varphi(t)}{1+t^2}\, dt > -\infty. \tag{1.2}$$

In connection with (1.2), see [G, Chap. II, Sec. 4].

Once (1.2) holds, we form the *outer function* $\mathcal{O}_\varphi$ with modulus φ by setting

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp\left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) \log \varphi(t)\, dt \right\}, \quad z \in \mathbb{C}_+,$$

and note that the above problem is equivalent to ascertaining when

$$\mathcal{O}_\varphi \in \Lambda^\alpha_{\mathrm{A}}. \tag{1.3}$$

Received January 31, 1996.

Supported in part by Grants R2D000 and R2D300 from the International Science Foundation and by a grant from Pro Mathematica (France).

Michigan Math. J. 44 (1997).

(The equivalence is due to the fact that the outer part of a Λ_A^α function must itself belong to Λ_A^α; see [H1].)

We now remark that conditions (1.1) and (1.2) alone are not at all sufficient for (1.3) to hold. In fact, according to [HS] and [H2] (as well as to an unpublished result of Carleson and Jacobs), (1.1) and (1.2) together imply merely that $\mathcal{O}_\varphi \in \Lambda_A^{\alpha/2}$, the exponent $\alpha/2$ being best possible.

In this paper we point out a new crucial condition on φ (stated in several equivalent ways) that provides, in conjunction with (1.1) and (1.2), a complete characterization of outer functions $\mathcal{O}_\varphi$ lying in Λ_A^α. This is contained in Theorem 1.

An alternative description of Λ_A^α moduli was obtained earlier by Shirokov (see [S, Chap. II]). His criterion, stated in terms of a certain maximal function associated with φ, looks somewhat more complicated than ours, and so does the proof. On the other hand, Shirokov's description works also in the case $\alpha \in (0, +\infty)\setminus\mathbb{Z}$, with the appropriate understanding of the spaces in question. Anyway, both the results and techniques of the present paper are different from (and independent of) those in [S]. Moreover, we have been unable to find a direct proof of the equivalence between the two characterizations.

Now let $d\mu_z$ stand for the *harmonic measure* on $\mathbb{R}$ associated with a point $z \in \mathbb{C}_+$; that is,

$$d\mu_z(t) \stackrel{\text{def}}{=} \frac{1}{\pi}\frac{\operatorname{Im} z}{|t-z|^2}\,dt, \qquad t \in \mathbb{R}.$$

Further, let $\operatorname{Lip}\alpha(\bar{\mathbb{C}}_+)$ denote the set of all (complex-valued) functions f living on $\bar{\mathbb{C}}_+ \stackrel{\text{def}}{=} \mathbb{C}_+ \cup \mathbb{R}$ and satisfying

$$|f(z_1) - f(z_2)| \le \text{const}|z_1 - z_2|^\alpha \quad \text{whenever } z_1, z_2 \in \bar{\mathbb{C}}_+.$$

Note that $\Lambda_A^\alpha = H^\infty \cap \operatorname{Lip}\alpha(\bar{\mathbb{C}}_+)$. Our main result is as follows.

THEOREM 1. *Let $0 < \alpha < 1$ and let φ be a nonnegative function on $\mathbb{R}$ satisfying* (1.1) *and* (1.2). *The following are equivalent.*

(i) $\mathcal{O}_\varphi \in \Lambda_A^\alpha$.

(ii) $|\mathcal{O}_\varphi| \in \operatorname{Lip}\alpha(\bar{\mathbb{C}}_+)$.

(iii) *As $z = x + iy$ ranges over $\mathbb{C}_+$, one has*

$$\varphi(x) - |\mathcal{O}_\varphi(z)| = O(y^\alpha).$$

(iv) *As $z = x + iy$ ranges over $\mathbb{C}_+$, one has*

$$\int \varphi\, d\mu_z - \exp\left(\int \log\varphi\, d\mu_z\right) = O(y^\alpha). \tag{1.4}$$

Yet another similar criterion, valid for $\alpha \in (0, 1/2)$ only (and playing an auxiliary role in what follows), is given by the next theorem.

THEOREM 2. *Let $0 < \alpha < 1/2$ and let $\varphi \ge 0$ be a function in $L^\infty(\mathbb{R})$ such that* (1.2) *holds true. The following are equivalent.*

(i) $\mathcal{O}_\varphi \in \Lambda^\alpha_{\mathrm{A}}$.
(ii) *For* $z = x + iy \in \mathbb{C}_+$,

$$\int \varphi^2 \, d\mu_z - \exp\biggl(2 \int \log \varphi \, d\mu_z\biggr) = O(y^{2\alpha}). \tag{1.5}$$

REMARK. Using the Garsia norm on the space BMO (see [G, Chap. VI] or [K, Chap. X]), one easily obtains the following supplement to Theorem 2: Given a nonnegative function $\varphi \in L^2(\mathbb{R}, dt/(1+t^2))$ satisfying (1.2), we have $\mathcal{O}_\varphi \in$ BMOA if and only if the left-hand side of (1.5) is bounded on $\mathbb{C}_+$.

Our further strategy is as follows. In Section 2, we give two useful characterizations of $\Lambda^\alpha_{\mathrm{A}}$ functions. One of these involves Poisson integrals; the other is a remarkable theorem of Dyn'kin on the so-called pseudoanalytic extension.

In Section 3, we first prove Theorem 2 and then use it to derive Theorem 1. The proof of Theorem 2 is quite elementary (it relies on the Poisson integral characterization from Section 2), but the passage to Theorem 1 seems to require more sophisticated reasoning. It is here that Dyn'kin's $\bar\partial$ techniques are brought into play. Finally, Section 4 contains a few concluding remarks.

2. Preliminaries on Lipschitz Spaces

The following lemma is essentially known (in connection with part (1), see e.g. [CS, Prop. 1]). Nonetheless, we include a short proof.

LEMMA 1. *Assume that* $f \in H^\infty$.

(1) *If* $0 < \alpha < 1$, *then conditions* (a) *and* (b) *are equivalent:*
  (a) $f \in \Lambda^\alpha_{\mathrm{A}}$;
  (b) *for* $z = x + iy \in \mathbb{C}_+$,

$$\int |f(t) - f(z)| \, d\mu_z(t) = O(y^\alpha). \tag{2.1}$$

(2) *If* $0 < \alpha < 1/2$, *then* (a) *and* (b) *are also equivalent to the following condition:*
  (c) *for* $z = x + iy \in \mathbb{C}_+$,

$$\int |f(t) - f(z)|^2 \, d\mu_z(t) = O(y^{2\alpha}). \tag{2.2}$$

Proof. If (a) holds then, for $0 < \alpha < 1$,

$$\int |f(t) - f(z)| \, d\mu_z(t) \le \text{const} \int |t - z|^\alpha \, d\mu_z(t) = O(y^\alpha).$$

Similarly, in the case $0 < \alpha < 1/2$, (a) yields

$$\int |f(t) - f(z)|^2 \, d\mu_z(t) \le \text{const} \int |t - z|^{2\alpha} \, d\mu_z(t) = O(y^{2\alpha}).$$

The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are thus established for the appropriate values of α.

Since (c) $\Rightarrow$ (b) by the Cauchy–Schwarz inequality, it remains to prove that (b) $\Rightarrow$ (a). To this end, we write

$$\begin{aligned}\int |f(t)-f(z)|\,d\mu_z(t) &= \frac{y}{\pi}\int \frac{|f(t)-f(z)|}{|t-z|^2}\,dt \\ &\geq 2y\left|\frac{1}{2\pi i}\int \frac{f(t)-f(z)}{(t-z)^2}\,dt\right| = 2y|f'(z)|.\end{aligned}$$

In conjunction with (b), this gives

$$f'(z) = O(y^{\alpha-1}),$$

which is but a well-known restatement of (a) (see e.g. [St, Chap. V]). □

As another auxiliary result, we cite an important theorem due to Dyn'kin. Before stating it, we introduce the notation $\mathbb{C}_-$ for the lower half-plane, so that

$$\mathbb{C}_- \stackrel{\text{def}}{=} \mathbb{C}\setminus(\mathbb{C}_+\cup\mathbb{R}),$$

and recall that the Cauchy–Riemann operator $\bar\partial$ is defined by

$$\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x+iy.$$

LEMMA 2 (cf. [Dyn1; Dyn2]). *Let f be an H^∞ function continuous up to $\mathbb{R}$. In order that $f \in \Lambda^\alpha_A$ $(0<\alpha<1)$, it is necessary and sufficient that there exist a function $F \in C^1(\mathbb{C}_-)$ such that*

$$\lim_{\substack{z\to t\\ z\in\mathbb{C}_-}} F(z) = f(t) \quad \text{for all } t\in\mathbb{R} \tag{2.3}$$

and

$$\bar\partial F(z) = O(|y|^{\alpha-1}), \quad z = x+iy \in \mathbb{C}_-. \tag{2.4}$$

3. Proofs of the Theorems

PROOF OF THEOREM 2. In view of Lemma 1(2), condition (i) of Theorem 2 is equivalent to the relation

$$\int |\mathcal{O}_\varphi(t) - \mathcal{O}_\varphi(z)|^2\,d\mu_z(t) = O(y^{2\alpha}), \quad z\in\mathbb{C}_+ \tag{3.1}$$

(this is precisely (2.2) with $f = \mathcal{O}_\varphi$). Rewriting the left-hand side of (3.1) as

$$\int |\mathcal{O}_\varphi|^2\,d\mu_z - |\mathcal{O}_\varphi(z)|^2 = \int \varphi^2\,d\mu_z - \exp\left(2\int \log\varphi\,d\mu_z\right), \tag{3.2}$$

we see that (3.1) coincides with (1.5). The desired equivalence relation is thus established. □

PROOF OF THEOREM 1. Since the modulus of a Lip α function is again a Lip α function, it is clear that (i) implies (ii). The implication (ii) $\Rightarrow$ (iii) is also obvious.

To see that (iii) implies (iv), we note that the left-hand side of (1.4) equals

$$\int \varphi\, d\mu_z - |\mathcal{O}_\varphi(z)| = \left\{\int \varphi\, d\mu_z - \varphi(x)\right\} + \{\varphi(x) - |\mathcal{O}_\varphi(z)|\}$$
$$\stackrel{\text{def}}{=} A(z) + B(z).$$

Since the Poisson integral of a Λ^α function belongs to Lip $\alpha(\bar{\mathbb{C}}_+)$ (cf. [St, Chap. V]), we have $A(z) = O(y^\alpha)$. On the other hand, condition (iii) says $B(z) = O(y^\alpha)$. The two estimates yield (iv).

It remains to prove that (iv) implies (i). Setting $\varphi_1 \stackrel{\text{def}}{=} \sqrt{\varphi}$ (here $\sqrt{}$ is the positive branch of the square root) and substituting $\varphi = \varphi_1^2$ into (1.4), we obtain

$$\int \varphi_1^2\, d\mu_z - \exp\left(2\int \log\varphi_1\, d\mu_z\right) = O(y^\alpha), \tag{3.3}$$

which is precisely condition (1.5) with φ replaced by φ_1 and α replaced by $\alpha/2$. By Theorem 2, this means that the outer function $f_1 \stackrel{\text{def}}{=} \mathcal{O}_{\varphi_1}$ belongs to $\Lambda_A^{\alpha/2}$.

Since our aim is to show that $f \stackrel{\text{def}}{=} \mathcal{O}_\varphi$ belongs to Λ_A^α, it now suffices to verify the following.

Claim. It f_1 is an outer function in $\Lambda_A^{\alpha/2}$ with $|f_1|^2 \in \Lambda^\alpha$, then $f_1^2 \in \Lambda_A^\alpha$.

The proof of this assertion will be based on Lemma 2. As before, we put $f = f_1^2$ and $\varphi = |f| = |f_1|^2$. (It is understood that f_1 and f live on $\mathbb{C}_+ \cup \mathbb{R}$, while φ lives on $\mathbb{R}$.) In order to show that $f \in \Lambda_A^\alpha$, we shall construct an appropriate *pseudoanalytic extension* of f into $\mathbb{C}_-$, that is, a function $F \in C^1(\mathbb{C}_-)$ satisfying (2.3) and (2.4).

Throughout the rest of this section, $z = x + iy$ will denote a point in $\mathbb{C}_-$, so that $y < 0$. Set

$$F_1(z) \stackrel{\text{def}}{=} f(\bar{z}), \quad \psi(z) \stackrel{\text{def}}{=} \int \varphi(t)\, d\mu_{\bar{z}}(t),$$

and

$$F_2(z) \stackrel{\text{def}}{=} \psi^2(z)/\overline{f(\bar{z})}.$$

Further, let $\chi \in C^1[0, +\infty)$ be a nondecreasing function such that

$$\chi(t) = 0 \quad \text{for } 0 \le t \le 1$$

and

$$\chi(t) = 1 \quad \text{for } 2 \le t < +\infty.$$

Finally, we define the desired pseudoanalytic extension by

$$F(z) \stackrel{\text{def}}{=} F_1(z)\left\{1 - \chi\left(\frac{|f(\bar{z})|}{|y|^\alpha}\right)\right\} + F_2(z)\chi\left(\frac{|f(\bar{z})|}{|y|^\alpha}\right). \tag{3.4}$$

In order to check (2.3) and (2.4), we introduce the sets

$$E_1 \stackrel{\text{def}}{=} \{ z \in \mathbb{C}_- : |f(\bar{z})| \le |y|^\alpha \},$$
$$E_2 \stackrel{\text{def}}{=} \{ z \in \mathbb{C}_- : |f(\bar{z})| \ge 2|y|^\alpha \},$$
$$E_3 \stackrel{\text{def}}{=} \mathbb{C}_- \backslash (E_1 \cup E_2).$$

We distinguish three cases.

Case 1: $z \in E_1$. It follows that $F(z) = F_1(z) = f(\bar{z})$, whence

$$\lim_{\substack{\zeta \to t \\ \zeta \in E_1}} F(\zeta) = f(t), \quad t \in \mathbb{R} \cap \operatorname{clos} E_1. \tag{3.5}$$

Also, since $f = f_1^2$, we have

$$\bar{\partial} F(z) = f'(\bar{z}) = 2 f_1(\bar{z}) f_1'(\bar{z}),$$

and so

$$|\bar{\partial} F(z)| = 2|f_1(\bar{z})||f_1'(\bar{z})| \le \text{const}|y|^{\alpha/2}|y|^{\alpha/2-1} = \text{const}|y|^{\alpha-1}.$$

(Here we have used the inequality $|f_1(\bar{z})| \le |y|^{\alpha/2}$, valid because $z \in E_1$, and the estimate $f_1'(\bar{z}) = O(|y|^{\alpha/2-1})$, which is due to the hypothesis that $f_1 \in \Lambda_A^{\alpha/2}$.) Thus

$$\bar{\partial} F(z) = O(|y|^{\alpha-1}), \quad z \in E_1. \tag{3.6}$$

Case 2: $z \in E_2$. We have then

$$F(z) = F_2(z) = \psi^2(z)/\overline{f(\bar{z})}. \tag{3.7}$$

Next, we observe that

$$\frac{\psi(z)}{|f(\bar{z})|} = O(1). \tag{3.8}$$

Indeed, one has

$$\begin{aligned} \frac{\psi(z)}{|f(\bar{z})|} &= \frac{\psi(z) - |f(\bar{z})|}{|f(\bar{z})|} + 1 \\ &= \frac{1}{|f(\bar{z})|} \left\{ \int \varphi \, d\mu_{\bar{z}} - \exp\left(\int \log \varphi \, d\mu_{\bar{z}} \right) \right\} + 1. \end{aligned}$$

By (1.4), the expression in {braces} is $O(|y|^\alpha)$, while $|f(\bar{z})| \ge 2|y|^\alpha$ because $z \in E_2$. Consequently, the ratio in question is bounded.

(We remark that condition (1.4), employed here with z replaced by $\bar{z}$, actually follows from the hypotheses of the Claim. In fact, we have seen that the inclusion $f_1 \in \Lambda_A^{\alpha/2}$ is equivalent to (3.3) with $\varphi_1 = |f_1|$, which in turn coincides with (1.4).)

It is now clear that

$$\lim_{\substack{\zeta \to t \\ \zeta \in E_2}} F(\zeta) = f(t), \quad t \in \mathbb{R} \cap \operatorname{clos} E_2. \tag{3.9}$$

Indeed, if $f(t) \ne 0$ then (3.9) is immediate from (3.7) and the fact that $\psi|_{\mathbb{R}} = \varphi = |f|$. In case $f(t) = 0$, one should also invoke (3.8).

Further, since $\varphi \in \Lambda^\alpha$ and ψ is the Poisson integral of φ, we have

$$|\nabla\psi(z)| = O(|y|^{\alpha-1})$$

(cf. [St, Chap. V, Sec. 4]), and hence also

$$\bar\partial\psi(z) = O(|y|^{\alpha-1}). \tag{3.10}$$

Using (3.7) and the fact that the function $z \mapsto 1/\overline{f(\bar z)}$ is holomorphic on $\mathbb{C}_-$, we write

$$\bar\partial F(z) = \frac{1}{\overline{f(\bar z)}} \cdot 2\psi(z) \cdot \bar\partial\psi(z)$$

and then conclude from (3.8) and (3.10) that

$$\bar\partial F(z) = O(|y|^{\alpha-1}), \quad z \in E_2. \tag{3.11}$$

Case 3: $z \in E_3$. In this case, we have

$$|y|^\alpha < |f(\bar z)| < 2|y|^\alpha. \tag{3.12}$$

The arguments pertaining to Cases 1 and 2 show also that

$$F_1(z) = O(|y|^\alpha), \qquad F_2(z) = O(|y|^\alpha) \tag{3.13}$$

and

$$\bar\partial F_1(z) = O(|y|^{\alpha-1}), \qquad \bar\partial F_2(z) = O(|y|^{\alpha-1}). \tag{3.14}$$

Given $t \in \mathbb{R} \cap \operatorname{clos} E_3$, it follows at once from (3.12) that $f(t) = 0$. Therefore, (3.13) yields

$$\lim_{\substack{\zeta\to t\\ \zeta\in E_3}} F(\zeta) = f(t), \quad t \in \mathbb{R} \cap \operatorname{clos} E_3. \tag{3.15}$$

Differentiating (3.4) gives

$$\begin{aligned}\bar\partial F(z) &= \bar\partial F_1(z)\left\{1 - \chi\left(\frac{|f(\bar z)|}{|y|^\alpha}\right)\right\} + \bar\partial F_2(z)\cdot\chi\left(\frac{|f(\bar z)|}{|y|^\alpha}\right)\\ &\quad + (F_2(z) - F_1(z))\cdot\chi'\left(\frac{|f(\bar z)|}{|y|^\alpha}\right)\cdot\bar\partial\left(\frac{|f(\bar z)|}{|y|^\alpha}\right).\end{aligned} \tag{3.16}$$

The first two terms on the right are $O(|y|^{\alpha-1})$, as is readily seen from (3.14). In order to obtain a similar estimate for the third term, we note that

$$F_2(z) - F_1(z) = O(|y|^\alpha)$$

(see (3.13)), while χ' is bounded; thus, it would suffice to check that

$$\bar\partial\left(\frac{|f(\bar z)|}{|y|^\alpha}\right) = O\left(\frac{1}{|y|}\right). \tag{3.17}$$

This can be done by a straightforward calculation:

$$\begin{aligned}\bar\partial\left(\frac{|f(\bar z)|}{|y|^\alpha}\right) &= |y|^{-\alpha}\bar\partial|f(\bar z)| + |f(\bar z)|\bar\partial(|y|^{-\alpha})\\ &= |y|^{-\alpha}\bar\partial\left(f_1(\bar z)\overline{f_1(\bar z)}\right) + \tfrac{1}{2}i\alpha|f(\bar z)||y|^{-\alpha-1}\\ &= |y|^{-\alpha}\overline{f_1(\bar z)}f_1'(\bar z) + \tfrac{1}{2}i\alpha|f(\bar z)||y|^{-\alpha-1}\\ &= O(|y|^{-1}).\end{aligned}$$

Here the final conclusion relies on the right-hand inequality in (3.12), which is also used in the form

$$|f_1(\bar{z})| < \sqrt{2}\,|y|^{\alpha/2},$$

and on the estimate

$$f_1'(\bar{z}) = O(|y|^{\alpha/2-1})$$

(recall that $f_1 \in \Lambda_A^{\alpha/2}$).

We eventually arrive at (3.17), which in turn implies, by virtue of (3.16) and the subsequent remarks, that

$$\bar{\partial}F(z) = O(|y|^{\alpha-1}), \quad z \in E_3. \tag{3.18}$$

Now that the three cases have been studied, a mere juxtaposition of (3.5), (3.9), and (3.15) yields (2.3), whereas a similar juxtaposition of (3.6), (3.11), and (3.18) yields (2.4). An application of Lemma 2 completes the proof of the Claim, as well as that of the theorem. □

4. Concluding Remarks

(1) In Theorem 1, one might as well deduce the implication (i) $\Rightarrow$ (iv) from Lemma 1(1). In fact, we see that condition (2.1), with f outer, is equivalent to the seemingly weaker condition where the integrand is replaced by $|f(t)| - |f(z)|$.

(2) This paper deals with outer functions only, but the interplay of inner and outer factors of Λ_A^α functions has also been studied. In this connection, we refer to [D1], [D2], [D3], and [S, Chap. I].

(3) Given $\alpha > 1$, $\alpha \notin \mathbb{N}$, set

$$\Lambda^\alpha \stackrel{\text{def}}{=} \{ f \in C^{[\alpha]}(\mathbb{R}) \cap L^\infty(\mathbb{R}) : f^{([\alpha])} \in \Lambda^{\alpha-[\alpha]} \}$$

(here $[\alpha]$ is the integral part of α), and let the spaces Λ_A^α and $\operatorname{Lip}\alpha(\bar{\mathbb{C}}_+)$ be introduced in a similar fashion. We remark that the (obvious) implication (i) $\Rightarrow$ (ii) in Theorem 1 becomes obviously false when $\alpha > 1$. For example, the function $f(z) = z/(z+i)$ is in $H^\infty \cap C^\infty(\bar{\mathbb{C}}_+)$, whereas $|f(x)|$ is nondifferentiable at the origin, and so $|f(z)|$ fails to belong to any $\operatorname{Lip}\alpha(\bar{\mathbb{C}}_+)$ with $\alpha > 1$. This feature (i.e., the failure of our criterion for higher-order smoothness classes) distinguishes our approach from Shirokov's [S]. On the other hand, it might be still possible to modify condition (iii) and/or (iv) of Theorem 1 so as to provide the requested characterization in the case $\alpha > 1$, $\alpha \notin \mathbb{N}$.

(4) In the near future, the author is planning to extend the current results to the Lipschitz-type spaces $\operatorname{Lip}\omega$ (generated by continuity moduli ω other than $\omega(t) = t^\alpha$) and to spaces of functions that are "smooth in the mean" (e.g. Besov and Sobolev spaces).

(5) The membership criterion given by Theorem 1 seems to be fairly manageable and efficient. For instance, it can be used to derive the Havin–Shamoyan–Carleson–Jacobs theorem saying that $\varphi \in \Lambda^\alpha$ implies $\mathcal{O}_\varphi \in \Lambda_A^{\alpha/2}$ (see Section 1). This will be presented elsewhere.

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