Structural Stability of Kleinian Groups

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0. Introduction

A dynamical system is called *structurally stable* if in general its topological structure is invariant under small perturbation. In the theory of complex dynamics of Kleinian groups and iterated rational maps, problems on structural stability have played an important role. Sullivan [17] proved that, if a finitely generated torsion-free Kleinian group is structurally stable, then its action on the limit set is expanding; however, this problem is still open for iterated rational maps. Expanding on the limit set means convex cocompactness of the Kleinian group. In this paper we remove the assumption "torsion-free" from Sullivan's result; we characterize structural stability of finitely generated Kleinian groups that may contain elliptic elements. The precise definition of structural stability and the statement of our main theorem are given in the next section.

We say a Kleinian group is *quasiconformally stable* if its small perturbations preserving the parabolic elements are all induced by quasiconformal automorphisms of the Riemann sphere. This property has been considered in the quasiconformal deformation theory of Kleinian groups. From the so-called λ-lemma, we see that if small perturbations are all isomorphic then they must be quasiconformal deformations. Hence quasiconformal stability follows from structural stability. Marden [9] proved that geometrically finite Kleinian groups are quasiconformally stable. The converse is also true for torsion-free groups, which is a corollary to Sullivan's theorem just mentioned. Namely, geometric finiteness and quasiconformal stability are equivalent in the torsion-free case. In the last section of this paper, the assumption "torsion-free" is removed also from this equivalence.

1. Preliminaries and the Statement of the Main Theorem

Let G be a finitely generated group with a fixed system of generators $G = \langle g_1, \dots, g_m \rangle$. We define $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ as the set of all $\mathrm{PSL}(2, \mathbb{C})$-representations of G. It is regarded as an algebraic subvariety of $V_G = \mathrm{PSL}(2, \mathbb{C})^m$ by the identification

$$\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C})) \ni \rho \mapsto (\rho(g_1), \dots, \rho(g_m)) \in V_G.$$

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We say that ρ (or $\rho(G)$) is *structurally stable* if there exists an open neighborhood U ($\subset V_G$) of ρ such that each representation in $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C})) \cap U$ is faithful.

Our main theorem is a generalization of Sullivan's result [17] about structural stability of torsion-free groups. The statement is as follows.

THEOREM 1. *Let G be a finitely generated nonsolvable group (possibly with torsion) and let $\rho: G \to \mathrm{PSL}(2, \mathbb{C})$ be a faithful representation (not necessarily discrete). Then ρ is structurally stable if and only if either ρ is rigid or $\rho(G)$ is a geometrically finite Kleinian group all of whose cusps are rigid.*

Here we should define the terms that appear in the above theorem. A faithful representation $\rho: G \to \mathrm{PSL}(2, \mathbb{C})$ is called *rigid* if there is a neighborhood $U \subset V_G$ of ρ such that all elements of $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C})) \cap U$ are given by conjugations of $\mathrm{PSL}(2, \mathbb{C})$.

A discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is called a Kleinian group. It is identified with an isometry group acting properly discontinuously on the upper half space model $(\mathbf{H}^3, ds)$ of the hyperbolic space, where

$$\mathbf{H}^3 = \{ (x, y, t) \in \mathbf{R}^3 \mid t > 0 \}, \qquad ds^2 = (dx^2 + dy^2 + dt^2)/t^2.$$

Thus the quotient space $N_\Gamma = \mathbf{H}^3/\Gamma$ admits hyperbolic orbifold structure. When Γ contains elements of finite order, N_Γ has an exceptional set (branch loci) where the hyperbolic structure is singular.

Let us denote by $\Omega(\Gamma)$ the region of discontinuity, which is the maximal open subset of $\hat{\mathbb{C}} = \partial \mathbf{H}^3$ where Γ acts properly discontinuously. It may be empty. We define the limit set $\Lambda(\Gamma)$ of Γ as the complement of $\Omega(\Gamma)$ in $\hat{\mathbb{C}}$. If the limit set consists of more than two points, we say that Γ is non-elementary. Otherwise, Γ is elementary. The assumption in Theorem 1 that G is nonsolvable implies that most discrete faithful representations are non-elementary. For a non-elementary Kleinian group, the limit set is the minimal non-empty invariant closed set in $\hat{\mathbb{C}}$.

We define $\mathrm{Hull}(\Lambda(\Gamma))$ ($\subset \mathbf{H}^3$) as the convex hull of the set of all the geodesics joining any two points of the limit set $\Lambda(\Gamma)$. It is the minimal invariant contractible closed convex set in $\mathbf{H}^3$. The convex kernel of N_Γ is $C_\Gamma = \mathrm{Hull}(\Lambda(\Gamma))/\Gamma$. We say that a finitely generated Kleinian group Γ or a hyperbolic orbifold N_Γ is *geometrically finite* if the convex kernel C_Γ has finite volume.

We should also define the notion of cusps. Let x be a fixed point of a parabolic element of a Kleinian group Γ. The stabilizer

$$\mathrm{Stab}_\Gamma(x) = \{ \gamma \in \Gamma \mid \gamma(x) = x \}$$

is an elementary subgroup of Γ containing the parabolic element. By conjugation, we may assume that $x = \infty$ and that $\mathrm{Stab}_\Gamma(x)$ contains a primitive parabolic transformation $a(z) = z + 1$ (cf. [10, Chap. I, B1]). Then it is known that

$$\tilde{P}_x = \{ (x, y, t) \in \mathbf{H}^3 \mid t > 1 \}$$

is Γ-equivariant and $\mathrm{Stab}_\Gamma(x)$-invariant. We regard parabolic fixed points x and x' as equivalent if there is $\gamma \in \Gamma$ such that $\gamma(x) = x'$. By this relation, we define an equivalence class $[x]$ of parabolic fixed points, and associate a subregion $P[x] = \tilde{P}_x / \mathrm{Stab}_\Gamma(x)$ of N_Γ with each class $[x]$. We call $\mathrm{Stab}_\Gamma(x)$ a *cusp* of Γ and $P[x]$ a cusp of N_Γ.

The types of cusps are completely classified as follows (cf. [10, Chap. V, D]). We say a cusp is of rank 2 if it contains an abelian group of rank 2 ($\cong \mathbf{Z} \oplus \mathbf{Z}$). Otherwise, it is of rank 1. We say a 2-dimensional orbifold is of $(g, n; \nu_1, \nu_2, \dots, \nu_n)$-type if it is a closed surface of genus g having n-singular points with the branch orders $2 \le \nu_1 \le \cdots \le \nu_n \le \infty$ (the order ∞ means that it is a puncture).

Rank 1

(i) $P[x] \cong$ annulus $\times$ I (open interval); $\mathrm{Stab}(x) =$ infinite cyclic group.
(ii) $P[x] \cong (0, 3; 2, 2, \infty)$-surface $\times$ I; $\mathrm{Stab}(x) =$ infinite dihedral group.

Rank 2

(iii) [torus cusp] $P[x] \cong$ torus $\times$ I;
Stab(x) is conjugate to $\langle a(z) = z + 1, b(z) = z + \tau \rangle$ ($\mathrm{Im}\, \tau > 0$).
(iv) [pillow cusp] $P[x] \cong (0, 4; 2, 2, 2, 2)$-surface $\times$ I;
Stab(x) is conjugate to $\langle a(z) = z+1, b(z) = z+\tau, e(z) = -z \rangle$ ($\mathrm{Im}\, \tau > 0$).
(v) [rigid cusp 3] $P[x] \cong (0, 3; 3, 3, 3)$-surface $\times$ I;
Stab(x) is conjugate to $W_3 = \langle a(z) = z + 1, e_3(z) = e^{2\pi i/3} z \rangle$.
(vi) [rigid cusp 4] $P[x] \cong (0, 3; 2, 4, 4)$-surface $\times$ I;
Stab(x) is conjugate to $W_4 = \langle a(z) = z + 1, e_4(z) = e^{2\pi i/4} z \rangle$.
(vii) [rigid cusp 6] $P[x] \cong (0, 3; 2, 3, 6)$-surface $\times$ I;
Stab(x) is conjugate to $W_6 = \langle a(z) = z + 1, e_6(z) = e^{2\pi i/6} z \rangle$.

We say that a cusp W is *rigid* if the identity representation $\iota \colon W \to \mathrm{PSL}(2, \mathbb{C})$ is rigid in the sense just described. The following proposition implies that cusps of types (v), (vi), and (vii) are rigid.

PROPOSITION 1. *Let α and ε be elements of* PSL(2, $\mathbb{C}$) *not of order* 2. *If α and $\varepsilon \circ \alpha \circ \varepsilon^{-1}$ commute but do not coincide, then α is parabolic with the fixed point in common with ε.*

Proof. Let L be the set of fixed points of α. It consists of one or two points. Since α and $\varepsilon \circ \alpha \circ \varepsilon^{-1}$ commute and α does not have order 2, L is also the set of fixed points of $\varepsilon \circ \alpha \circ \varepsilon^{-1}$ (cf. [10, Chap. I, D3]). This implies that L is invariant under ε, and since it is not of order 2, each point of L must be fixed by ε. If α is not parabolic then L consists of two points. Then L is also the set of fixed points of ε, and thus α and ε commute. But this contradicts our assumption. Therefore we know that α is parabolic. We have already seen that the fixed point of α is fixed by ε. □

We take a representation ρ of W_n ($n = 3, 4, 6$) so close to ι that $\rho(a)$ is not of order 2, $\rho(e_n)$ is elliptic of order n, and $\rho(e_n \circ a \circ e_n^{-1} \circ a^{-1})$ is not the identity.

From Proposition 1, we see that $\rho(a)$ is parabolic with a fixed point in common with $\rho(e_n)$. By conjugation, we can transfer the common fixed point to ∞ and the other fixed point of $\rho(e_n)$ to 0. This implies that the elliptic element is $z \mapsto e^{2\pi i/n}z$. Further, we can conjugate $\rho(a)$ to $a(z) = z + 1$. Therefore ρ is obtained by conjugation. Hence the cusps of types (v), (vi), and (vii) are rigid.

On the other hand, cusps of types (i), (ii), (iii), and (iv) are not rigid. We prove this only for the cusp $W = \langle a, b, e\rangle$ of type (iv). Let

$$\rho_r(a)(z) = \frac{\sqrt{1+r^2}z+1}{r^2z+\sqrt{1+r^2}}, \quad \rho_r(b)(z) = \frac{\sqrt{1+r^2\tau^2}z+\tau}{r^2\tau z+\sqrt{1+r^2\tau^2}}, \quad \rho_r(e)(z) = -z$$

for any constant r in a neighborhood of 0. We can see that such correspondence extends to a homomorphism ρ_r of W and that ρ_r converges to the identity representation ι as $r \to 0$. Since the traces $\operatorname{tr}\rho_r(a) = 2\sqrt{1+r^2}$ and $\operatorname{tr}\rho_r(b) = 2\sqrt{1+r^2\tau^2}$ vary as r does, W is not rigid. Notice that W contains cusps of types (i), (ii), and (iii) as subgroups; one can similarly check that the restriction of $\{\rho_r\}$ to these subgroups demonstrates that these cusps are also nonrigid.

2. Sullivan's Result

We shall show Theorem 1 by improving Sullivan's argument [17]. He proved that under the extra assumption that G is torsion-free, if ρ is structurally stable and nonrigid then $\rho(G)$ is convex cocompact—in other words, $\rho(G)$ is a geometrically finite Kleinian group without cusps. However, a part of his proof does not require that G be torsion-free. We can abstract the following result in general, not only for torsion-free groups. A similar result was also obtained by Riley [14].

LEMMA 1 (Sullivan). *If a* PSL(2, $\mathbb{C}$)*-representation ρ of a nonsolvable finitely generated group G is structurally stable and nonrigid, then $\rho(G)$ is Kleinian with non-empty region of discontinuity. Moreover, any representation ρ' close to ρ is a quasiconformal deformation of ρ; that is, there is a quasiconformal automorphism f of $\hat{\mathbb{C}}$ such that $\rho'(g) = f\circ\rho(g)\circ f^{-1}$ for every $g \in G$.*

Although this lemma was not stated in [17] exactly as it is here, we can easily verify it as follows. For an arbitrary representation $\rho' \in \operatorname{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ close to ρ, there exists a complex disk with center at ρ containing ρ' and contained in the analytic set $\operatorname{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ (cf. [18, Chap. 3, Thm. 3D]). We identify this disk with the unit disk D. Since ρ is structurally stable, we have a holomorphic 1-parameter family of isomorphisms of G defined over D. Then we have Lemma 1 from Theorem 2 in [17].

By this lemma, we see that the dimension of $\operatorname{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ at ρ is equal to that of the space of quasiconformal deformations of $\rho(G)$. It is known that the quasiconformal deformation space has dimension that is three greater than the dimension of the Teichmüller space of the orbifold $\Omega(\rho(G))/\rho(G)$ (see e.g. [7]). We shall next estimate the dimension of $\operatorname{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ by using a core of the Kleinian orbifold.

3. Relative Cores of Indecomposable Kleinian Orbifolds

In order to investigate 3-dimensional topological manifolds with finitely generated fundamental groups, we often utilize their cores. Here a compact submanifold is called a *core* if the inclusion induces an isomorphism between the fundamental groups. Feighn and Mess [4] showed the existence of an orbifold core under a certain assumption on the orbifold fundamental group. In this section, we construct a so-called relative core of a Kleinian orbifold under an assumption that is similar to theirs.

First of all, we consider borders of finite ends of the orbifold by using the region of discontinuity in the sphere at infinity. This means that we attach $\Omega(\Gamma)/\Gamma$ to the orbifold and regard $M_\Gamma = (\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$ as a topological orbifold with boundary. Its orbifold fundamental group is the same as N_Γ. We will find a core in M_Γ.

We remove all the cusps of N_Γ. Further, when a cusp determines punctures of $\Omega(\Gamma)/\Gamma$, we also remove neighborhoods of the punctures from M_Γ. Precisely, the removed regions are described as follows. For an equivalence class $[x]$ where $\mathrm{Stab}_\Gamma(x)$ is of rank 2, we do nothing; we just set $P^*[x] = P[x]$. But in certain cases where $\mathrm{Stab}_\Gamma(x)$ is of rank 1, we need an expansion of $P[x]$. Let $x = \infty$ and let $a(z) = z + 1$ be a primitive parabolic element of $\mathrm{Stab}_\Gamma(\infty)$. A cusped region for a is a subregion of $\Omega(\Gamma)$ of the form $\{ z \mid \mathrm{Im}\, z > M_1 \text{ or } \mathrm{Im}\, z < M_2 \}$ that is precisely invariant under $\mathrm{Stab}_\Gamma(\infty)$. When we can choose such M_1 and M_2 to be finite, we always do so. Then the cusped region for ∞ consists of two components. When we cannot choose M_1 or M_2 to be finite, we define $M_1 = \infty$ or $M_2 = -\infty$. In this case, the cusped region consists of only one component or it may be empty. We define

$$\tilde{P}_x^* = \{ (x, y, t) \in \mathbf{H}^3 \mid t > 1 \} \cup \{ (x, y, t) \in \mathbf{H}^3 \cup \mathbb{C} \mid y < M_2 \text{ or } M_1 < y \}.$$

This set is equivariant under Γ. Put $P^*[x] = \tilde{P}_x^*/\mathrm{Stab}_\Gamma(x)$, which we call an *extended cusp* of M_Γ. Now we remove $\bigcup P^*[x]$ from M_Γ, where the union is taken over all the equivalence classes $[x]$, and denote the resulting orbifold with boundary by $(M_\Gamma)_0$.

We consider a finite-sheeted branch covering manifold of the orbifold $(M_\Gamma)_0$. By Selberg's lemma [16], a finitely generated group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ contains a torsion-free normal subgroup Γ' of finite index. There corresponds the covering diagram

$$\mathbf{H}^3 \cup \Omega(\Gamma) \xrightarrow{\Psi} M_{\Gamma'} \xrightarrow{\eta} M_\Gamma.$$

Thus we can consider a manifold $(M_{\Gamma'})_0 := \eta^{-1}((M_\Gamma)_0)$ that covers $(M_\Gamma)_0$, as well as the covering transformation group H, which is isomorphic to the finite group Γ/Γ'. The fundamental group of $(M_{\Gamma'})_0$ is isomorphic to Γ'. Since Γ' contains no elliptic elements, its cusps are of type (i) or (iii) of the classification in Section 1. The cusps of $N_{\Gamma'}$ have a one-to-one correspondence with the components of $M_{\Gamma'} - (M_{\Gamma'})_0$ that are homeomorphic to (annulus)$\times$(interval) or (torus)$\times$(interval).

In the manifold $(M_{\Gamma'})_0$ with boundary, we will construct an H-invariant core. Then, taking the quotient by H, we obtain an orbifold core in $(M_\Gamma)_0$. We further require that the core contain a certain compact subset of $\partial(M_{\Gamma'})_0$. Such a core is called a *relative* core.

The compact subset T of $\partial(M_{\Gamma'})_0$ that should be contained in our relative core is taken as follows. By the finiteness theorems due to Ahlfors (cf. [8]) and Sullivan (cf. [3]), we know that $\partial(M_{\Gamma'})_0$ consists of a finite number of components of finite topological type; they are closed surfaces or they have infinite cylinders coming from cusps of rank 1. Also, $\partial(M_\Gamma)_0 = \eta(\partial(M_{\Gamma'})_0)$ is a finite union of 2-dimensional orbifolds. Let $\{\hat{T}_1, \hat{T}_2, \dots, \hat{T}_k\}$ be the set of the components of $\partial(M_\Gamma)_0$. If $\hat{T}_i$ is not closed, we choose a compact suborbifold T_i with boundary such that the inclusion $T_i \hookrightarrow \hat{T}_i$ induces an isomorphism between the orbifold fundamental groups. If $\hat{T}_i$ is closed, we just set $T_i = \hat{T}_i$. Then we define $T := \bigcup_{i=1}^{k} \eta^{-1}(T_i)$, which is an H-invariant, finite union of compact surfaces.

McCullough [11] proved the relative core theorem in general; however, it is not easy to make such a core H-invariant. Here we will follow an argument due to Kulkarni and Shalen [8, Prop. 2.12], applying it to the indecomposable case in the following sense.

DEFINITION. We say a finitely generated Kleinian group Γ is *decomposable* if Γ splits as a nontrivial amalgamated free product (case I) $\Gamma = \Gamma_1 *_C \Gamma_2$ or HNN extension (case II) $\Gamma = \Gamma_0 *_C$ over a finite cyclic group C (possibly trivial) that satisfies the following two conditions:

(*) each parabolic element of Γ is contained in a conjugate of Γ_1 or Γ_2 in case I and in a conjugate of Γ_0 in case II;

(**) for each connected component Δ of the region of discontinuity $\Omega(\Gamma)$, the component subgroup $\mathrm{Stab}_\Gamma(\Delta)$ is contained in a conjugate of Γ_1 or Γ_2 in case I and in a conjugate of Γ_0 in case II.

We say Γ is *indecomposable* if it is not decomposable.

REMARK. The condition (*) was introduced by Bonahon [1, p. 73]. Marden [9, Sec. 12] called Γ reducible if it satisfies the condition (**). According to a definition by Kulkarni and Shalen [8, p. 161], Γ is decomposable relative to the cusps and the component subgroups if it is decomposable in our sense.

For each $i = 1, 2, \dots, k$, the inclusion $T_i \hookrightarrow (M_\Gamma)_0$ induces a homomorphism of the orbifold fundamental group $\varphi_i \colon \pi_1^{\mathrm{orb}}(T_i) \to \Gamma$. We denote the image of φ_i by G_i ($\subset \Gamma$). Then all component subgroups and all parabolic elements of Γ are contained in conjugates of $G_1, G_2, \dots, G_k$. Hence, if Γ is indecomposable, then there is no nontrivial decomposition of Γ such that $G_1, G_2, \dots, G_k$ are contained in conjugates of the factors.

Now we precisely state the result we will prove in this section.

LEMMA 2. *Let Γ be a finitely generated indecomposable Kleinian group. Then there is a compact suborbifold MC_Γ of $(M_\Gamma)_0$ such that the inclusion map induces* *an isomorphism of the orbifold fundamental group,* $MC_\Gamma \cap \partial(M_\Gamma)_0 = \eta(T)$, *and* $\widetilde{MC}_\Gamma = (\eta \circ \Psi)^{-1}(MC_\Gamma)$ *is contractible.*

Proof. First we construct an H-ample submanifold L in $(M_{\Gamma'})_0$. Here we say L is *H-ample* if it satisfies the following properties:

(a) L is compact, connected, and H-invariant;
(b) $L \cap \partial(M_{\Gamma'})_0 = T$;
(c) the inclusion $\eta(L) \hookrightarrow (M_\Gamma)_0$ induces a surjective homomorphism of the orbifold fundamental group $\alpha\colon \pi_1^{\mathrm{orb}}(\eta(L)) \to \Gamma$;
(d) there is an injective homomorphism $\beta\colon \Gamma \to \pi_1^{\mathrm{orb}}(\eta(L))$ such that $\alpha \circ \beta =$ id; and
(e) for each $i = 1, 2, \dots, k$, the composite $\beta \circ \varphi_i$ coincides with a homomorphism induced by the inclusion $T_i \hookrightarrow \eta(L)$.

We can show the existence of an H-ample submanifold by changing the proof of [8, Lemma 2.14] into the H-equivariant version. For the reader's convenience we include a sketch of the proof.

It is known that a finitely generated fundamental group of a 3-manifold is finitely presented by Scott's theorem [15], and a finite index extension of a finitely presented group is also finitely presented. Hence we know Γ is finitely presented, and we can choose a finite 2-complex K and an isomorphism $J\colon \pi_1(K) \to \Gamma$. Considering the universal covers of K and $(M_\Gamma)_0$, we can construct a map $f\colon K \to (M_\Gamma)_0$ that induces J. We can also take a finite 2-complex K_i, an isomorphism $\theta_i\colon \pi_1(K_i) \to \pi_1^{\mathrm{orb}}(T_i)$, and a map $g_i\colon K_i \to T_i$ inducing θ_i for each $i = 1, 2, \dots, k$. Moreover, there is a map $f_i\colon K_i \to K$ that induces $J^{-1} \circ \varphi_i \circ \theta_i$. Let $\hat{K}$ be the union of K and $\{K_i \times [0,1]\}_{i=1,\dots,k}$ by identifying $(x, 1)$ for $x \in K_i$ with $f_i(x)$ for each i. Then f extends to a map $g\colon \hat{K} \to (M_\Gamma)_0$ inducing J. Since $g|_{K_i}$ induces $\varphi_i \circ \theta_i$, we see that $g|_{K_i}$ is homotopic to $g_i\colon K_i \to T_i$ ($\hookrightarrow (M_\Gamma)_0$). Thus we may assume $g|_{K_i} = g_i$ and $g(\hat{K}) \cap \partial(M_\Gamma)_0 = \bigcup_{i=1}^k T_i$ after modifying g by a homotopy.

Let R be a regular neighborhood of $g(\hat{K})$ such that $R \cap \partial(M_\Gamma)_0 = \bigcup_{i=1}^k T_i$, and let $L = \eta^{-1}(R)$. Then we can see that L is an H-ample submanifold of $(M_{\Gamma'})_0$. Indeed, setting $\beta = g_* \circ J^{-1}$, where $g_*\colon \pi_1(\hat{K}) \to \pi_1^{\mathrm{orb}}(R)$ is a homomorphism induced by $g\colon \hat{K} \to R$, we see that β is a right inverse to a surjective homomorphism $\alpha\colon \pi_1^{\mathrm{orb}}(R) \to \Gamma$ induced by the inclusion $R \hookrightarrow (M_\Gamma)_0$. By construction, the other conditions (a), (b), and (e) are easily seen.

Next we find an H-ample submanifold L such that the frontier

$$\mathrm{Fr}\, L = \overline{L} \cap \overline{(M_{\Gamma'})_0 - L}$$

is incompressible. Then $\alpha\colon \pi_1(L) \to \Gamma'$ is injective, and since α is also surjective, it is an isomorphism—namely, L is a core of $(M_{\Gamma'})_0$ (and hence $\eta(L)$ is an orbifold core of $(M_\Gamma)_0$).

If $\mathrm{Fr}\, L$ is compressible, then by the equivariant loop theorem due to Meeks and Yau [13] there exists an H-invariant family of mutually disjoint compressing disks $\{D_i\}_{i=1,\dots,n}$ in $(M_{\Gamma'})_0$ such that $D_i \cap \mathrm{Fr}\, L = \partial D_i$ and ∂D_i does not bound a disk in Fr L for each i. The stabilizer $\mathrm{Stab}_H(D_i)$ of each D_i is finite cyclic or dihedral. In case $\mathrm{Stab}_H(D_i)$ contains an element e of order 2 that maps one side of D_i to another, we slightly move D_i to a disk D_i' parallel to D_i and replace D_i with the two disks D_i' and $e(D_i')$. By such a change, we may assume that, for each D_i in the H-invariant family, $\mathrm{Stab}_H(D_i)$ is finite cyclic (possibly trivial) and $D_i/\mathrm{Stab}_H(D_i)$ is a disk with at most one branch point. Let D be one of the compressing disks D_i. It is contained either in $(M_{\Gamma'})_0 - L$ or in L. Let E be an equivariant regular neighborhood of D in $(M_{\Gamma'})_0 - L$ or in L respectively. In the first case, we attach the images $\{h(E)\}_{h\in H}$ to L. The resulting compact submanifold is clearly H-ample. In the second case, we remove $\{h(E)\}_{h\in H}$ from L and take the closure. The resulting submanifold L_0 is not necessarily connected. We will show that a component of L_0 is H-ample.

First we consider the case where L_0 is not connected. Let L_1 and L_2 be the components of L_0 adjoining E. All the components of L_0 are equivalent to either L_1 or L_2 under H. Consider the orbifolds $\eta(L)$, $\eta(L_1)$, and $\eta(L_2)$. The orbifold fundamental group splits as the amalgamated free product

$$\pi_1^{\mathrm{orb}}(\eta(L)) = \pi_1^{\mathrm{orb}}(\eta(L_1)) *_{\tilde{C}} \pi_1^{\mathrm{orb}}(\eta(L_2))$$

over $\tilde{C}$, where $\tilde{C}$ is isomorphic to $\mathrm{Stab}_H(D)$. The fact that D is disjoint from T combined with property (e) of the H-ample submanifold implies that the $\beta(G_i)$ ($\subset \beta(\Gamma)$) are contained in conjugates of either $\pi_1^{\mathrm{orb}}(\eta(L_1))$ or $\pi_1^{\mathrm{orb}}(\eta(L_2))$. By the Kurosh subgroup theorem generalized to the amalgamated free product (cf. [5]), we see that the subgroup $\beta(\Gamma)$ of $\pi_1^{\mathrm{orb}}(\eta(L))$ is represented by either an amalgamated free product $A *_C B$ or an HNN extension $A *_C$, where C is a finite cyclic subgroup, and that the $\beta(G_i)$ are contained in conjugates of either A or B. However, $\beta(\Gamma) \cong \Gamma$ is indecomposable by hypothesis, which implies that the decomposition of $\beta(\Gamma)$ is trivial; that is, $\beta(\Gamma)$ is conjugate to a subgroup of $\pi_1^{\mathrm{orb}}(\eta(L_1))$ or $\pi_1^{\mathrm{orb}}(\eta(L_2))$, say of $\pi_1^{\mathrm{orb}}(\eta(L_1))$. This provides the required homomorphism $\beta\colon \Gamma \to \pi_1^{\mathrm{orb}}(\eta(L_1))$, making L_1 H-ample.

Next we consider the case where L_0 is connected. Then L_0 is H-invariant. Consider the orbifolds $\eta(L)$ and $\eta(L_0)$. In this case, the orbifold fundamental group splits as the HNN extension

$$\pi_1^{\mathrm{orb}}(\eta(L)) = \pi_1^{\mathrm{orb}}(\eta(L_0)) *_{\tilde{C}}.$$

We apply the Kurosh subgroup theorem generalized to the HNN extension (cf. [6]) to the subgroup $\beta(\Gamma)$. As in the first case, we can see that $\beta(\Gamma)$ is conjugate to a subgroup of $\pi_1^{\mathrm{orb}}(\eta(L_0))$. This provides $\beta\colon \Gamma \to \pi_1^{\mathrm{orb}}(\eta(L_0))$, and thus L_0 is H-ample.

Finally we produce an irreducible H-ample submanifold with incompressible frontier. We define complexity for an H-ample submanifold L as

$$c(L) = \sum_S \{1 + (\text{genus } S)^2\},$$

where S runs over the components of ∂L. Let L_* be an H-ample submanifold with the least complexity. If L_* contains embedded non-trivial spheres, they bound balls in $(M_{\Gamma'})_0$ since $(M_{\Gamma'})_0$ is irreducible. By joining all the balls to L_*, we have an H-ample submanifold whose complexity is less than $c(L_*)$. This is a contradiction, and hence L_* is irreducible. If Fr L_* is compressible, then by thickening or cutting along disks we can obtain a new H-ample submanifold as before. It is easy to see that such surgery decreases the complexity. This again contradicts the least complexity of L_*, and thus Fr L_* is incompressible.

As we have noted, this L_* turns out to be an irreducible core of $(M_{\Gamma'})_0$. Take the quotient $MC_\Gamma := \eta(L_*)$. This is an orbifold core of M_Γ, that is, a compact suborbifold of M_Γ such that the inclusion induces an isomorphism of $\pi_1^{\mathrm{orb}}(MC_\Gamma)$ onto Γ. Since L_* is irreducible, $\widetilde{MC}_\Gamma = \Psi^{-1}(L_*)$ is contractible. This completes the proof of Lemma 2. □

REMARK. From the preceding argument or from Section 1.6 of the preprint of [1] (Prépublication Orsay no. 85T08, 1985), we see that if Γ is indecomposable then its torsion-free normal subgroup Γ' of finite index is also indecomposable. Hence, by the main theorem in [1], we know that $N_{\Gamma'}$ is homeomorphic to the interior of a compact 3-manifold, not only the existence of the relative core.

4. Branch Loci

In this section, we investigate singular loci in the orbifold. For a maximal elliptic cyclic subgroup $\langle e\rangle$ of Γ, the fixed point set of e in $\mathbf{H}^3$ is called the *axis* of $\langle e\rangle$. We consider the image of the axis by the projection $\eta\circ\Psi\colon \mathbf{H}^3 \to N_\Gamma$. We call it a *branch locus* associated with $\langle e\rangle$. The branch loci are in a one-to-one correspondence with the conjugacy classes of maximal elliptic cyclic subgroups in Γ. Each branch locus is either a circle, a segment, a ray, or an infinite line in the hyperbolic orbifold N_Γ. The end points of rays and segments are always shared by other branch loci. These singularities are caused by dihedral or polyhedral finite subgroups, which are completely classified (cf. [10, Chap. V, C]). The shared points are always trivalent; there three branch loci join or one locus connects with a circle locus. We call them *Y-singularities.*

We observe the branch loci in the orbifold core MC_Γ. By considering the intersection of the axes with $\widetilde{MC}_\Gamma$, the following claim is apparent.

LEMMA 3.

(a) *The intersection of every branch locus with MC_Γ is non-empty. Hence the total number of the branch loci in N_Γ is finite.*
(b) *The intersection of every branch locus with MC_Γ is connected. Moreover, any point shared by two branch loci is always contained in MC_Γ.*
(c) *Any branch locus which is a circle or segment lies entirely in MC_Γ.*

Proof. (a) Suppose there is an axis of $\langle e\rangle$ in the complement of $\widetilde{MC}_\Gamma$. Since $\widetilde{MC}_\Gamma$ is connected and Γ-invariant, there is a curve α joining a point p and $e(p)$ in $\widetilde{MC}_\Gamma$, and the union of the images of α under $\langle e\rangle$ constitutes a nontrivial loop in $\widetilde{MC}_\Gamma$. But this contradicts the contractibility of $\widetilde{MC}_\Gamma$. If there were infinitely many branch loci in N_Γ, they would accumulate to a point in MC_Γ. But this contradicts the proper discontinuity of the action of Γ at a lift of the point.

(b) Suppose there is an axis of $\langle e\rangle$ whose intersection with $\widetilde{MC}_\Gamma$ is not connected. Then there is a simple curve β in $\widetilde{MC}_\Gamma$ joining an end point of a component of the axis in $\widetilde{MC}_\Gamma$ and an end point of another component. Consider a simple closed curve $\beta \cup e(\beta)$ in $\widetilde{MC}_\Gamma$. It is spanned by a disk D in $\widetilde{MC}_\Gamma$, and the union of the images of D under $\langle e\rangle$ constitutes a nontrivial sphere in $\widetilde{MC}_\Gamma$. But this contradicts the contractibility of $\widetilde{MC}_\Gamma$. If a common point p of two axes is not in $\widetilde{MC}_\Gamma$, then by using the stabilizer of p we can also make a nontrivial sphere in $\widetilde{MC}_\Gamma$ bounding p. Thus we have again obtained a contradiction.

(c) A branch locus that is a circle is the image of an axis which is also invariant under a loxodromic cyclic group. In this case, the whole axis is contained in $\widetilde{MC}_\Gamma$; otherwise the intersection of the axis with $\widetilde{MC}_\Gamma$, which is invariant under the translation on the axis, would not be connected. A branch locus that is a segment is the image of an axis which has two distinct points shared with other axes. Then the portion between the two points must be in $\widetilde{MC}_\Gamma$ by (b), and thus the whole branch locus must be in MC_Γ. □

Branch loci that are rays or infinite lines pass through or go toward ∂MC_Γ. Thus their intersections with MC_Γ are regarded as segments. In sum, the branch loci in MC_Γ are circles or segments that have a one-to-one correspondence with the conjugacy classes of maximal elliptic cyclic subgroups of Γ.

5. The Euler Characteristic of the Orbifold Core

We remove the tubular neighborhoods of the branch loci from MC_Γ, and denote the resulting topological manifold by O_Γ. It is compact and irreducible. We will calculate the Euler characteristic χ of this manifold.

We classify the boundary of O_Γ into four parts. Hereafter we say a compact surface is of (g, n)-type if it has genus g and n boundary components. The intersection with $\Omega(\Gamma)/\Gamma$ is called the *ordinary part*. It consists of compact surfaces with negative Euler characteristic. The intersection with the boundary of the extended cusps is called the *cuspidal part*. It consists of annuli, tori, $(0, 3)$-surfaces or $(0, 4)$-surfaces. Annuli come from (i), tori from (iii), $(0, 3)$-surfaces from (ii), (v), (vi), and (vii), and $(0, 4)$-surfaces from (iv) of the classification of cusps in Section 1. The boundary of the tubular neighborhoods of the branch loci is called the *singular part*. With each Y-singularity, we may associate a pair of pants over it as a subregion of the singular part. The remainder of ∂O_Γ is called the *frontier part*.

We use the following notation (# indicates the number of the elements):

$$p_1 = \#\{\text{cusps of type (i) or (ii) (rank 1)}\}, \quad p_2 = \#\{\text{cusps of type (iii) (torus)}\}$$
$$q = \#\{\text{cusps of type (iv) (pillow)}\}, \quad y_1 = \#\{\text{cusps of type (ii), (v), (vi), or (vii)}\}$$
$$e = \#\{\text{branch loci}\}, \quad e_2 = \#\{\text{isolated branch loci of circle}\}$$
$$y_2 = \#\{Y\text{-singularities}\}$$
$$t = \#\{\text{toral components of } \partial O_\Gamma\}$$

Moreover, set $p = p_1 + p_2$, $y = y_1 + y_2$, and $e_1 = e - e_2$. Note that $t = p_2 + e_2$, because a toral component of ∂O_Γ bounds a component of $M_\Gamma - O_\Gamma$ having an abelian orbifold fundamental group.

Now we begin to compute $\chi(O_\Gamma)$. By Poincaré duality for the double of O_Γ with respect to ∂O_Γ, we know $\chi(O_\Gamma)$ is equal to half of $\chi(\partial O_\Gamma)$. Let $\{S_i\}$ $(i = 1, \dots, k)$ be the components of the ordinary part, and let $\{S_i\}$ $(i = k+1, \dots, m)$ be the components of the frontier part. We suppose they have topological type (g_i, n_i). In addition, there are y_1 number of (0, 3)-surfaces and q number of (0, 4)-surfaces in the cuspidal part, and y_2 number of pairs of pants over Y-singularities. The remainder of these surfaces in ∂O_Γ consists of annuli or tori, which have no contribution to the Euler characteristic of ∂O_Γ. Then we see

$$-\chi(\partial O_\Gamma) = \sum_{i=1}^{m}(2g_i - 2 + n_i) + y + 2q.$$

By using this equality, we obtain

$$-3\chi(O_\Gamma) = -\frac{3}{2}\chi(\partial O_\Gamma) = \sum_{i=1}^{m}(3g_i - 3 + n_i) + \frac{1}{2}\left(\sum_{i=1}^{m} n_i + 3y + 4q\right) + q.$$

Here $\sum_{i=1}^{m} n_i + 3y + 4q$ is the total number of the boundary components of the above surfaces. Two of them are connected each directly or by an annulus. (For example, a surface in the cuspidal part caused by a cusp of type (ii) has three boundary components: one of them is a loop corresponding to the parabolic element; the others are round branch loci of order 2, and they are connected by two annuli over the branch loci to distinct surfaces or by an annulus to each other.) We can see that the number of the parabolic connections is the same as the number of the cusps of rank 1 (recall that a cusp of type (ii) counts exactly one parabolic connection), and the number of the elliptic connections is the same as the number of the branch loci that are not isolated circles. Summing up, we have $p_1 + e_1$ connections. Therefore

$$-3\chi(O_\Gamma) = \sum_{i=1}^{m}(3g_i - 3 + n_i) + (p_1 + e_1) + q. \tag{1}$$

6. Structural Stability Implies Geometric Finiteness

In this section, we prove the "only if" part of Theorem 1. Let $\rho: G \to \mathrm{PSL}(2, \mathbb{C})$ be a nonrigid structurally stable representation. Then by Sullivan's result (Lemma 1 in Section 2), $\rho(G) = \Gamma$ is a Kleinian group with the region of discontinuity $\Omega(\Gamma)$.

First, we assume that Γ is indecomposable and take O_Γ constructed as in the previous section. The inclusion map $O_\Gamma \hookrightarrow M_\Gamma$ induces a surjective homomorphism $\tilde{\rho}: \pi_1(O_\Gamma) \to \Gamma$. Letting $\varepsilon_i \in \pi_1(O_\Gamma)$ $(i = 1, \dots, e)$ correspond to the simple loops around the branch loci with the order n_i, we see that the kernel of $\tilde{\rho}$ is the normal closure of the elements $\{\varepsilon_1^{n_1}, \dots, \varepsilon_e^{n_e}\}$. Set $\pi = \pi_1(O_\Gamma)$ in brief. Let

$\iota\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be the identity representation. The surjective homomorphism $\tilde{\rho}$ defines an injection

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \hookrightarrow \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{C}))$$

such that ι is mapped to $\tilde{\rho}$. The isomorphism ρ identifies $\mathrm{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ with $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$. In this manner, $\mathrm{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$ is regarded as a subvariety of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{C}))$.

By Thurston's result (cf. [2, Prop. 3.2.1]), we can estimate the local dimension $\dim_{\tilde{\rho}}$ of the algebraic variety $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{C}))$ at $\tilde{\rho}$ as

$$\dim_{\tilde{\rho}} \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{C})) \geq -3\chi(O_\Gamma) + t + 3.$$

We fix here a system of generators of π. We also fix a representation of each ε_i $(i = 1, \dots, e)$ by the generators. Then we can define $\theta(\varepsilon_i)$ for $\theta \in V_\pi$, and $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ is the intersection of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{C}))$ with the algebraic variety

$$\{\theta \in V_\pi \mid \theta(\varepsilon_i)^{n_i} = \mathrm{id}\ (i = 1, \dots, e)\}.$$

Hence (cf. [18, Chap. 2, Thm. 12C]), we have

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \geq -3\chi(O_\Gamma) + t + 3 - e.$$

Substitution of (1) and $t = p_2 + e_2$ yields

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \geq \sum_{i=1}^{m} (3g_i - 3 + n_i) + 3 + (p+q). \tag{2}$$

On the other hand, by Lemma 1 in Section 2 we know that

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) = \dim T(\Omega(\Gamma)/\Gamma) + 3, \tag{3}$$

where $T(\Omega(\Gamma)/\Gamma)$ is the Teichmüller space of the complex structure on the orbifold $\Omega(\Gamma)/\Gamma = \partial M_\Gamma$, and that

$$\dim T(\Omega(\Gamma)/\Gamma) = \sum_{i=1}^{k} (3g_i - 3 + n_i). \tag{4}$$

By the formulas (2), (3), and (4), we have the following conditions:

(a) the sum $\sum_{i=k+1}^{m} (3g_i - 3 + n_i)$ over the surfaces in the frontier part is zero;
(b) the number $P(\Gamma) := p + q$ of the nonrigid cusps of N_Γ is zero.

Condition (a) means that the frontier part consists only of (0, 3)-surfaces.

We will show that there is no (0, 3)-surface in the frontier part. Suppose such a surface $\mathcal{S}$ exists. We consider the image F of a homomorphism $\pi_1(\mathcal{S}) \to \Gamma$ induced by the inclusion $\mathcal{S} \hookrightarrow (M_\Gamma)_0$. Then F must be Fuchsian (cf. [10, Chap. IX, C]). Let E be a component of $(M_\Gamma)_0 - MC_\Gamma$ adjoining $\mathcal{S}$. Since MC_Γ is a core, we can see that the image of $\pi_1^{\mathrm{orb}}(E)$ in Γ coincides with F. Hence the stabilizer of a connected component $\tilde{E}$ of the lift of E to the universal cover $\mathbf{H}^3 \cup \Omega(\Gamma)$ is a Fuchsian group, which we may assume to be F. In particular, $\tilde{E}$ must contain a component Δ of $\Omega(\Gamma)$. However, the relative core MC_Γ contains Δ/F except for cusp neighborhoods. This is a contradiction.

Therefore the condition (a) actually implies that the frontier part does not exist and $(M_\Gamma)_0$ itself is compact. Hence so is $(M_{\Gamma'})_0$. This is equivalent to the geometric finiteness of Γ' by [9, Prop. 4.2], and equivalent to the geometric finiteness of Γ. We have proved the "only if" part of Theorem 1 under the assumption that Γ is indecomposable.

Next we treat the general case. We will show that an arbitrary finitely generated Kleinian group Γ satisfies

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \geq \dim T(\Omega(\Gamma)/\Gamma) + P(\Gamma) + 3. \tag{5}$$

Formulas (2) and (4) imply that an indecomposable Γ satisfies (5).

Suppose that Γ is decomposable in the sense of the definition in Section 3 as

$$\text{(I)}\ \Gamma = \Gamma_1 *_C \Gamma_2 \quad \text{or} \quad \text{(II)}\ \Gamma = \Gamma_0 *_C.$$

Let $S = \Delta/\mathrm{Stab}_\Gamma(\Delta)$ be any component of $\Omega(\Gamma)/\Gamma$. By (**) of the definition of a decomposable Kleinian group, there is $\gamma \in \Gamma$ such that $\gamma\,\mathrm{Stab}_\Gamma(\Delta)\gamma^{-1} = \mathrm{Stab}_\Gamma(\gamma(\Delta))$ is contained in some Γ_i $(i = 0, 1, 2)$. Then $\mathrm{Stab}_\Gamma(\gamma(\Delta)) = \mathrm{Stab}_{\Gamma_i}(\gamma(\Delta))$, and S is isometrically equivalent to $\gamma(\Delta)/\mathrm{Stab}_{\Gamma_i}(\gamma(\Delta))$, which is a component of $\Omega(\Gamma_i)/\Gamma_i$. Thus, to any S in $\Omega(\Gamma)/\Gamma$ there corresponds a unique component of $\Omega(\Gamma_1)/\Gamma_1 \cup \Omega(\Gamma_2)/\Gamma_2$ in case I or that of $\Omega(\Gamma_0)/\Gamma_0$ in case II. Hence

$$\dim T(\Omega(\Gamma_1)/\Gamma_1) + \dim T(\Omega(\Gamma_2)/\Gamma_2) \geq \dim T(\Omega(\Gamma)/\Gamma); \tag{6}$$

$$\dim T(\Omega(\Gamma_0)/\Gamma_0) \geq \dim T(\Omega(\Gamma)/\Gamma). \tag{7}$$

In case I, we have

$$\begin{aligned}&\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))\\ &\qquad \geq \dim_\iota \mathrm{Hom}(\Gamma_1, \mathrm{PSL}(2,\mathbb{C})) + \dim_\iota \mathrm{Hom}(\Gamma_2, \mathrm{PSL}(2,\mathbb{C})) - 2,\end{aligned} \tag{8}$$

because the amalgamating group C is elliptic cyclic (possibly trivial) and C restricts the representations of the free product $\Gamma_1 * \Gamma_2$ by at most two dimensions. Suppose Γ_1 and Γ_2 satisfy (5). Then, using (8), (5), (6), and $P(\Gamma_1) + P(\Gamma_2) = P(\Gamma)$ in order, we obtain

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \geq \dim T(\Omega(\Gamma)/\Gamma) + P(\Gamma) + 4. \tag{9}$$

In particular, we see that Γ satisfies (5).

In case II, though the fixed points of the loxodromic element $f \in \Gamma$ conjugating the elliptic cyclic group $C \subset \Gamma_0$ is fixed once Γ_0 is given, there remains the ambiguity of the multiplier of f. Hence f gains at least one dimension in addition to the representations of Γ_0, which yields

$$\dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C})) \geq \dim_\iota \mathrm{Hom}(\Gamma_0, \mathrm{PSL}(2,\mathbb{C})) + 1. \tag{10}$$

Suppose Γ_0 satisfies (5). Then, using (10), (5), (7), and $P(\Gamma_0) = P(\Gamma)$ in order, we also obtain the inequality (9), and (5) for Γ.

Any finitely generated Kleinian group Γ can be decomposed into indecomposable ones after a finite number of steps. This is seen as follows. For a finitely generated Kleinian group Γ, we define an index

$$r(\Gamma) = \dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) - 3.$$

Formulas (8) and (10) mean that the index is strictly decreasing in each decomposition. However, the index must be nonnegative. This implies that such decompositions are possible only finitely many times.

Therefore, we see that any Γ satisfies (5) by induction, and (9) implies that, once Γ is decomposable, it does not satisfy the equality in (5). If a nonrigid representation $\rho: G \to \Gamma$ is structurally stable, the formula (3) is satisfied and thus the equality in (5) must be satisfied for such Γ. Hence we need only consider the indecomposable case. We have already shown the assertion in this case, and thus our proof of the "only if" part of Theorem 1 is complete.

7. Quasiconformal Stability

The "if" part of Theorem 1 is a consequence of Marden's result [9, Prop. 9.1] about quasiconformal stability. First we give its definition here. For a finitely generated Kleinian group Γ, we define another algebraic subvariety of V_Γ as

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{p}}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) \\ &\qquad = \{\theta \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) \mid \mathrm{tr}^2\,\theta(\gamma) = 4 \ (\forall\gamma : \text{parabolic})\}.\end{aligned}$$

We say that Γ is *quasiconformally stable* if there exists an open neighborhood U ($\subset V_\Gamma$) of the identity representation ι such that each element of

$$\mathrm{Hom}_{\mathrm{p}}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) \cap U$$

is a quasiconformal deformation. Marden proved that geometrically finite torsion-free Kleinian groups are quasiconformally stable. This result is easily generalized to the case where they have torsion [11].

Proof ("if" part of Theorem 1). Suppose that $\rho(G) = \Gamma$ is geometrically finite without nonrigid cusps. In particular, Γ is quasiconformally stable by Marden's theorem. Since all the cusps of Γ are rigid, we know that $\mathrm{Hom}_{\mathrm{p}}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ coincides with $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ near the identity representation ι. Thus the quasiconformal stability implies the structural stability of Γ. □

Finally, we show a parallel result with respect to quasiconformal stability. For torsion-free groups, this was proved by Sullivan [17].

THEOREM 2. *Quasiconformally stable Kleinian groups are geometrically finite. Hence quasiconformal stability and geometric finiteness are equivalent in general.*

Proof. The methods are similar to those used in proving Theorem 1; in addition we need to estimate the dimension of $\mathrm{Hom}_{\mathrm{p}}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$: it is the restriction of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ to the subvariety $\{\theta\}$ of V_Γ where $\mathrm{tr}^2 \theta(\gamma) = 4$ for parabolic elements $\gamma \in \Gamma$. (To define this variety precisely, we carry out the same discussion as in the previous section.) However, we need not consider the elements from all the conjugacy classes of maximal parabolic subgroups in Γ; we note the following simple fact.

PROPOSITION 2. *If two nontrivial elements α and β of* $\mathrm{PSL}(2, \mathbb{C})$ *commute and α is parabolic, then so is β.*

By this proposition, we can decide that a homomorphism of Γ close to the identity is type-preserving if we verify that just one parabolic element of each conjugacy class of nonrigid cusps in Γ remains parabolic under the homomorphism. Nonrigid cusps are rank-1, torus, or pillow cusps. Their total number is $P(\Gamma)$. Hence, denoting the local dimension of $\mathrm{Hom}_{\mathrm{p}}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ at the origin ι by d, we have

$$d \geq \dim_\iota \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) - P(\Gamma). \tag{11}$$

As in the previous section, we need only consider the case where Γ is indecomposable. By the inequalities (2) and (11), we see that

$$d \geq \sum_{i=1}^{m} (3g_i - 3 + n_i) + 3,$$

where the notation is the same as in the proof of Theorem 1. On the other hand, quasiconformal stability implies $d = \dim T(\Omega(\Gamma)/\Gamma) + 3 = \sum_{i=1}^{k} (3g_i - 3 + n_i) + 3$. Hence we have the condition (a) in the previous section, which means that Γ is geometrically finite. This finishes a proof of Theorem 2. □

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References

[1] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3,* Ann. of Math. (2) 124 (1986), 71–158.

[2] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds,* Ann. of Math. (2) 117 (1983), 109–146.

[3] M. Feighn and D. McCullough, *Finiteness conditions for 3-manifolds with boundary,* Amer. J. Math. 109 (1987), 1155–1169.

[4] M. Feighn and G. Mess, *Conjugacy classes of finite subgroups of Kleinian groups,* Amer. J. Math. 113 (1991), 179–188.

[5] A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroup,* Trans. Amer. Math. Soc. 150 (1970), 227–255.

[6] ———, *Subgroups of HNN groups and groups with one defining relation,* Canad. J. Math. 23 (1971), 627–643.

[7] I. Kra and B. Maskit, *The deformation space of a Kleinian group,* Amer. J. Math. 103 (1981), 1065–1102.
[8] R. S. Kulkarni and P. B. Shalen, *On Ahlfors' finiteness theorem,* Adv. in Math. 76 (1989), 155–169.
[9] A. Marden, *The geometry of finitely generated Kleinian groups,* Ann. of Math. (2) 99 (1974), 383–462.
[10] B. Maskit, *Kleinian groups,* Springer, New York, 1988.
[11] K. Matsuzaki, *Geometric finiteness, quasiconformal stability and surjectivity of the Bers map for Kleinian groups,* Tôhoku Math. J. 43 (1991), 327–336.
[12] D. McCullough, *Compact submanifolds of 3-manifolds with boundary,* Quart. J. Math. Oxford Ser. (2) 37 (1986), 299–307.
[13] W. H. Meeks and S. T. Yau, *The equivariant Dehn's lemma and loop theorem,* Comment. Math. Helv. 56 (1981), 225–239.
[14] R. Riley, *Holomorphically parameterized families of subgroups of* $SL(2, \mathbb{C})$, Mathematika 32 (1985), 248–264.
[15] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented,* J. London Math. Soc. (2) 6 (1973), 437–440.
[16] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces,* Contributions to function theory, pp. 147–164, Tata Institute of Fundamental Research, Bombay, 1960.
[17] D. Sullivan, *Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups,* Acta Math. 155 (1985), 243–260.
[18] H. Whitney, *Complex analytic varieties,* Addison–Wesley, Reading, MA, 1972.

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