On Iteration in Planar Domains

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Introduction

Let $G \subset \hat{\mathbb{C}}$ be a domain in the complex plane, and let $f: G \to G$ be an analytic function mapping G into itself. By f^n we denote the nth iterate $f^n = f \circ \cdots \circ f$ (n times) of f. The behavior of the sequence $(f^n)_n$ as $n \to \infty$ is of great interest and has already been studied in depth for the most important choices of G. If $G = \hat{\mathbb{C}}$, then f is a rational function. Many significant results have been proved during the last years (see e.g. [B2; CG; Mi; S]). In the case $G = \mathbb{C}$, the function f is an entire function; see [Be] for an excellent overview. If G is the unit disk, $G = \mathbb{D}$, then the situation becomes easier, since in this case the family $\{ f^n \mid n \in \mathbb{N}\}$ is normal. Initial results have been discovered by Julia, Wolff, and Valiron (see e.g. [V]); further results have been found by Pommerenke and Baker [P; BaP] and by Cowen [C].

Owing to the uniformization theorem of Koebe and Poincaré, any other domain $G \subset \hat{\mathbb{C}}$ (and even every Riemann surface) is conformally equivalent to the quotient of one of these standard domains and a discrete, fixed-point free subgroup of the automorphism group associated with that domain. Hence, it seems obvious that the behavior of the iterates of an analytic function in an arbitrary domain can be deduced from one of the cases mentioned above. If f possesses a fixed point in G then this statement is true, but if not then the boundary of G becomes significant and the boundary behavior of the universal covering map must be examined. The cases where G is covered by the whole complex plane are represented by the plane $\mathbb{C}$ itself with entire functions being iterated and by the punctured plane $\mathbb{C}^*$, the latter case being reduced to the first one by means of the exponential map that covers $\mathbb{C}^*$ by $\mathbb{C}$ and whose boundary behavior is well known. The only case where $G \subset \hat{\mathbb{C}}$ is covered by $\hat{\mathbb{C}}$ is the case $G = \hat{\mathbb{C}}$, so that this case does not introduce new problems compared with the iteration of rational functions.

From now on, let $G \subset \hat{\mathbb{C}}$ be a planar domain covered by the unit disk $\mathbb{D}$, that is, G has at least three boundary points in $\hat{\mathbb{C}}$. We call such domains *hyperbolic*, since they carry a hyperbolic metric (see below). Note that some authors use the term "hyperbolic" differently: they call a domain G hyperbolic if it possesses a Green's function. In [Hei] the iteration of analytic functions in hyperbolic domains has already been studied, in particular the case of f having a fixed point in G is dealt

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with. Hence, we are only interested in analytic functions $f: G \to G$ without fixed points. Marden and Pommerenke [MP] proved some results for this case; the authors work on hyperbolic Riemann surfaces and extend some of the results known in the unit disk to the surface. Their principal result gives a semiconjugation of f to a covering map of an auxiliary surface.

In this paper we prove that f is semiconjugated to a Möbius transformation of an auxiliary domain onto itself. Furthermore, we are interested in a classification of the behavior of the iterates f^n in terms of purely geometric conditions concerning G and f. The sequence of hyperbolic distances $(\lambda_G(f^n(z), f^{n+1}(z)))_n$ takes a central position in this classification, and we prove a relation between this sequence and the corresponding sequence of quasihyperbolic distances.

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1. Statement of Results

Let $G \subset \hat{\mathbb{C}}$ be a hyperbolic domain, and let $f: G \to G$ be an analytic function mapping G into itself. By conjugation with a Möbius transformation we may assume that $\infty \notin G$, hence $G \subset \mathbb{C}$. Let $\mathbb{D} = \{|\zeta| < 1\}$ be the unit disk. Since G is hyperbolic, there is a universal covering map $p: \mathbb{D} \to G$ and a Fuchsian group $\Gamma \subset \text{Möb}(\mathbb{D})$ without elliptic elements with $p \circ \gamma = p$ for all $\gamma \in \Gamma$ so that G is conformally equivalent to $\mathbb{D}/\Gamma$. The analytic map $f: G \to G$ lifts to an analytic map $\tilde{f}: \mathbb{D} \to \mathbb{D}$ with $p \circ \tilde{f} = f \circ p$. Note that there are many different choices of $\tilde{f}$. Under our assumptions we have $\tilde{f}(\zeta) \neq \zeta$ for all $\zeta \in \mathbb{D}$.

Letting ρ_G denote the density of the Poincaré (hyperbolic) metric in G, we have $\rho_G(p(\zeta)) = \rho_{\mathbb{D}}(\zeta)/|p'(\zeta)|$ for $\zeta \in \mathbb{D}$, where $\rho_{\mathbb{D}}(\zeta) = 1/(1-|\zeta|^2)$. Note that some authors use $2/(1-|\zeta|^2)$ instead. By $\lambda_G(\cdot,\cdot)$ we denote the hyperbolic distance in G; see [A] for further details. The Schwarz lemma yields $\lambda_G(f(z_1), f(z_2)) \leq \lambda_G(z_1, z_2)$ for $z_1, z_2 \in G$, so that the sequence $(\lambda_G(f^n(z), f^{n+1}(z)))$ converges for every $z \in G$ as $n \to \infty$.

THEOREM 1.1. *Let $f: G \to G$ map the hyperbolic domain $G \subset \mathbb{C}$ analytically and without fixed points into itself. Let $\tilde{f}: \mathbb{D} \to \mathbb{D}$ be any lift of f under any universal covering map and let $z \in G$ be an arbitrary point. Then*

$$\lambda_G(f^n(z), f^{n+1}(z)) \xrightarrow{n\to\infty} 0 \quad \text{if and only if} \quad \lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \xrightarrow{n\to\infty} 0.$$

This theorem makes precise the intuitively obvious fact that the behavior of f is somehow strongly related to the behavior of any lift $\tilde{f}$ of f under a universal covering map $p: \mathbb{D} \to G$. Note that from the definition of λ_G we only have

$$\lambda_G(f^n(p(\zeta)), f^{n+1}(p(\zeta))) = \min\{\lambda_{\mathbb{D}}(\tilde{f}^n(\zeta), \gamma(\tilde{f}^{n+1}(\zeta))) \mid \gamma \in \Gamma\}$$
$$\leq \lambda_{\mathbb{D}}(\tilde{f}^n(\zeta), \tilde{f}^{n+1}(\zeta)) \quad \text{for } \zeta \in \mathbb{D}, \tag{1}$$

where "$<$" can actually occur (e.g. $G = \mathbb{D}\backslash\{0\}$, $f(z) = -z^2$).

Observe that, by Theorem 1.1, the sequence $(\lambda_G(f^n(z), f^{n+1}(z)))$ converges to zero for each $z \in G$ or does not converge to zero for any $z \in G$ as $n \to \infty$. Note that the theorem also applies if G is the unit disk. Furthermore, we obtain from Theorem 1.1 that different lifts $\tilde{f}: \mathbb{D} \to \mathbb{D}$ of the function f under arbitrary universal covering maps $p: \mathbb{D} \to G$ behave similarly.

We remark that, for multiply connected domains G, the hyperbolic metric λ_G has no simple geometric meaning, whereas the quasihyperbolic metric λ_G^* defined by the quasihyperbolic density $\rho_G^*(z) = 1/\text{dist}(z, \partial G)$ is purely geometric. If G is simply connected, then by the Koebe distortion theorem and the Schwarz lemma we have

$$\frac{1}{4} \leq \frac{\rho_G(z)}{\rho_G^*(z)} \leq 1 \quad \text{for all } z \in G,$$

that is, $\rho_G(z) \approx \rho_G^*(z)$. This is no longer true for multiply connected domains G; in fact, $\rho_G(z)/\rho_G^*(z) \leq 1$ remains true but in general we cannot find a lower bound $c > 0$ for this quotient (example: $G = \mathbb{D}\backslash\{0\}$; see [BP] for a detailed discussion).

We say that $f: G \to G$ has an *isolated boundary fixed point* $a \in \partial_{\hat{\mathbb{C}}} G$ if f extends analytically to the isolated boundary point a and fixes a; thus $f(a) = a$, with a suitable definition of "analytic" if $a = \infty$.

From now on we shall assume that f has no isolated boundary fixed point. For otherwise we can take $\tilde{G} = G \cup \{a\}$. Then the function $f: \tilde{G} \to \tilde{G}$ has an interior fixed point in $a \in \tilde{G}$. This case has already been examined: Kœnigs [K] proved in 1884 that for $f'(a) \neq 0$ there is a local holomorphic change of coordinates $w = g(z)$ with $g(a) = 0$ so that locally $g \circ f \circ g^{-1}$ is the map $w \mapsto f'(a) \cdot w$. Böttcher [Bö] dealt with the case $f'(a) = 0$ and proved that such an f is locally conjugate to the map $w \mapsto w^k$ for some $k \in \mathbb{N}$.

THEOREM 1.2. *Let $f: G \to G$ map the hyperbolic domain $G \subset \mathbb{C}$ analytically, without fixed points, and without isolated boundary fixed points into itself. If $z_0 \in G$ is an arbitrary point, then there exists a constant $c > 0$ (depending on z_0 and f but not on n) with*

$$c \leq \frac{\rho_G(f^n(z_0))}{\rho_G^*(f^n(z_0))} \leq 1 \quad \textit{for all } n \in \mathbb{N}. \tag{2}$$

The theorem shows that we have $\rho_G(f^n(z)) \approx \rho_G^*(f^n(z))$ for all $n \in \mathbb{N}$ and fixed $z \in G$ even though G is multiply connected. It is then easy to see the following.

COROLLARY 1.3. *Let the assumptions of Theorem 1.2 hold. Then*

$$\lambda_G(f^n(z), f^{n+1}(z)) \xrightarrow{n\to\infty} 0 \quad \textit{if and only if} \quad \lambda_G^*(f^n(z), f^{n+1}(z)) \xrightarrow{n\to\infty} 0$$

for arbitrary $z \in G$.

Our main result makes Theorem 1 in [MP] more precise.

THEOREM 1.4. *Let $f: G \to G$ map the hyperbolic domain $G \subset \mathbb{C}$ analytically, without fixed points, and without isolated boundary fixed points into itself. Then there exist a domain $H \subset \mathbb{C}$, an analytic function $g: G \to H$ mapping G into H, and a Möbius transformation $\varphi: H \to H$ with $\varphi(H) = H$ such that the following functional equation holds in G:*

$$g \circ f = \varphi \circ g.$$

The domain H is hyperbolic if (any) $z \in G$ satisfies $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$; otherwise, $H = \mathbb{C}$.

Furthermore, we have:

(i) *g is injective if and only if f is injective;*
(ii) *if $f \in \operatorname{Aut}(G)$ then H is conformally equivalent to G and g maps G biholomorphically onto H, where $\operatorname{Aut}(G)$ denotes the group of all analytic automorphisms of G; and*
(iii) *if $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$ and if G is finitely connected and $f \notin \operatorname{Aut}(G)$, then we can choose $H = \mathbb{D}$.*

We shall prove Theorem 1.1 in Section 3. Section 4 contains the proofs of Theorem 1.2 and its Corollary 1.3; Section 5 is devoted to the proof of Theorem 1.4. In Section 2 we prove some geometric results that will be very useful later on. Finally, in Section 6 we consider the case where f is a Möbius transformation, and we explicitly calculate some of the quantities appearing in Theorem 1.4 and its proof.

2. Some Geometric Results

In this section we state some probably known geometric results; for the reader's convenience, we give proofs.

LEMMA 2.1. *Let c_0 and K be positive constants. Then there is a constant $c > 0$ (depending on c_0 and K) such that the following holds: If $G \subset \mathbb{C}$ is a hyperbolic domain and $z_0, z_1 \in G$ satisfy $\rho_G(z_0) \geq c_0 \cdot \rho_G^*(z_0)$ and $\lambda_G(z_0, z_1) \leq K$, then*

$$\rho_G(z_1) \geq c \cdot \rho_G^*(z_1).$$

Proof. Without loss of generality we can assume $z_0 = 0 \in G$ and $1 \in \partial G$ with $\operatorname{dist}(0, \partial G) = 1$. Our assumptions then are

$$\rho_G(0) \geq c_0, \qquad \lambda_G(0, z_1) \leq K,$$

and we shall prove that

(i) $\rho_G(z_1) \geq c_1$ and
(ii) $\operatorname{dist}(z_1, \partial G) \geq c_2$,

where c_1 and c_2 are positive constants depending only on c_0 and K. This yields $\rho_G(z_1) \geq c \cdot \rho_G^*(z_1)$ with $c = c_1 \cdot c_2 > 0$ depending only on c_0 and K.

We can find positive constants d and $\tilde{d}$ depending only on c_0 (and not on G) such that there is some $a \in \mathbb{C}\setminus G$ with $d \leq |1-a| \leq \tilde{d}$ (see e.g. [BP, Thm. 1]). With $\operatorname{dist}(0, \partial G) = 1$ we have $|a| \geq 1$.

For the proof of (i) let $H_a := \mathbb{C}\setminus\{1, a\}$. Then $G \subset H_a$ and so

$$\rho_G(z) \geq \rho_{H_a}(z)$$

holds for $z \in G$. Furthermore,

$$\lambda_{H_a}(0, z_1) \leq \lambda_G(0, z_1) \leq K.$$

There is a constant c_1, not depending on the point a but depending only on the bounds $d, \tilde{d}$ and on K, such that

$$\rho_{H_a}(z) \geq c_1$$

holds for all $z \in H_a$ satisfying $\lambda_{H_a}(0, z) \leq K$. This implies $\rho_G(z_1) \geq c_1$.

For the proof of (ii), let

$$B = B(r, R) := \{|z| \geq 1,\ r \leq |z-1| \leq R\},$$

and for $b \in B$ define $H_b := \mathbb{C}\setminus\{1, b\}$. Then there is some $\tilde{c} > 0$ depending only on r, R, and K such that all $b \in B$ and all $z \in H_b$ with $\lambda_{H_b}(0, z) \leq K$ satisfy

$$|z-1| \geq \tilde{c}, \qquad |z-b| \geq \tilde{c},$$

that is,

$$\operatorname{dist}(z, \partial H_b) \geq \tilde{c}.$$

In particular, for $r = d$ and $R = \tilde{d}$, this gives us some constant $\tilde{c}_1 > 0$ depending only on d, $\tilde{d}$ and K, and thus only on c_0 and K, with

$$|z_1 - 1| \geq \tilde{c}_1, \qquad |z_1 - a| \geq \tilde{c}_1$$

(note that $a \in B$). Let now $b \in \partial G$ with $|z_1 - b| = \operatorname{dist}(z_1, \partial G)$. We have $|b| \geq 1$, and we show that $|z_1 - b| \geq c_2$ for a constant $c_2 > 0$ depending only on c_0 and K.

(1) If $|b-1| < \tilde{c}_1/2$ then $|z_1 - b| \geq |z_1 - 1| - |b-1| \geq \tilde{c}_1/2$.

(2) If $|b-1| \geq \tilde{c}_1/2$ then, since $\lambda_{H_a}(0, z_1) \leq K$ and $d \leq |1-a| \leq \tilde{d}$, we find $\tilde{R} > 0$ depending only on d and $\tilde{d}$ with $|z_1| < \tilde{R}$. Using $|b - z_1| = \operatorname{dist}(z_1, \partial G) \leq |1 - z_1|$, this implies

$$|b-1| \leq |b-z_1| + |z_1 - 1| \leq 2|z_1 - 1| \leq 2|z_1| + 2 \leq 2\tilde{R} + 2.$$

From the above with $r = \tilde{c}_1/2$ and $R = 2\tilde{R} + 2$ we therefore have $b \in B(r, R)$ and find $\tilde{c}_2 > 0$, depending only on r, R, and K and thus only on c_0 and K, with $|z_1 - b| \geq \tilde{c}_2$. Note, that $z_1 \in H_b$ and $\lambda_{H_b}(0, z_1) \leq \lambda_G(0, z_1) \leq K$ since $G \subset H_b$.

Let $c_2 = \min\{\tilde{c}_1/2, \tilde{c}_2\}$. Then $\operatorname{dist}(z_1, \partial G) = |z_1 - b| \geq c_2$. □

The following lemma will prove to be very useful.

LEMMA 2.2. *Let c_0 be a positive constant. Then there is a constant $c > 0$ (depending on c_0) such that the following holds: If $G \subset \mathbb{C}$ is a hyperbolic domain and $z_0 \in G$ satisfies $\rho_G(z_0) \geq c_0 \cdot \rho_G^*(z_0)$ then*

$$D_{\rho_G}(z_0, c) \subset K(z_0, \operatorname{dist}(z_0, \partial G)/2),$$

where $D_{\rho_G}(z_0, c)$ denotes the hyperbolic disk with (hyperbolic) center z_0 and (hyperbolic) radius c; $K(\cdot, \cdot)$ denotes the Euclidean disk with Euclidean center and Euclidean radius.

Proof. As in the proof of Lemma 2.1, we may assume without loss of generality that $z_0 = 0 \in G$ and $1 \in \partial G$ with $\operatorname{dist}(0, \partial G) = 1$ and $\rho_G(0) \geq c_0$. We show that there is a constant $c > 0$ depending only on c_0 such that

$$D_{\rho_G}(0, c) \subset K(0, 1/2).$$

As in the proof of Lemma 2.1, we find some $a \in \mathbb{C} \backslash G$ with $0 < d \leq |1 - a| \leq \tilde{d}$ and $d, \tilde{d} > 0$ depending only on c_0; furthermore, $|a| \geq 1$. This shows that there is a constant $c > 0$ with $\lambda_{H_a}(0, z) \geq c$ for all $z \in H_a$ with $|z| = 1/2$. Since $G \subset H_a$, we conclude that $\lambda_G(0, z) \geq c$ for all $z \in G$ with $|z| = 1/2$, which proves that $D_{\rho_G}(0, c) \subset K(0, 1/2)$. □

COROLLARY 2.3. *Let c_0 be a positive constant. Then there is a constant $c > 0$ (depending on c_0) such that the following holds: If $G \subset \mathbb{C}$ is a hyperbolic domain and $z_0 \in G$ satisfies $\rho_G(z_0) \geq c_0 \cdot \rho_G^*(z_0)$, then any universal covering map $p: \mathbb{D} \to G$ is injective in $D_{\rho_{\mathbb{D}}}(\zeta_0, c)$ for any $\zeta_0 \in \mathbb{D}$ with $p(\zeta_0) = z_0$.*

Proof. From Lemma 2.2 we obtain a constant $c > 0$ depending only on c_0 such that $D_{\rho_G}(z_0, c) \subset K(z_0, \operatorname{dist}(z_0, \partial G)/2)$ holds. Since the latter (Euclidean) disk is simply connected, we find that $D_{\rho_{\mathbb{D}}}(\zeta_0, c)$ is mapped biholomorphically onto $D_{\rho_G}(z_0, c)$ by any universal covering map $p: \mathbb{D} \to G$ for any $\zeta_0 \in \mathbb{D}$ with $p(\zeta_0) = z_0$. □

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we must study the behavior of the iterates $\tilde{f}^n$ of any lift $\tilde{f}$ of f under a universal covering map $p: \mathbb{D} \to G$. Since $\tilde{f}$ has no fixed point in $\mathbb{D}$, the fundamental theorem of Wolff, Denjoy, and Valiron (see [V]) asserts that the sequence of iterates $(\tilde{f}^n)$ converges to the Denjoy–Wolff point $a \in \partial\mathbb{D}$ of $\tilde{f}$ uniformly on compact subsets in $\mathbb{D}$. Theorem 1 in [P] shows: If $\tau_n \in \text{Möb}(\mathbb{D})$ keeps the point a fixed and sends $\tilde{f}^n(0)$ to the origin, and if $\tilde{g}_n: \mathbb{D} \to \mathbb{D}$ is defined by $\tilde{g}_n := \tau_n \circ \tilde{f}^n$, then the limit

$$\tilde{g}(z) = \lim_{n \to \infty} \tilde{g}_n(z)$$

exists locally uniformly in $\mathbb{D}$ and satisfies $\tilde{g}(0) = 0$, $\tilde{g}(\mathbb{D}) \subset \mathbb{D}$, where $\tilde{g} \equiv 0$ if and only if $\lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \xrightarrow{n \to \infty} 0$. If $\tilde{g} \not\equiv 0$, then

$$\tilde{\varphi}(z) = \lim_{n\to\infty} (\tau_n \circ \tau_{n+1}^{-1})(z)$$

also exists, $\tilde{\varphi} \neq \mathrm{id}$, $\tilde{\varphi} \in \text{Möb}(\mathbb{D})$, and satisfies

$$\tilde{g} \circ \tilde{f} = \tilde{\varphi} \circ \tilde{g}.$$

Furthermore, since $p \circ \tilde{f} = f \circ p$ holds in $\mathbb{D}$, the map $\tilde{f}: \mathbb{D} \to \mathbb{D}$ induces a homomorphism $\tilde{f}_* : \Gamma \to \Gamma$ with

$$\tilde{f} \circ \gamma = \tilde{f}_*(\gamma) \circ \tilde{f} \quad \text{for all } \gamma \in \Gamma, \tag{3}$$

where Γ is the Fuchsian group associated with p. Since $\tilde{f}_*$ is a homomorphism, we have $(\tilde{f}^n)_* = (\tilde{f}_*)^n$ and simply write $\tilde{f}_*^n$. From this we obtain $\tilde{g}_n \circ \gamma = \tau_n \circ \tilde{f}_*^n(\gamma) \circ \tau_n^{-1} \circ \tilde{g}_n$. Let now $\lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \not\to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \tau_n \circ \tilde{f}_*^n(\gamma) \circ \tau_n^{-1} =: \tilde{g}_*(\gamma)$ exists for all $\gamma \in \Gamma$, and satisfies $\tilde{g}_*(\gamma) \in \text{Möb}(\mathbb{D})$ and

$$\tilde{g} \circ \gamma = \tilde{g}_*(\gamma) \circ \tilde{g}.$$

Let

$$B_* := \tilde{g}_*(\Gamma). \tag{4}$$

LEMMA 3.1. *Let $f: G \to G$ map the hyperbolic domain G analytically and without fixed points into itself, let $p: \mathbb{D} \to G$ be a universal covering map of G with Fuchsian group Γ, let $\tilde{f}: \mathbb{D} \to \mathbb{D}$ be a lift of f under p, and let*

$$\lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \not\to 0 \quad \textit{as } n \to \infty.$$

Define $\tilde{g}$, $\tilde{\varphi}$, and B_* as above. Then we have:

(i) *the group B_* is a Fuchsian group, and if B_* is not cyclic then the extended group $\langle B_*, \tilde{\varphi}\rangle$ is also Fuchsian;*

(ii) *B_* does not contain elliptic elements; and*

(iii) *if f is of infinite order (that is, if $f^n \neq \mathrm{id}$ for all n) and if B_* is not cyclic, then $\langle B_*, \tilde{\varphi}\rangle$ does not contain elliptic elements.*

Proof. (i) This statement is contained in Lemma 4 in [MP]. Note however that the authors work under the general assumption that f is of infinite order. If f is of finite order then $\tilde{f} \in \text{Möb}(\mathbb{D})$, and from the definition of $\tau_n, \tilde{g}_n, \tilde{g}$ one sees that $\langle B_*, \tilde{\varphi}\rangle = \langle \Gamma, \tilde{f}\rangle$. If $m \in \mathbb{N}$ is minimal with $f^m = \mathrm{id}$, then $\tilde{f}^m \in \Gamma$ and thus, using (3), every element α of $\langle B_*, \tilde{\varphi}\rangle$ can be written in the form

$$\alpha = \gamma \circ \tilde{f}^{\iota(\alpha)}, \quad \gamma \in \Gamma, \ \iota(\alpha) \in \{0, \dots, m-1\};$$

this representation is unique. Now assume that $(\alpha_n) \subset \langle B_*, \tilde{\varphi}\rangle$ is a sequence converging in $\mathbb{D}$ to the identity with $\alpha_n \neq \mathrm{id}$ for all $n \in \mathbb{N}$. Then there is a subsequence (n_k) and $\iota \in \{0, \dots, m-1\}$ such that $\iota(\alpha_{n_k}) = \iota$ for all k. Hence

$$\alpha_{n_k} = \gamma_{n_k} \circ \tilde{f}^{\iota} \quad \text{for all } k \in \mathbb{N},$$

so that $\gamma_{n_k} = \alpha_{n_k} \circ \tilde{f}^{-\iota} \to \tilde{f}^{-\iota}$ as $k \to \infty$. Because of the discreteness of Γ and using the minimal choice of m, we obtain $\iota = 0$ and $\gamma_{n_k} = \mathrm{id}$ for k large enough;

hence $\alpha_{n_k} = \mathrm{id}$, which contradicts the assumption. Therefore $\langle B_*, \tilde{\varphi}\rangle$ is discrete and thus a Fuchsian group.

(ii) Looking at the trace, we see that B_* does not contain elliptic elements, since every element of B_* is some limit of nonelliptic elements.

(iii) If B_* is not cyclic, then we know from (i) that $\langle B_*, \tilde{\varphi}\rangle$ is a discrete group. Thus we need only show that it does not contain elliptic elements of finite order. Note that since f is of infinite order, we have $\tilde{f}^n \notin \Gamma$ for all $n \in \mathbb{N}$. In this case, Lemma 5 in [MP] shows that $\tilde{\varphi}^n \notin B_*$ for all $n \in \mathbb{N}$. We define a homomorphism j of $\langle B_*, \tilde{\varphi}\rangle$ onto $\mathbb{Z}$ such that $j(\beta) = 0$ for $\beta \in B_*$ and $j(\tilde{\varphi}) = 1$ as follows: the elements $\alpha \in \langle B_*, \tilde{\varphi}\rangle$ have the form

$$\alpha = \beta_1 \circ \tilde{\varphi}^{n_1} \circ \cdots \circ \beta_p \circ \tilde{\varphi}^{n_p}, \quad \beta_\nu \in B_*, \quad n_\nu \in \mathbb{Z},$$

for some $p \in \mathbb{N}$. Since $\tilde{\varphi} \circ \beta \circ \tilde{\varphi}^{-1} \in B_*$ for $\beta \in B_*$ (see (5.4) in [MP]) and $\tilde{\varphi}^n \neq \mathrm{id}$ for all $n \in \mathbb{Z}$, we conclude that we can uniquely define the homomorphism j by the exponent sum

$$j(\alpha) := n_1 + n_2 + \cdots + n_p.$$

Furthermore, we see that α can be written in the form

$$\alpha = \tilde{\varphi}^{-m} \circ \beta \circ \tilde{\varphi}^{m+j(\alpha)}$$

with some $\beta \in B_*$ and $m \in \mathbb{N}_0$.

Now suppose that $\alpha \in \langle B_*, \tilde{\varphi}\rangle$ is elliptic of order k. Then $kj(\alpha) = j(\alpha^k) = j(\mathrm{id}) = 0$ and we have

$$\alpha = \tilde{\varphi}^{-m} \circ \beta \circ \tilde{\varphi}^{m}$$

for some $\beta \in B_*$, $m \in \mathbb{N}_0$. But this gives $\beta^k = \mathrm{id}$, which is a contradiction since B_* does not contain elliptic elements. □

Proof of Theorem 1.1. Let $p: \mathbb{D} \to G$ be any universal covering map of G, and let $\tilde{f}: \mathbb{D} \to \mathbb{D}$ be any lift of f under the map p. If f is of finite order, then $f \in \mathrm{Aut}(G)$, hence $\lambda_G(f^n(z), f^{n+1}(z)) = \lambda_G(z, f(z)) \not\to 0$ as $n \to \infty$ for all $z \in G$, and the same holds for the lift $\tilde{f} \in \mathrm{Möb}(\mathbb{D})$, so that there is nothing to show in this case. Let now f be of infinite order. We shall consider two cases: in (a) we consider the special case where $z = p(0)$ holds; in (b) we will see that the behavior of the sequence of iterates of one point in G determines the behavior of the sequence of iterates of each point in G, and this will complete the proof of the theorem.

Case (a): We first consider the case where $z = p(0)$. Let $z_n := f^n(z)$ and $\zeta_n = \tilde{f}^n(0)$ for $n \in \mathbb{N}_0$. Since $z_n = p(\zeta_n)$ we see from (1) that $\lambda_{\mathbb{D}}(\zeta_n, \zeta_{n+1}) \xrightarrow{n\to\infty} 0$ implies $\lambda_G(z_n, z_{n+1}) \xrightarrow{n\to\infty} 0$. For the other direction we assume that

$$\lambda_G(z_n, z_{n+1}) \to 0 \quad \text{and} \quad \lambda_{\mathbb{D}}(\zeta_n, \zeta_{n+1}) \not\to 0 \quad \text{as } n \to \infty.$$

Recall the definitions of τ_n, $\tilde{g}_n$, $\tilde{g}$, $\tilde{\varphi}$ and B_* (see the beginning of this section). Let $H = \mathbb{D}/B_*$, and let $q: \mathbb{D} \to H$ be a universal covering map with Fuchsian group B_* (see Lemma 3.1). Then $\tilde{g} = \lim_{n\to\infty} \tau_n \circ \tilde{f}^n$ induces an analytic map $g: G \to H$ that satisfies

$$g \circ p = q \circ \tilde{g}.$$

In [MP, 5.4] it is shown that $\tilde{\varphi} \circ B_* \circ \tilde{\varphi}^{-1} \subset B_*$, so that $\tilde{\varphi}$ is projected to a covering map $\varphi: H \to H$ with $\varphi \circ q = q \circ \tilde{\varphi}$. We have

$$g \circ f = \varphi \circ g,$$

so that

$$\lambda_G(z_n, z_{n+1}) \geq \lambda_H(g(z_n), g(z_{n+1})) = \lambda_H(\varphi^n(g(z_0)), \varphi^{n+1}(g(z_0))) \quad (5)$$

for all $n \in \mathbb{N}$. We distinguish three possibilities for B_* as follows.

(i) $B_* = \{\mathrm{id}\}$. Then H is simply connected and $q: \mathbb{D} \to H$ maps the unit disk conformally onto H; hence $\varphi \in \mathrm{Aut}(H)$. Since $\tilde{\varphi}$ has no fixed point in $\mathbb{D}$, the map φ has no fixed point in H. Thus, together with (5) we conclude

$$\lambda_G(z_n, z_{n+1}) \geq \lambda_H(g(z_0), \varphi(g(z_0))) > 0 \quad \text{for all } n \in \mathbb{N},$$

which contradicts our assumption.

(ii) $B_* = \langle \alpha \rangle$, $\alpha \in \text{Möb}(\mathbb{D})$ parabolic or hyperbolic. If α is parabolic, then we can choose $H = \mathbb{D} \backslash \{0\}$ and hence $\varphi(w) = e^{i\beta} w^k$ for some $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$, which together with (5) yields a contradiction to our assumption. If α is hyperbolic then we can choose H to be an annulus, which implies that φ is a rotation, $\varphi \neq \mathrm{id}$, which together with (5) again contradicts our assumption.

(iii) B_* is not cyclic. We obtain from Lemma 3.1 that $\langle B_*, \tilde{\varphi} \rangle$ is a Fuchsian group without elliptic elements. Let $w_0 \in \mathbb{D}$ with $q(w_0) = g(z_0)$. Using that $q \circ \tilde{\varphi} = \varphi \circ q$ we find $\beta_n \in B_*$ such that

$$\lambda_H(\varphi^n(g(z_0)), \varphi^{n+1}(g(z_0))) = \lambda_{\mathbb{D}}(\tilde{\varphi}^n(w_0), \beta_n(\tilde{\varphi}^{n+1}(w_0))).$$

Thus (5) and the assumption $\lambda_G(z_n, z_{n+1}) \xrightarrow{n \to \infty} 0$ yield

$$\lambda_{\mathbb{D}}(w_0, (\tilde{\varphi}^{-n} \circ \beta_n \circ \tilde{\varphi}^{n+1})(w_0)) \to 0 \quad \text{as } n \to \infty,$$

and this implies $\tilde{\varphi}^{-n} \circ \beta_n \circ \tilde{\varphi}^{n+1} = \mathrm{id}$ for $n \geq n_0$ since $\tilde{\varphi}^{-n} \circ \beta_n \circ \tilde{\varphi}^{n+1}$ is an element of the discontinuous group $\langle B_*, \tilde{\varphi} \rangle$ that does not contain elliptic elements. Hence $\tilde{\varphi} = \beta_n^{-1} \in B_*$; this contradicts Lemma 5 in [MP], which asserts that $\tilde{\varphi}^k \notin B_*$ for all $k \in \mathbb{N}$.

Case (b): Let now $v_0, w_0 \in G$ be arbitrary, with $v_n = f^n(v_0)$ and $w_n = f^n(w_0)$. We show that

$$\lambda_G(v_n, v_{n+1}) \xrightarrow{n \to \infty} 0 \quad \text{if and only if } \lambda_G(w_n, w_{n+1}) \xrightarrow{n \to \infty} 0,$$

which will complete the proof of our theorem.

It suffices to show one direction of the implication. Suppose $\lambda_G(v_n, v_{n+1}) \xrightarrow{n \to \infty} 0$. Let $\hat{p}: \mathbb{D} \to G$ be a universal covering map with $\hat{p}(0) = v_0$, and let $\hat{f}: \mathbb{D} \to \mathbb{D}$ be a lift of f under $\hat{p}$. From (a) we know that $\lambda_{\mathbb{D}}(\zeta_n, \zeta_{n+1}) \xrightarrow{n \to \infty} 0$ with $\zeta_n = \hat{f}^n(0)$; thus Theorem 1 in [P] shows that

$$\hat{g}_n(\zeta) = \hat{\tau}_n(\hat{f}^n(\zeta)) \xrightarrow{n \to \infty} 0,$$

where $\hat{\tau}_n \in \text{Möb}(\mathbb{D})$ sends $\hat{\zeta}_n$ to the origin and keeps the Denjoy–Wolff point $\hat{a} \in \partial\mathbb{D}$ of $\hat{f}$ fixed. Let $\hat{s}_0 \in \mathbb{D}$ with $\hat{p}(\hat{s}_0) = w_0$, $\hat{s}_n := \hat{f}^n(\hat{s}_0)$. Then we have $\hat{g}_n(\hat{s}_0) \xrightarrow{n\to\infty} 0$ and so $\lambda_{\mathbb{D}}(\hat{s}_n, \hat{\zeta}_n) \xrightarrow{n\to\infty} 0$. This yields

$$\begin{aligned} \lambda_G(w_n, w_{n+1}) &\leq \lambda_{\mathbb{D}}(\hat{s}_n, \hat{s}_{n+1}) \\ &\leq \lambda_{\mathbb{D}}(\hat{s}_n, \hat{\zeta}_n) + \lambda_{\mathbb{D}}(\hat{\zeta}_n, \hat{\zeta}_{n+1}) + \lambda_{\mathbb{D}}(\hat{\zeta}_{n+1}, \hat{s}_{n+1}) \\ &\xrightarrow{n\to\infty} 0. \end{aligned}$$

□

4. Proof of Theorem 1.2

In this section we give the proofs of Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. We show that if (2) is not true, then the iterates f^n converge to an isolated boundary fixed point of f, and this contradicts our assumption. Let $p\colon \mathbb{D} \to G$ be a universal covering map with $p(0) = z_0$, let $z_n = f^n(z_0)$, choose a lift $\tilde{f}$ of f, and let $\zeta_n = \tilde{f}^n(0)$, $\zeta_0 = 0$. Assume that (2) is not true.

Part (A): There is no $R > 0$ such that p is injective in $D_{\rho_{\mathbb{D}}}(\zeta_n, R)$ for all $n \in \mathbb{N}$. Indeed, if there is an $R > 0$ such that p is injective in $D_{\rho_{\mathbb{D}}}(\zeta_n, R)$ for all $n \in \mathbb{N}$, then the Koebe distortion theorem yields

$$\frac{\rho_G(z_n)}{\rho_G^*(z_n)} \geq \frac{1}{4}\tanh(R).$$

Thus (2) holds, which contradicts our assumption.

Part (B): We show that there are $\gamma_n \in \Gamma\backslash\{\text{id}\}$ such that $\lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) \to 0$ as $n \to \infty$. From (A) we find that $n_k \in \mathbb{N}, n_k \nearrow \infty$ with p not being injective in $D_{\rho_{\mathbb{D}}}(\zeta_{n_k}, 1/k)$. Thus there are $v_k \in D_{\rho_{\mathbb{D}}}(\zeta_{n_k}, 1/k)$ with $\alpha_k(v_k) \in D_{\rho_{\mathbb{D}}}(\zeta_{n_k}, 1/k)$ for some $\alpha_k \in \Gamma$, $\alpha_k \neq \text{id}$. This implies

$$\lambda_{\mathbb{D}}(\zeta_{n_k}, \alpha_k(\zeta_{n_k})) \leq \lambda_{\mathbb{D}}(\zeta_{n_k}, \alpha_k(v_k)) + \lambda_{\mathbb{D}}(\alpha_k(v_k), \alpha_k(\zeta_{n_k})) < 2/k \qquad (6)$$

for all $k \in \mathbb{N}$. Without loss of generality, n_0 can be chosen so large that from [P, Thm. 2] we obtain that $\tilde{f}$ is injective in $\bigcup_{n\geq n_0} D_{\rho_{\mathbb{D}}}(\zeta_n, 2)$. This together with (6) gives $\tilde{f}(\zeta_{n_k}) \neq \tilde{f}(\alpha_k(\zeta_{n_k}))$ for all $k \in \mathbb{N}$; hence

$$\zeta_{n_k+1} \neq \tilde{f}_*(\alpha_k)(\zeta_{n_k+1}) \quad \text{for all } k \in \mathbb{N},$$

that is, $\tilde{f}_*(\alpha_k) \neq \text{id}$ for all $k \in \mathbb{N}$ and similarly

$$\tilde{f}_*^j(\alpha_k) \neq \text{id} \quad \text{for all } j \in \mathbb{N}, k \in \mathbb{N}.$$

Together with (6) this shows that

$$0 < \lambda_{\mathbb{D}}(\zeta_{n_k+j}, \tilde{f}_*^j(\alpha_k)(\zeta_{n_k+j})) < 2/k \quad \text{for all } k \in \mathbb{N}, j \in \mathbb{N}. \qquad (7)$$

Because of the discreteness of Γ we can choose $\gamma_n \in \Gamma$, $\gamma_n \neq \text{id}$, in such a way that

$$\lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) \leq \lambda_{\mathbb{D}}(\zeta_n, \gamma(\zeta_n)) \quad \text{for all } \gamma \in \Gamma, \gamma \neq \text{id}, \qquad (8)$$

and this together with (7) yields $\lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) \xrightarrow{n\to\infty} 0$.

Part (C): Next we show that we can find $n_1 \in \mathbb{N}$ so that for $n \geq n_1$ we can choose $\gamma_n = \gamma^* \in \Gamma$ in (8), where γ^* does not depend on n. We use the following fact (see e.g. [B1, Thm. 8.3.1]): There is a universal constant $K > 0$ such that

$$\max\{\lambda_{\mathbb{D}}(\zeta, \gamma(\zeta)), \lambda_{\mathbb{D}}(\zeta, \tilde{\gamma}(\zeta))\} > K \tag{9}$$

for all $\gamma, \tilde{\gamma} \in \Gamma$ with $\langle \gamma, \tilde{\gamma} \rangle$ not cyclic and arbitrary $\zeta \in \mathbb{D}$. (Note that Γ is a discrete group belonging to a planar domain, hence every elementary subgroup of Γ is cyclic.)

(a) Let $w_0 \in \mathbb{D}$ with $\lambda_{\mathbb{D}}(w_0, \zeta_0) \leq K/10$, $w_n = \tilde{f}^n(w_0)$, and $n_1 \geq n_0$ such that $\lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) < K/10$ for all $n \geq n_1$. We show that

$$\lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)) \leq \lambda_{\mathbb{D}}(w_n, \gamma(w_n)) \quad \text{for all } \gamma \in \Gamma,\ \gamma \neq \text{id},\ n \geq n_1. \tag{10}$$

Otherwise we find $\alpha \in \Gamma$, $\alpha \neq \text{id}$, and $n \geq n_1$ with

$$\lambda_{\mathbb{D}}(w_n, \alpha(w_n)) < \lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)),$$

which yields

$$\begin{aligned} \lambda_{\mathbb{D}}(\zeta_n, \alpha(\zeta_n)) &\leq 2\lambda_{\mathbb{D}}(\zeta_n, w_n) + \lambda_{\mathbb{D}}(w_n, \alpha(w_n)) \\ &< 2\lambda_{\mathbb{D}}(\zeta_n, w_n) + \lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)) \\ &\leq 4\lambda_{\mathbb{D}}(\zeta_n, w_n) + \lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) < K. \end{aligned}$$

But this, together with (9) and the fact that $\lambda_{\mathbb{D}}(\zeta_n, \gamma_n(\zeta_n)) < K/10$, implies that $\langle \gamma_n, \alpha \rangle$ is cyclic, that is, $\gamma_n = \beta^k$ and $\alpha = \beta^l$ for some $\beta \in \Gamma$ and $k, l \in \mathbb{Z}$. Because of (8) we conclude $k = \pm 1$ so that $\alpha = \gamma_n^{kl}$, contradicting the inequality $\lambda_{\mathbb{D}}(w_n, \alpha(w_n)) < \lambda_{\mathbb{D}}(w_n, \gamma_n(w_n))$ which followed from the assumption that (10) is not true. Hence we have shown that (10) holds.

Furthermore, there is no $R > 0$ with p being injective in $D_{\rho_{\mathbb{D}}}(w_n, R)$ for all $n \in \mathbb{N}$. For otherwise, as in part (A), we find that $\rho_G(w_n)/\rho_G^*(w_n) \geq \tanh(R)/4$ which together with Lemma 2.1 yields a contradiction to the assumption that (2) is not true. As in part (B), from this we obtain that

$$\lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)) \xrightarrow{n \to \infty} 0.$$

(b) Let $w_0 \in \mathbb{D}$ be arbitrary and $w_n = \tilde{f}^n(w_0)$. Then we find $k \in \mathbb{N}$ and $w_0^1, \ldots, w_0^k \in \mathbb{D}$ with $w_0^1 = \zeta_0$, $w_0^k = w_0$, and $\lambda_{\mathbb{D}}(w_0^j, w_0^{j+1}) \leq K/10$ for $j = 1, \ldots, k-1$. Applying (a) recursively to the iterates of w_0^j and w_0^{j+1} for $j = 1, \ldots, k-1$, we finally obtain

$$\lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)) \xrightarrow{n \to \infty} 0$$

and

$$\lambda_{\mathbb{D}}(w_n, \gamma_n(w_n)) \leq \lambda_{\mathbb{D}}(w_n, \gamma(w_n)) \quad \text{for all } \gamma \in \Gamma,\ \gamma \neq \text{id},\ n \geq n_1$$

with some $n_1 \in \mathbb{N}$.

(c) Let now $w_0 = \tilde{f}(\zeta_0)$ and $w_n = \tilde{f}^n(w_0) = \zeta_{n+1}$. Applying (b) with $\gamma = \gamma_{n+1}$, we conclude that

$$\lambda_{\mathbb{D}}(\zeta_{n+1}, \gamma_n(\zeta_{n+1})) \leq \lambda_{\mathbb{D}}(\zeta_{n+1}, \gamma_{n+1}(\zeta_{n+1})) \quad \text{for all } n \geq n_1.$$

Because of the minimal choice of γ_n and γ_{n+1} (see (8)) and because of (9), this yields $\gamma_n = \gamma_{n+1}^{\pm 1}$ for all $n \geq n_1$.

Part (D): Thus, we have shown that there is $\gamma^* \in \Gamma$ with

$$\lambda_{\mathbb{D}}(\zeta_n, \gamma^*(\zeta_n)) \xrightarrow{n\to\infty} 0$$

and

$$\lambda_{\mathbb{D}}(\zeta_n, \gamma^*(\zeta_n)) \leq \lambda_{\mathbb{D}}(\zeta_n, \gamma(\zeta_n)) \quad \text{for all } \gamma \in \Gamma,\ \gamma \neq \mathrm{id},\ n \geq n_1. \tag{11}$$

In particular, this shows that γ^* is parabolic. Let now $\tilde{C}_n$ be a geodesic in $\mathbb{D}$ from ζ_n to $\gamma^*(\zeta_n)$ and $C_n = p(\tilde{C}_n)$. Then C_n is a closed curve in G and, for $n \geq n_1$, it is a Jordan curve because of (11) and (9). Furthermore, C_n and C_{n+1} are freely homotopic and $\ell_{\rho_G}(C_n) \xrightarrow{n\to\infty} 0$, where ℓ_{ρ_G} denotes the hyperbolic length in G. From this one easily concludes that there is an isolated boundary point $a \in \partial_{\hat{\mathbb{C}}} G$ with $z_n \xrightarrow{n\to\infty} a$ (note that $z_n \in C_n$). Now, because of Picard's theorem, the point a is not an essential singularity of f since f maps G into G and G possesses at least two boundary points in $\mathbb{C}$. With $f(z_n) = z_{n+1} \xrightarrow{n\to\infty} a$ we conclude that f has an analytic extension to a with $f(a) = a$ (with a suitable definition of "analytic" if $a = \infty$). □

Now, together with Lemma 2.1, the Corollary 1.3 is an easy consequence of Theorem 1.2.

5. Proof of Theorem 1.4

We consider two cases: One is $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$ for all $z \in G$; the other is $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$ for all $z \in G$. From Theorem 1.1 we know that no other cases can occur.

5.1. The Case $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$

We first assume that $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$. Let $p\colon \mathbb{D} \to G$ be a universal covering map with Fuchsian group Γ, and let $\tilde{f}\colon \mathbb{D} \to \mathbb{D}$ be a lift of f under p. Then we have $\lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \not\to 0$ as $n \to \infty$ (see Theorem 1.1) and with $\tau_n \in \text{Möb}(\mathbb{D})$ as in Section 3 we have

$$\tilde{g}_n = \tau_n \circ \tilde{f}^n \xrightarrow{n\to\infty} \tilde{g} \not\equiv 0, \tag{12}$$

$$\tilde{g} \circ \gamma = \tilde{g}_*(\gamma) \circ \tilde{g} \quad \text{for all } \gamma \in \Gamma, \tag{13}$$

$$\tilde{g} \circ \tilde{f} = \tilde{\varphi} \circ \tilde{g} \quad \text{with } \tilde{\varphi} = \lim_{n\to\infty} (\tau_n \circ \tau_{n+1}^{-1}) \in \text{Möb}(\mathbb{D}). \tag{14}$$

Let $B_n := \tau_n \circ \Gamma \circ \tau_n^{-1} = \{\, \tau_n \circ \gamma \circ \tau_n^{-1} \mid \gamma \in \Gamma \,\}$.

LEMMA 5.1. *Let $f\colon G \to G$ map the hyperbolic domain $G \subset \mathbb{C}$ analytically, without fixed points, and without isolated boundary fixed points into itself, and let $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$. Choose a universal covering map $p\colon \mathbb{D} \to G$, denote its Fuchsian group by Γ, let $\tilde{f}\colon \mathbb{D} \to \mathbb{D}$ be a lift of f, and define τ_n and B_n as above. Then the family $\{\, B_n \mid n \in \mathbb{N} \,\}$ is uniformly discrete; that is, there is an open neighborhood U of the identity* id *in* Möb$(\mathbb{D})$ *such that $U \cap B_n = \{\mathrm{id}\}$ for all $n \in \mathbb{N}$.*

Proof. Let $z_0 = p(0)$, $z_n = f^n(z_0)$, and $\zeta_n = \tilde{f}^n(0)$. Then Theorem 1.2 shows that $\rho_G(z_n) \geq c_0 \cdot \rho_G^*(z_n)$ for all $n \in \mathbb{N}$ and some constant $c_0 > 0$. Thus Corollary 2.3 yields that p is injective in $D_{\rho_{\mathbb{D}}}(\zeta_n, c)$ for all $n \in \mathbb{N}$ and some $c > 0$ depending only on c_0. Hence $p \circ \tau_n^{-1}$ is injective in $D_{\rho_{\mathbb{D}}}(0, c)$ for all $n \in \mathbb{N}$, so that $\beta(D_{\rho_{\mathbb{D}}}(0, c)) \cap D_{\rho_{\mathbb{D}}}(0, c) = \varnothing$ for all $\beta \in B_n \backslash \{\mathrm{id}\}$, $n \in \mathbb{N}$. Let $U := \{\alpha \in \text{Möb}(\mathbb{D}) \mid \alpha(D_{\rho_{\mathbb{D}}}(0, c)) \cap D_{\rho_{\mathbb{D}}}(0, c) \neq \varnothing\}$. Then U is an open neighborhood of the identity in Möb($\mathbb{D}$) and satisfies $U \cap B_n = \{\mathrm{id}\}$ for all $n \in \mathbb{N}$. □

As in Section 3, let $B_* := \tilde{g}_*(\Gamma)$. We introduce a new group

$$B := \bigcup_{k=0}^{\infty} \tilde{\varphi}^{-k} \circ B_* \circ \tilde{\varphi}^k. \tag{15}$$

In [MP] the group B_* is of main interest, whereas we shall work with the group B from now on; this is a major difference from [MP]. The following four lemmas state the most important tools needed for the proof of Theorem 1.4.

LEMMA 5.2. *Let the assumptions of Lemma 5.1 hold, and define $\tilde{g}_n$, $\tilde{g}$, B_*, and B as before. Then, for every $\gamma \in \Gamma$ and $k \in \mathbb{N}_0$, the limit $\lim_{n\to\infty} \tau_{n+k} \circ \tilde{f}_*^n(\gamma) \circ \tau_{n+k}^{-1}$ exists and is equal to $\tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^k$. We have*

$$B = \left\{ \lim_{n\to\infty} \tau_{n+k} \circ \tilde{f}_*^n(\gamma) \circ \tau_{n+k}^{-1} \;\middle|\; \gamma \in \Gamma,\, k \in \mathbb{N}_0 \right\}; \tag{16}$$

B is a Fuchsian group without elliptic elements.

Proof. Let $\gamma \in \Gamma$ and $k \in \mathbb{N}_0$. Then we have

$$\begin{aligned} &\tau_{n+k} \circ \tilde{f}_*^n(\gamma) \circ \tau_{n+k}^{-1} \\ &\quad = \tau_{n+k} \circ \tau_{n+k-1}^{-1} \circ \tau_{n+k-1} \circ \cdots \circ \tau_n^{-1} \circ \tau_n \circ \tilde{f}_*^n(\gamma) \circ \tau_n^{-1} \circ \tau_n \circ \cdots \circ \tau_{n+k}^{-1} \\ &\quad \xrightarrow{n\to\infty} \tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^k. \end{aligned} \tag{17}$$

Let now $\beta \in B$, that is, $\beta = \tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^k$ for some $\gamma \in \Gamma$, $k \in \mathbb{N}_0$. Then $\beta = \lim_{n\to\infty} \tau_{n+k} \circ \tilde{f}_*^n(\gamma) \circ \tau_{n+k}^{-1}$ by (17) and the other inclusion in (16) is also clear from (17). With this and the knowledge that $\tau_{n+k} \circ \tilde{f}_*^n(\gamma) \circ \tau_{n+k}^{-1} \in B_{n+k}$ for $\gamma \in \Gamma$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, the discreteness of B is an immediate consequence of Lemma 5.1.

We now show that B is a group: Since $B \subset \text{Möb}(\mathbb{D})$, it suffices to show that $\beta_1 \circ \beta_2^{-1} \in B$ for $\beta_1, \beta_2 \in B$. Let $\beta_1, \beta_2 \in B$, $\beta_1 = \tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma_1) \circ \tilde{\varphi}^k$, and $\beta_2 = \tilde{\varphi}^{-l} \circ \tilde{g}_*(\gamma_2) \circ \tilde{\varphi}^l$. Since $(\tilde{f}_*^n(\gamma_2))^{-1} = \tilde{f}_*^n(\gamma_2^{-1})$, we obtain

$$(\tilde{g}_*(\gamma_2))^{-1} = \tilde{g}_*(\gamma_2^{-1})$$

and thus

$$\beta_2^{-1} = \tilde{\varphi}^{-l} \circ \tilde{g}_*(\gamma_2^{-1}) \circ \tilde{\varphi}^l.$$

We now use that $\tilde{\varphi} \circ B_* \circ \tilde{\varphi}^{-1} \subset B_*$ (see (5.4) in [MP]; note that in general this is a proper inclusion). Then we have:

$$\begin{aligned}
\text{If } k>l:\quad \beta_1\circ\beta_2^{-1} &= \tilde{\varphi}^{-k}\circ\tilde{g}_*(\gamma_1)\circ(\tilde{\varphi}^{k-l}\circ\tilde{g}_*(\gamma_2^{-1})\circ\tilde{\varphi}^{-(k-l)})\circ\tilde{\varphi}^k\\
&= \tilde{\varphi}^{-k}\circ\tilde{g}_*(\gamma_1)\circ\tilde{g}_*(\gamma_3)\circ\tilde{\varphi}^k \quad \text{for some } \gamma_3\in\Gamma\\
&= \tilde{\varphi}^{-k}\circ\tilde{g}_*(\gamma_1\circ\gamma_3)\circ\tilde{\varphi}^k\in B;\\
\text{If } k=l:\quad \beta_1\circ\beta_2^{-1} &= \tilde{\varphi}^{-k}\circ\tilde{g}_*(\gamma_1\circ\gamma_2^{-1})\circ\tilde{\varphi}^k\in B;\\
\text{If } k<l:\quad \beta_1\circ\beta_2^{-1} &= \tilde{\varphi}^{-l}\circ(\tilde{\varphi}^{l-k}\circ\tilde{g}_*(\gamma_1)\circ\tilde{\varphi}^{-(l-k)})\circ\tilde{g}_*(\gamma_2^{-1})\circ\tilde{\varphi}^l\in B.
\end{aligned}$$

Hence B is a Fuchsian group.

Since every element of B is the limit of nonelliptic Möbius transformations (see (16)), looking at the trace we conclude that the group B does not contain elliptic elements. This completes the proof of the lemma. □

REMARK. Using the notation of the proof of Lemma 3.1, we see that the group B is the kernel of the homomorphism j defined there.

Next we show that the group B has some other representation, too. Let us define the convergence of groups in the sense of Chabauty: A sequence (Δ_n) of discrete subgroups of Möb($\mathbb{D}$) is said to *converge in the sense of Chabauty* to a discrete subgroup Δ of Möb($\mathbb{D}$) if the following two conditions hold:

(a) for any sequence $(\delta_{n_k})_k$ with $\delta_{n_k}\in\Delta_{n_k}$ converging to some $\delta\in$ Möb($\mathbb{D}$) as $k\to\infty$, we have $\delta\in\Delta$; and

(b) for each $\delta\in\Delta$ there is a sequence (δ_n) with $\delta_n\in\Delta_n$ such that $\delta_n\to\delta$ as $n\to\infty$.

See [H] for a detailed discussion of this topology.

LEMMA 5.3. *Let the assumptions of Lemma 5.1 hold, and define B as in* (15). *Then the sequence* (B_n) *of groups converges in the sense of Chabauty to the group* B.

Proof. Using Lemma 5.2, we need only check part (a) of the definition of the Chabauty convergence. For $\beta_{n_k}\in B_{n_k}$, suppose $\beta_{n_k}\to\beta\in$ Möb($\mathbb{D}$) as $k\to\infty$. Then β_{n_k} can be written as $\beta_{n_k}=\tau_{n_k}\circ\gamma_{n_k}\circ\tau_{n_k}^{-1}$, where $\gamma_{n_k}\in\Gamma$. We will show

$$\tilde{\varphi}^{-n_k}\circ\tilde{g}_*(\gamma_{n_k})\circ\tilde{\varphi}^{n_k}\to\beta \quad \text{as } k\to\infty. \tag{18}$$

To see this, note the following:

$$\begin{aligned}
\tilde{\varphi}^{-n_k}\circ\tilde{g}_*(\gamma_{n_k})\circ\tilde{\varphi}^{n_k}\circ\tilde{g} &= \tilde{\varphi}^{-n_k}\circ\tilde{g}_*(\gamma_{n_k})\circ\tilde{g}\circ\tilde{f}^{n_k} \quad \text{by (14)}\\
&= \tilde{\varphi}^{-n_k}\circ\tilde{g}\circ\gamma_{n_k}\circ\tilde{f}^{n_k} \quad \text{by (13)}\\
&= \tilde{\varphi}^{-n_k}\circ\tilde{g}\circ\tau_{n_k}^{-1}\circ\beta_{n_k}\circ\tilde{g}_{n_k}.
\end{aligned}$$

Now, $\tilde{\varphi}^{-n_k}\circ\tilde{g}\circ\tau_{n_k}^{-1}$ converges to the identity as $k\to\infty$, since by (12) and (14) we have

$$\tilde{\varphi}^{-n_k}\circ\tilde{g}\circ\tau_{n_k}^{-1}\circ\tilde{g}_{n_k}=\tilde{g} \quad \text{for all } k\in\mathbb{N}$$

and $\tilde{g}_{n_k}\to\tilde{g}\not\equiv$ const. as $k\to\infty$, the convergence being locally uniform in $\mathbb{D}$. Note that the family $\{\tilde{\varphi}^{-n_k}\circ\tilde{g}\circ\tau_{n_k}^{-1}\mid k\in\mathbb{N}\}$ is normal in $\mathbb{D}$. Using this result, we obtain from the above calculation that

$$\tilde{\varphi}^{-n_k} \circ \tilde{g}_*(\gamma_{n_k}) \circ \tilde{\varphi}^{n_k} \circ \tilde{g} \to \mathrm{id} \circ \beta \circ \tilde{g} \quad \text{as } k \to \infty;$$

again considering normality and using that $\tilde{g} \not\equiv$ const., this shows (18).

Since B is a discrete subgroup of $\text{Möb}(\mathbb{D})$ by Lemma 5.2, and since $\tilde{\varphi}^{-n_k} \circ \tilde{g}_*(\gamma_{n_k}) \circ \tilde{\varphi}^{n_k} \in B$ for every $k \in \mathbb{N}$, we obtain from (18) that there is some $k_0 \in \mathbb{N}$ such that

$$\tilde{\varphi}^{-n_k} \circ \tilde{g}_*(\gamma_{n_k}) \circ \tilde{\varphi}^{n_k} = \beta \quad \text{for all } k \geq k_0.$$

Hence $\beta \in B$, and this completes the proof of the lemma. □

Now we want to show that the group B belongs to a planar domain—that the Riemann surface $\mathbb{D}/B$ is conformally equivalent to a hyperbolic domain $H \subset \mathbb{C}$. We use the following classical fact (see [N, Chap. 9]).

THEOREM 5.4. *Let $\Delta \subset \text{Möb}(\mathbb{D})$ be a Fuchsian group without elliptic elements. Then the Riemann surface $\mathbb{D}/\Delta$ is conformally equivalent to a planar domain (i.e., a subdomain of the complex plane) if and only if every closed Jordan curve $J \subset \mathbb{D}/\Delta$ divides $\mathbb{D}/\Delta$.*

A consequence of this theorem is the following lemma.

LEMMA 5.5. *Let $\Delta \subset \text{Möb}(\mathbb{D})$ be a Fuchsian group without elliptic elements. Then the Riemann surface $\mathbb{D}/\Delta$ is not conformally equivalent to a planar domain if and only if the following condition holds: There are two simple hyperbolic elements $\alpha, \beta \in \Delta$ whose axes A_α, A_β intersect in a point $w_0 \in \mathbb{D}$, and if $v \in A_\alpha$ and $w \in A_\beta$ are equivalent points with respect to Δ then $v = \alpha^m(w_0)$ and $w = \beta^n(w_0)$ for some $m, n \in \mathbb{Z}$.*

Proof. Let $q: \mathbb{D} \to \mathbb{D}/\Delta$ be a universal covering map of the Riemann surface $\mathbb{D}/\Delta$.

(i) Assume that there are two simple hyperbolic elements $\alpha, \beta \in \Delta$ whose axes intersect and that do not have other nontrivial equivalent points with respect to Δ. Then the subarcs $[w_0, \alpha(w_0)]$ of A_α and $[w_0, \beta(w_0)]$ of A_β map to simple closed geodesics in $\mathbb{D}/\Delta$ that intersect exactly once. If $\mathbb{D}/\Delta$ were conformally equivalent to a planar domain then this could not happen (by the Jordan curve theorem); thus, $\mathbb{D}/\Delta$ is not planar.

(ii) Let, on the other hand, $\mathbb{D}/\Delta$ be not conformally equivalent to a planar domain. Then there is a closed Jordan curve $\tilde{C}$ in $\mathbb{D}/\Delta$ that does not divide $\mathbb{D}/\Delta$ (see Theorem 5.4), so we can find another Jordan curve $\tilde{D}$ in $\mathbb{D}/\Delta$ that intersects $\tilde{C}$ exactly once. Choose lifts $C_0, D_0: [0, 1) \to \mathbb{D}$ of $\tilde{C}, \tilde{D}$ with common initial point $C_0(0) = D_0(0)$. Since $\tilde{C}, \tilde{D}$ are closed curves there exist $\alpha, \beta \in \Delta$ such that C_0, D_0 extend continuously to 1 by $C_0(1) = \alpha(C_0(0))$ and $D_0(1) = \beta(D_0(0))$. Since the algebraic intersection number of curves is invariant under free homotopies, we conclude from the fact that $\tilde{C}, \tilde{D}$ intersect exactly once that $\langle \alpha, \beta \rangle$ is not cyclic.

Denote by A_α and A_β the axes of α and β. Since $\langle \alpha, \beta \rangle$ is not cyclic we have $A_\alpha \neq A_\beta$. Consider the curves $C := \bigcup_{n \in \mathbb{Z}} \alpha^n(C_0)$ and $D := \bigcup_{n \in \mathbb{Z}} \beta^n(D_0)$ that intersect exactly once and that do not have equivalent points with respect to Δ except for the images under α and β of their intersection point $C_0(0)$. Since A_α and

C have the same endpoints and so do A_β and D, we conclude that A_α and A_β intersect in a point $w_0 \in \mathbb{D}$ and that α and β are simple hyperbolic elements of Δ. From this we see that A_α and A_β do not have equivalent points with respect to Δ except for the images of w_0 under α and β, and this proves the lemma. □

Lemma 5.5 enables us to prove that the property of planarity persists to the limit under Chabauty convergence.

LEMMA 5.6. *Let $\Delta_n \subset$ Möb$(\mathbb{D})$ be Fuchsian groups without elliptic elements such that $\mathbb{D}/\Delta_n$ is conformally equivalent to a planar domain for every $n \in \mathbb{N}$. Assume furthermore that the groups (Δ_n) are uniformly discrete and converge in the sense of Chabauty to a Fuchsian group Δ. Then Δ is a Fuchsian group without elliptic elements and $\mathbb{D}/\Delta$ is conformally equivalent to a planar domain.*

Proof. Since every element of Δ is some limit of elements of the groups Δ_n, we see by looking at the trace that Δ does not contain elliptic elements. Observe that the uniform discreteness of (Δ_n) implies that two sequences (δ_n), (η_n) with $\delta_n, \eta_n \in \Delta_n$, $\delta_n \neq \eta_n$, $\delta_n \to \delta \in \Delta$, and $\eta_n \to \eta \in \Delta$ cannot have the same limit, since otherwise $(\delta_n^{-1} \circ \eta_n)$ would converge to the identity.

Assume that $\mathbb{D}/\Delta$ is not conformally equivalent to a planar domain. Then there are two simple hyperbolic elements $\alpha, \beta \in \Delta$ whose axes intersect in a point $w_0 \in \mathbb{D}$ such that the subarcs $[w_0, \alpha(w_0))$ of A_α and $[w_0, \beta(w_0))$ of A_β do not contain equivalent points with respect to Δ (see Lemma 5.5). By our assumption there are $\alpha_n \in \Delta_n$ converging to α and $\beta_n \in \Delta_n$ converging to β; by looking at the trace, we may assume that α_n and β_n are hyperbolic. Since the axes $A_{\alpha_n}, A_{\beta_n}$ converge to A_α, A_β, we conclude that A_{α_n} and A_{β_n} intersect in a point w_n for n large enough and that $w_n \to w_0$ as $n \to \infty$. Since α, β are simple we may also assume that α_n, β_n are simple. Since $\mathbb{D}/\Delta_n$ is conformally equivalent to a planar domain for every $n \in \mathbb{N}$, we conclude from Lemma 5.5 that there is a point $x_n \in (w_n, \alpha_n(w_n))$, a point $y_n \in (w_n, \beta_n(w_n))$, and an element $\delta_n \in \Delta_n$ such that $\delta_n(x_n) = y_n$. Note that since α_n, β_n are simple, we have $\delta_n \neq \mathrm{id}, \beta_n, \alpha_n^{-1}, \beta_n \circ \alpha_n^{-1}$. By choosing convergent subsequences we obtain $x_{n_k} \to x \in [w_0, \alpha(w_0)]$, $y_{n_k} \to y \in [w_0, \beta(w_0)]$, and $\delta_{n_k} \to \delta \in \Delta$, where $\delta(x) = y$. Thus from our assumption we conclude that: $x = y = w_0$ and $\delta = \mathrm{id}$; or $x = w_0$, $y = \beta(w_0)$, and $\delta = \beta$; or $x = \alpha(w_0)$, $y = w_0$, and $\delta = \alpha^{-1}$; or $x = \alpha(w_0)$, $y = \beta(w_0)$, and $\delta = \beta \circ \alpha^{-1}$. This contradicts the uniform discreteness of (Δ_n). □

For the proof of Theorem 1.4 we consider two cases. The first case now follows; the second case is dealt with in Section 5.2.

Proof of Theorem 1.4 (Case 1). Let $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$. Choose a universal covering map $p: \mathbb{D} \to G$ from $\mathbb{D}$ onto G, denote its Fuchsian group by Γ, let $\tilde{f}: \mathbb{D} \to \mathbb{D}$ be a lift of f, and define $\tau_n, \tilde{g}_n, \tilde{g}, \tilde{\varphi}, B_*, B_n$ as in the beginning of Section 5.1. Then from Lemma 5.2 we obtain that $B = \bigcup_{k=0}^{\infty} \tilde{\varphi}^{-k} \circ B_* \circ \tilde{\varphi}^k$ is a Fuchsian group without elliptic elements, and Lemma 5.3 gives that B is the limit in the sense of Chabauty of the sequence of uniformly discrete groups (B_n) (see

Lemma 5.1 for the uniform discreteness). Using Lemma 5.6 we obtain that $\mathbb{D}/B$ is conformally equivalent to a planar domain $H \subset \mathbb{C}$, since $B_n = \tau_n \circ \Gamma \circ \tau_n^{-1}$ is a Fuchsian group belonging to a planar domain. From Theorem A in [Ma] we know that we can choose H in such a way that $\mathrm{Aut}(H) \subset$ Möb. Let $q: \mathbb{D} \to H$ be a universal covering map with Fuchsian group B. Since $\tilde{g} \circ \gamma = \tilde{g}_*(\gamma) \circ \tilde{g}$ for $\gamma \in \Gamma$ and $\tilde{g}_*(\gamma) \in B$, the map $\tilde{g}: \mathbb{D} \to \mathbb{D}$ induces an analytic map $g: G \to H$ satisfying

$$g \circ p = q \circ \tilde{g}. \tag{19}$$

Note that by the definition of B we have

$$\tilde{\varphi} \circ B \circ \tilde{\varphi}^{-1} = B.$$

Indeed, let $\beta \in B$, $\beta = \tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^k$. Then $\tilde{\varphi} \circ \beta \circ \tilde{\varphi}^{-1} = \tilde{\varphi}^{-k} \circ \tilde{\varphi} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^{-1} \circ \tilde{\varphi}^k \in B$ since $\tilde{\varphi} \circ B_* \circ \tilde{\varphi}^{-1} \subset B_*$. The other inclusion is clear from the definition of B. Hence, the Möbius transformation $\tilde{\varphi}$ is projected to a covering map $\varphi: H \to H$ with

$$\varphi \circ q = q \circ \tilde{\varphi}. \tag{20}$$

With $\tilde{g} \circ \tilde{f} = \tilde{\varphi} \circ \tilde{g}$, we conclude

$$g \circ f = \varphi \circ g.$$

We show that φ is injective. Let $w_1, w_2 \in G$ with $\varphi(w_1) = \varphi(w_2)$. Choose $v_1, v_2 \in \mathbb{D}$ with $q(v_1) = w_1$ and $q(v_2) = w_2$. Then we have $q(\tilde{\varphi}(v_1)) = q(\tilde{\varphi}(v_2))$ by (20), hence $\tilde{\varphi}(v_1) = \beta(\tilde{\varphi}(v_2))$ for some $\beta \in B$. Since $\tilde{\varphi} \circ B = B \circ \tilde{\varphi}$, we find $\tilde{\beta} \in B$ with $\beta \circ \tilde{\varphi} = \tilde{\varphi} \circ \tilde{\beta}$, and this leads to $\tilde{\varphi}(v_1) = \tilde{\varphi}(\tilde{\beta}(v_2))$, but the Möbius transformation $\tilde{\varphi}$ is injective so that we conclude $v_1 = \tilde{\beta}(v_2)$, which implies $w_1 = w_2$. Hence $\varphi \in \mathrm{Aut}(H)$ and from the choice of H we conclude $\varphi \in$ Möb.

(i) Let g be injective and let $f(w_1) = f(w_2)$ for some $w_1, w_2 \in G$. Choose $v_1, v_2 \in \mathbb{D}$ with $p(v_1) = w_1$ and $p(v_2) = w_2$. Then we have $\tilde{f}(v_1) = \gamma(\tilde{f}(v_2))$ for some $\gamma \in \Gamma$, hence $\tilde{f}^n(v_1) = (\tilde{f}_*^{n-1}(\gamma) \circ \tilde{f}^n)(v_2)$ so that we conclude $\tilde{g}(v_1) = \beta(\tilde{g}(v_2))$ with $\beta = \tilde{\varphi}^{-1} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi} \in B$. Thus (19) yields $g(w_1) = g(w_2)$, that is, $w_1 = w_2$. Therefore f is injective.

Let, on the other hand, f be injective and $g(w_1) = g(w_2)$ for some $w_1, w_2 \in G$. Choose $v_1, v_2 \in \mathbb{D}$ as above. Then we have $\tilde{g}(v_1) = \beta(\tilde{g}(v_2))$ for some $\beta \in B$, $\beta = \tilde{\varphi}^{-k} \circ \tilde{g}_*(\gamma) \circ \tilde{\varphi}^k$ where $\gamma \in \Gamma$ and $k \in \mathbb{N}_0$. Hence $\tilde{\varphi}^k(\tilde{g}(v_1)) = (\tilde{g}_*(\gamma) \circ \tilde{\varphi}^k \circ \tilde{g})(v_2)$. With $\tilde{g} \circ \tilde{f} = \tilde{\varphi} \circ \tilde{g}$ we obtain $\tilde{g}(\tilde{f}^k(v_1)) = \tilde{g}(\gamma(\tilde{f}^k(v_2)))$. From [MP, Lemma 2] we know that $\tilde{g} = \tilde{h}_n \circ \tilde{g}_n$, where $\tilde{h}_n$ is injective in compact sets $K \subset \mathbb{D}$ for $n \geq n_0$ (n_0 depending on K). Hence we conclude that there is an $n_0 \in \mathbb{N}$ such that $\tilde{g}_n(\tilde{f}^k(v_1)) = (\tilde{g}_n \circ \gamma \circ \tilde{f}^k)(v_2)$ for $n \geq n_0$, and this implies that $\tilde{f}^{n+k}(v_1) = (\tilde{f}_*^n(\gamma) \circ \tilde{f}^{n+k})(v_2)$. From this we see $f^{n+k}(w_1) = f^{n+k}(w_2)$ which gives $w_1 = w_2$, so that g is injective.

(ii) We remark that, from the construction of $\tilde{f}$ and $\tilde{g}$, the assumption $f \in \mathrm{Aut}(G)$ implies $\tilde{f} \in \text{Möb}(\mathbb{D})$ and $\tilde{g} = \mathrm{id}$, $\tilde{\varphi} = \tilde{f}$. Hence $B_* = \Gamma$, and this gives $\tilde{\varphi} \circ B_* \circ \tilde{\varphi}^{-1} = B_*$ (see [MP, Lemma 6]), hence $B = \Gamma$. Thus, H is conformally equivalent to G and $g: G \to H$ is a biholomorphic map.

(iii) Note that, under the given assumptions, the iterates f^n converge to the boundary of G locally uniformly. Hence, from the assumption that G is finitely connected, we conclude that every closed curve $C \subset G$ with compact trace becomes null-homotopic under iteration with f^n for some $n \in \mathbb{N}$. This shows that there is some $n \in \mathbb{N}$ such that $\tilde{f}_*^n(\Gamma) = \{\mathrm{id}\}$ (note that Γ is finitely generated), hence $B_* = B = \{\mathrm{id}\}$ so that H is simply connected, and we can choose $H = \mathbb{D}$. □

REMARK. As already mentioned, the use of group B instead of B_* is the main difference between our treatment and that in [MP]. When using the group B_* instead of B, the map $\tilde{\varphi}$ lifts only to a covering map of the surface $\mathbb{D}/B_*$ and not to an automorphism, and the surface is not planar in general. Furthermore, in [MP] a statement about injectivity similar to (i) is made which seems not to be true, since $\tilde{\varphi} \circ B_* = B_* \circ \tilde{\varphi}$ does not hold in general.

5.2. *The Case* $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ *as* $n \to \infty$

THEOREM 5.7. *Let $f: G \to G$ map the hyperbolic domain $G \subset \mathbb{C}$ analytically, without fixed points, and without isolated boundary fixed points into itself. Let $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$ and let $z_0 \in G$. Define*

$$h_n: G \to \mathbb{C}, \qquad h_n(z) := \frac{f^n(z) - f^n(z_0)}{f^{n+1}(z_0) - f^n(z_0)}.$$

Then the sequence (h_n) converges locally uniformly in G to an analytic function $h: G \to \mathbb{C}$. Furthermore, $h(f(z)) = h(z) + 1$ holds for all $z \in G$.

Proof. Let $p: \mathbb{D} \to G$ be a universal covering map with $p(0) = z_0$, let $\tilde{f}: \mathbb{D} \to \mathbb{D}$ be a lift of f, and let $\tilde{f}^n \to 1$ as $n \to \infty$; the last requirement can be achieved by a suitable choice of p. Then Theorem 1.1 yields $\lambda_{\mathbb{D}}(\tilde{f}^n(0), \tilde{f}^{n+1}(0)) \xrightarrow{n\to\infty} 0$. Hence we know from the theorem in [BaP] that

$$\tilde{h}_n(\zeta) := \frac{\tilde{f}^n(\zeta) - \tilde{f}^n(0)}{\tilde{f}^{n+1}(0) - \tilde{f}^n(0)} \xrightarrow{n\to\infty} \tilde{h}(\zeta)$$

locally uniformly in $\zeta \in \mathbb{D}$, where $\tilde{h}$ is analytic and nonconstant in the unit disk. Furthermore, $\tilde{h}(\tilde{f}(\zeta)) = \tilde{h}(\zeta) + 1$ holds for $\zeta \in \mathbb{D}$. From Theorem 1.2 and Corollary 2.3 we conclude that p is injective in $D_{\rho_{\mathbb{D}}}(\tilde{f}^n(0), c)$ for all $n \in \mathbb{N}$ and some $c > 0$. Define $p_n: \mathbb{D} \to G$, $p_n := p \circ \tau_n^{-1} \circ \psi$, where $\tau_n \in \mathrm{Möb}(\mathbb{D})$ as in Section 3, and where $\psi: \mathbb{D} \to D_{\rho_{\mathbb{D}}}(0, c)$, $\psi(\zeta) = \zeta \cdot R$, $R = \tanh(c)$. With $\tilde{g}_n = \tau_n \circ \tilde{f}^n$ as previously used and $\hat{h}_n(\zeta) := h_n(p(\zeta))$, we have

$$\begin{aligned}\hat{h}_n(\zeta) &= \frac{(p_n \circ \psi^{-1} \circ \tilde{g}_n)(\zeta) - p_n(0)}{(p_n \circ \psi^{-1} \circ \tilde{g}_n)(\tilde{f}(0)) - p_n(0)} \\ &= \frac{(p_n \circ \psi^{-1} \circ \tilde{g}_n)(\zeta) - p_n(0)}{(\psi^{-1} \circ \tilde{g}_n)(\zeta) \cdot p_n'(0)} \\ &\quad \cdot \frac{(\psi^{-1} \circ \tilde{g}_n)(\tilde{f}(0)) \cdot p_n'(0)}{(p_n \circ \psi^{-1} \circ \tilde{g}_n)(\tilde{f}(0)) - p_n(0)} \cdot \frac{\tilde{g}_n(\zeta)}{\tilde{g}_n(\tilde{f}(0))},\end{aligned}$$

where $(\psi^{-1} \circ \tilde{g}_n)(\zeta) = \tilde{g}_n(\zeta)/R \to 0$ locally uniformly in $\mathbb{D}$ as $n \to \infty$ (see [P, Thm. 1]). The family $\{ (p_n(s) - p_n(0))/p_n'(0) \mid n \in \mathbb{N} \}$ of (normalized) conformal maps in the unit disk is normal, and each of its limit functions is again a normalized conformal map in the unit disk. From this and from the fact that $\psi^{-1}(\tilde{g}_n(\zeta)) \to 0$ locally uniformly in $\mathbb{D}$ as $n \to \infty$, we conclude that

$$\frac{(p_n \circ \psi^{-1} \circ \tilde{g}_n)(\zeta) - p_n(0)}{(\psi^{-1} \circ \tilde{g}_n)(\zeta) \cdot p_n'(0)} \xrightarrow{n\to\infty} 1$$

locally uniformly in $\mathbb{D}$. It is easy to see that $\tilde{g}_n(\zeta)/\tilde{g}_n(\tilde{f}(0)) \to \tilde{h}(\zeta)$ locally uniformly in $\mathbb{D}$ as $n \to \infty$ (see e.g. [Bo, 4.10]), and this gives

$$\hat{h}_n(\zeta) \xrightarrow{n\to\infty} \tilde{h}(\zeta)$$

locally uniformly in $\mathbb{D}$; recall that $\hat{h}_n(\zeta) = h_n(p(\zeta))$. Hence we conclude

$$\tilde{h} \circ \gamma = \tilde{h} \quad \text{for all } \gamma \in \Gamma$$

and $h_n(z) \to h(z)$ locally uniformly in G as $n \to \infty$ with $h \circ p = \tilde{h}$. Now $\tilde{h}(\tilde{f}(\zeta)) = \tilde{h}(\zeta) + 1$ for $\zeta \in \mathbb{D}$, and therefore

$$h(f(z)) = h(z) + 1 \quad \text{for all } z \in G. \qquad \square$$

Proof of Theorem 1.4 (Case 2). Let $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$. With Theorem 5.7, we have already shown the first part of the theorem by choosing $H := \mathbb{C}$ and $g := h$. For the proof of (i) recall that $g = h = \lim_{n\to\infty} h_n$ and $h_n(z) = (f^n(z) - f^n(z_0))/(f^{n+1}(z_0) - f^n(z_0))$. Since $h(f(z)) = h(z) + 1$ we conclude that $h \not\equiv 0$; hence the injectivity of f (which gives injectivity of each h_n) implies the injectivity of h. Let now h be injective and $f(w_1) = f(w_2)$ for some $w_1, w_2 \in G$. Then $h_n(w_1) = h_n(w_2)$ for all $n \in \mathbb{N}$ and so $h(w_1) = h(w_2)$, that is, $w_1 = w_2$. Case (ii) cannot occur since $f \in \mathrm{Aut}(G)$ implies $\lambda_G(f^n(z), f^{n+1}(z)) = \lambda_G(z, f(z)) > 0$ for all $n \in \mathbb{N}$. $\square$

6. Examples and Final Remarks

As always, let $G \subset \mathbb{C}$ be a hyperbolic domain. In this section we consider the special case where f is a Möbius transformation mapping G into itself without fixed points. Without loss of generality, let $f(\infty) = \infty$. Consider the increasing sequence of domains

$$f^{-n}(G) = \{ z \in \mathbb{C} \mid f^n(z) \in G \}.$$

THEOREM 6.1. *Let f be a Möbius transformation mapping the hyperbolic domain $G \subset \mathbb{C}$ into itself without fixed points, where $f(\infty) = \infty$. Then*

$$\lambda_G(f^n(z), f^{n+1}(z)) \to 0 \quad \textit{as } n \to \infty \quad \textit{for (any) } z \in G$$

if and only if

$$\bigcup_{n\in\mathbb{N}} f^{-n}(G) = \mathbb{C} \quad \textit{or} \quad \bigcup_{n\in\mathbb{N}} f^{-n}(G) = \mathbb{C}\backslash\{b\},$$

where $b \in \mathbb{C}\backslash G$ is a fixed point of f.

Proof. If f is of finite order (i.e., if $f^n = \mathrm{id}$ for some $n \in \mathbb{N}$), then f is an automorphism of G and there is nothing to show. Hence let f be of infinite order. Define $H_n := f^{-n}(G)$; thus (H_n) is an increasing sequence of domains.

Part (a): Let $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$, and suppose that $H := \bigcup_{n\in\mathbb{N}} H_n$ has at least two boundary points in $\mathbb{C}$ (i.e., suppose that H is hyperbolic). Since $H_n \subset H$ for all $n \in \mathbb{N}$, for $z \in G$ we obtain

$$\lambda_G(f^n(z), f^{n+1}(z)) = \lambda_{H_n}(z, f(z)) \geq \lambda_H(z, f(z)) > 0.$$

Thus $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$, which is a contradiction. Hence $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}$ or $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}\backslash\{b\}$ for some $b \in \mathbb{C}\backslash G$. In the latter case $f(b) = b$ holds since $f(b) \notin \bigcup_{n\in\mathbb{N}} H_n$ by definition of H_n and $f(b) \neq \infty$ by the assumption $f(\infty) = \infty$.

Part (b): For the other implication we must consider two cases.

(i) Let $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}\backslash\{b\}$ and, without loss of generality, $b = 0$. Then $f(0) = 0$ and $f(\infty) = \infty$, so $f(z) = cz$ for some $c \neq 0$. If $|c| = 1$ then $c^n \neq 1$ follows for all $n \in \mathbb{N}$ since f is of infinite order. Thus, since G is a domain and $f(G) \subset G$ holds, in this case for any $r > 0$ either the circle $\{|z| = r\}$ is entirely contained in G or it does not meet G at all. If $\{|z| = r\} \cap G = \varnothing$ for some $r > 0$ then $\{|z| = r\} \cap H_n = \varnothing$ for all $n \in \mathbb{N}$, which is impossible since $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}\backslash\{0\}$. Hence we obtain $G = \mathbb{C}\backslash\{0\}$, which contradicts the hyperbolicity of G. Therefore we have $|c| \neq 1$ and, without loss of generality, $|c| < 1$. We prove that 0 is an isolated boundary point of G, that is, we find $r > 0$ with $\{0 < |z| < r\} \subset G$. Otherwise we find $w_k \in \mathbb{C}\backslash G$ with $w_k \to 0$ as $k \to \infty$. With $f(G) \subset G$ we then have $f^{-n}(w_k) \notin G$ for all $n \in \mathbb{N}$ and for $|w_k| \leq |c|$ we find $n_k \in \mathbb{N}$ with $|c| < |f^{-n_k}(w_k)| \leq 1$. Let now w be a limit point of the sequence $(f^{-n_k}(w_k))_k$. We show that w does not belong to $\bigcup_{n\in\mathbb{N}} H_n$: Either w is the limit of a constant subsequence, that is, $f^{-n_k}(w_k) = w$ for all k of the subsequence, in which case $f^{n_k}(w) \notin G$ and so $f^n(w) \notin G$ for all $n \in \mathbb{N}$, which gives $w \notin \bigcup_{n\in\mathbb{N}} H_n$. Or we have $w = \lim_{k\to\infty} f^{-n_k}(w_k)$ for some nonconstant subsequence, whence $w \notin G$. But in this case for $n \in \mathbb{N}$ and $n_k \geq n$ we have $f^{-n_k+n}(w_k) \notin G$, which yields $f^n(w) \notin G$; again we conclude that $w \notin \bigcup_{n\in\mathbb{N}} H_n$. With $|c| \leq |w| \leq 1$ this contradicts $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}\backslash\{0\}$.

Thus, we have shown that 0 is an isolated boundary point of G. Hence, for every $R > 0$ we find $n_0(R) \in \mathbb{N}$ such that $\{0 < |z| < R\} \subset H_n$ for all $n \geq n_0(R)$. Let $R > \max\{|z_0|, |z_1|\}$. Then for $n \geq n_0(R)$ we have

$$\lambda_G(f^n(z), f^{n+1}(z)) = \lambda_{H_n}(z, f(z)) \leq \lambda_{\{0<|z|<R\}}(z_0, z_1);$$

letting $R \to \infty$, we conclude that $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$.

(ii) Let $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{C}$. Then the only fixed point of f is the point ∞ and hence f is a translation: $f(z) = z + d$ for some $d \in \mathbb{C}$. As in (i) it is easy to prove that for $R > 0$ we find $n_0(R) \in \mathbb{N}$ such that $\{|z| < R\} \subset H_n$ for all $n \geq n_0(R)$. A similar calculation as above yields that $\lambda_G(f^n(z), f^{n+1}(z)) \to 0$ as $n \to \infty$. □

It is clear that in the case $\lambda_G(f^n(z), f^{n+1}(z)) \not\to 0$ as $n \to \infty$, the domain H mentioned in Theorem 1.4 can be chosen as $H = \bigcup_{n\in\mathbb{N}} f^{-n}(G)$.

REMARK. In [MP] the example mentioned above is also considered, and the authors state that the Fuchsian group belonging to the domain $H = \bigcup_{n \in \mathbb{N}} f^{-n}(G)$ is the group B_*. This seems to be false because we only have $B_* \subset B$, which in general is a proper inclusion.

A wide class of examples is given by iteration of rational or meromorphic functions f in $\mathbb{C}$ if we take G as an invariant component of the Fatou set of the given function. Our Theorem 1.4 shows that we can find a semiconjugation in G that transforms f to a Möbius transformation, which makes the situation easier and more regular.

References

[A] L. V. Ahlfors, *Conformal invariants,* McGraw-Hill, New York, 1973.

[BaP] I. N. Baker and Ch. Pommerenke, *On the iteration of analytic functions in a halfplane II,* J. London Math. Soc. (2) 20 (1979), 255–258.

[B1] A. F. Beardon, *The geometry of discrete groups,* Springer, New York, 1983.

[B2] ———, *Iteration of rational functions,* Springer, New York, 1991.

[BP] A. F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains,* J. London Math. Soc. (2) 18 (1978), 475–483.

[Be] W. Bergweiler, *Iteration of meromorphic functions,* Bull. Amer. Math. Soc. (N.S.) 29 (1993), 151–188.

[Bo] P. Bonfert, *Iteration mit Fixpunkt am Rande,* Diplomarbeit, Tech. Univ. Berlin, 1993.

[Bö] L. E. Böttcher, *The principal laws of convergence of iterates and their application to analysis* (Russian), Izv. Kazan. Fiz.-Mat. Obshch. 14 (1904), 155–234.

[CG] L. Carleson and T. W. Gamelin, *Complex dynamics,* Springer, New York, 1993.

[C] C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk,* Trans. Amer. Math. Soc. 265 (1981), 69–95.

[D] A. Denjoy, *Sur l'itération des fonctions analytiques,* C. R. Acad. Sci. Paris Sér. I Math. 182 (1926), 255–257.

[H] W. Harvey, *Chabauty spaces of discrete groups,* Discontinuous groups and Riemann surfaces, (College Park, MD, 1973), pp. 239–246, Princeton Univ. Press, Princeton, NJ, 1974.

[Hei] M. H. Heins, *On the iteration of functions which are analytic and single-valued in a given multiply-connected region,* Amer. J. Math. 63 (1941), 461–480.

[Hej] D. A. Hejhal, *Universal covering maps for variable regions,* Math. Z. 137 (1974), 7–20.

[K] G. Kœnigs, *Recherches sur les intégrales de certaines équations fonctionelles,* Ann. Sci. École Norm. Sup. (3) 1 (1884), 1–41.

[MP] A. Marden and Ch. Pommerenke, *Analytic self-mappings of infinite order of Riemann surfaces,* J. Analyse Math. 37 (1980), 186–207.

[Ma] B. Maskit, *The conformal group of a plane domain,* Amer. J. Math. 90 (1968), 718–722.

[Mi] J. Milnor, *Dynamics in one complex variable: introductory lectures,* preprint, SUNY, Stony Brook, 1990/5.

[N] R. Nevanlinna, *Uniformisierung,* Springer, New York, 1967.

[P] Ch. Pommerenke, *On the iteration of analytic functions in a halfplane I,* J. London Math. Soc. (2) 19 (1979), 439–447.
[S] N. Steinmetz, *Rational iteration,* de Gruyter, Berlin, 1993.
[V] G. Valiron, *Fonctions analytiques,* Presses Univ. France, Paris, 1954.

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