The Real Part of Entire Functions

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1. Introduction

Given an entire function

$$f(z) = \sum a_n z^n = u + iv,$$

let us write, as usual, $M(r)$ for the maximum modulus of f, and $A(r)$ and $B(r)$ for the minimum and maximum of u, the real part of f. We always have

$$-M(r) \leq A(r) \leq B(r) \leq M(r),$$

but in fact the outer inequalities are, for most values of r, almost equalities. Wiman [14] showed that

$$-A(r) \sim B(r) \sim M(r)$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure, that is, outside a set E such that

$$\text{logmeas } E = \int_{E\cap(1,\infty)} d\log t < \infty.$$

Hayman [10] obtained refinements of these estimates at the expense of a larger exceptional set, measured in terms of upper logarithmic density. The *upper* and *lower logarithmic densities* of E are defined by

$$\overline{\text{logdens}}\, E = \overline{\lim_{r\to\infty}} \frac{\text{logmeas } E_{(1,r)}}{\log r}, \qquad \underline{\text{logdens}}\, E = \underline{\lim_{r\to\infty}} \frac{\text{logmeas } E_{(1,r)}}{\log r},$$

where $E_{(1,r)}$ denotes the part of E contained in the interval $(1, r)$. Upper and lower $\log\log$ *densities* also arise in what follows and are defined analogously. Hayman proved the following.

THEOREM 1 [10, Thm. 10]. *Suppose that $f(z)$ is a transcendental entire function, and set*

$$P = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log\log r}. \tag{1}$$

Given $\varepsilon > 0$,

$$B(r) > M(r)\left(1 - \frac{\pi^2(\sigma(P)+\varepsilon)}{2\log M(r)}\right), \quad -A(r) > M(r)\left(1 - \frac{\pi^2(\sigma(P)+\varepsilon)}{2\log M(r)}\right), \tag{2}$$

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outside a set of r of lower logarithmic density less than 1, *where* $\sigma(P) = 0$ *if* $P < 2$, $\sigma(P) = (P-1)/P$ *if* $2 \le P < \infty$, *and* $\sigma(P) = 1$ *if* $P = \infty$.

Hayman showed that these inequalities are sharp, but since the hypothesis concerns upper growth, one would expect the estimate for the exceptional set to be given in terms of upper rather than lower logarithmic density. Hayman's proof involves Wiman–Valiron techniques in the form developed finally by Kövari [11]. A lower-order version of these methods [5; 6] can be used to prove the following theorem.

THEOREM 2 [6]. *Suppose that* $f(z)$ *is a transcendental entire function, and set*

$$p = \underline{\lim}_{r\to\infty} \frac{\log\log M(r)}{\log\log r}.$$

Given $\varepsilon > 0$, *the inequalities* (2) *hold, with P replaced by p, outside a set of lower logarithmic density no more than* $\sigma(p)/(\sigma(p)+\varepsilon)$.

Although this result has the right form, in that a restriction on lower growth gives rise to a conclusion outside a set of restricted lower logarithmic density, it skirts the issue of improving Theorem 1. Such an improvement is not in itself a matter of great moment, but seems to test the efficacy of Wiman–Valiron methods. The intention here is to prove the following.

THEOREM 3. *Suppose that* $f(z)$ *is a transcendental entire function, and set*

$$p = \underline{\lim}_{r\to\infty} \frac{\log\log M(r)}{\log\log r}, \qquad P = \overline{\lim}_{r\to\infty} \frac{\log\log M(r)}{\log\log r}.$$

Given $\varepsilon > 0$, *the inequalities* (2) *hold outside a set of r of:*

(i) *upper logarithmic density* 0 *if* $P < 2$,
(ii) *upper* log log *density at most* $\sigma(P)/(\sigma(P)+\varepsilon)$ *if* $2 \le P < \infty$,
(iii) *upper logarithmic density at most* $\sigma(P)/(\sigma(P)+\varepsilon)$ *if* $P = \infty$.

If P is replaced throughout by p, conclusions (i) *and* (ii) *hold with* upper *replaced by* lower.

Note that when $p = \infty$, $P = \infty$ also, and then (iii) applies. Since [1, p. 447]

$$\underline{\text{logdens}}\, E \le \underline{\text{loglogdens}}\, E \le \overline{\text{loglogdens}}\, E \le \overline{\text{logdens}}\, E, \tag{3}$$

Theorem 3 is an improvement of Theorem 2, and suggests that log log density is the appropriate measure when $2 \le P < \infty$.

2. The Wiman–Valiron Method

The characteristic feature of the Wiman–Valiron method is the analysis of an entire function $f(z) = \sum a_n z^n$ by means of auxiliary functions associated with its Taylor series: the *maximum term*, $\mu(r) = \max_{n\ge 0} |a_n| r^n$, and the *central index*, $N = N(r)$, which is the largest integer for which

$$\mu(r) = |a_N| r^N.$$

Kövari [11] established certain inequalities for the general term of the series, involving a decreasing, negative function $\alpha(t)$ on $[0, \infty)$. With

$$\alpha_n = \exp\left(\int_0^n \alpha(t)\,dt\right), \qquad \rho_n = \exp(-\alpha(n)), \tag{4}$$

it may be shown that

$$\frac{|a_n| r^n}{\mu(r)} \le \frac{\alpha_n}{\alpha_N} \rho_N^{n-N}$$

for all r outside an exceptional set depending on α and the growth of f. In general, useful results follow only when α matches the growth of f in some sense, and Hayman developed three classes of results, covering the cases of arbitrary growth, finite order, and zero order. These, however, can be unified.

THEOREM 4 [8]. *Suppose that $f(z) = \sum a_n z^n$ is an entire function and that $\alpha(t)$ is a decreasing, negative function on $[0, \infty)$. Define α_n and ρ_n by (4). Given $K > 1$ and $R > 0$, we have*

$$\frac{|a_n| r^n}{\mu(r)} \le \frac{\alpha_n}{\alpha_N} \rho_N^{n-N}, \quad 0 \le n \le KN, \tag{5}$$

and

$$\frac{|a_n| r^n}{\mu(r)} \le \left(\frac{\rho_N}{\rho_{KN}}\right)^{(1-K^{-1})n}, \quad n > KN, \tag{6}$$

for all $r \in (0, R)$ outside a subset of logarithmic measure at most

$$-\alpha(KN_0) + \alpha(0), \tag{7}$$

where $N = N(r)$ and $N_0 = N(R)$.

Given $K > 1$, a value of r for which (5) and (6) hold will be referred to here as *normal* with respect K and α, all other values being *exceptional.*

Theorem 4 provides estimates for the terms of the Taylor series without the need for assumptions concerning the growth of f. It is true that the estimates are useful only when α is related to the growth of f, for otherwise the exceptional set may be too large, but there is a gain in that it is possible to proceed with an analysis of f even though nothing may be known about its growth.

To prove the results leading to Theorem 3, it is necessary to make further restrictions on $\alpha(t)$. In all subsequent discussion, $\alpha(t)$ will be called a *comparison function* if it is decreasing and negative on $[0, \infty)$ and satisfies in addition the following conditions:

$$\alpha(t) \text{ is differentiable for all large } t, \text{ and } |\alpha'(t)| \text{ is nonincreasing to zero;} \tag{8}$$

and

$$|\alpha'(t)|^{-1} = o(t^2/\log t) \quad \text{as } t \to \infty. \tag{9}$$

Since Hayman's three comparison functions—chosen to suit the categories of arbitrary growth, finite order, and zero order—satisfy (8) and (9), the effect is to broaden somewhat the scope of his results. Let us note here a trivial but useful consequence of (9):

$$|\alpha'(t)|^{1/2}/t = o(|\alpha'(t)|) \quad \text{as } t \to \infty. \tag{10}$$

The initial steps in the proof of Theorem 3, up to certain fundamental inequalities for $A(r)$ and $B(r)$, follow those of Theorem 1, and a significant part of the work here is to adapt Hayman's results to the new (i.e., growth-restriction–free) circumstances. This is of some intrinsic interest. The proofs are abbreviated wherever possible with references to Hayman's work, but paradoxically the increased generality in α tends to simplify the considerations. The main novelty of the paper lies in the analysis of the inequalities for $A(r)$ and $B(r)$, where an iterative procedure exploits the advantages of the new comparison method. A scheme of the proof was presented at a conference at the Nankai Institute of Mathematics, Tianjin [9]. A complete proof is contained in the first author's thesis [4].

3. Preparatory Results

The first of these involves estimating the contribution made by terms in the Taylor series that are far from the maximum term. Earlier variants are due to Kövari [11], Clunie [3], and others.

THEOREM 5 (cf. [10, Lemma 2]). *Suppose that $f(z) = \sum a_n z^n$ is a transcendental entire function. Suppose also that $K > 1$, that $\alpha(t)$ is a comparison function, and that r_0 is normal with respect to K and α. Given a positive constant γ, define*

$$k = \operatorname{int}\left\{\frac{\gamma}{|\alpha'(KN_0)|}\log\left(\frac{1}{|\alpha'(KN_0)|}\right)\right\}^{1/2}, \tag{11}$$

where $N_0 = N(r_0)$ and int *denotes integral part. For r satisfying*

$$r_0 e^{-2/k} \le r \le r_0 e^{2/k}, \tag{12}$$

write

$$\mu_0(r) = |a_{N_0}|r^{N_0}.$$

Then, for any fixed real numbers q and $\gamma_1 < \gamma$,

$$\sum_{|n-N|\ge k} n^q |a_n| r^n = o\{\mu_0(r) N_0^q |\alpha'(KN_0)|^{(\gamma_1-1)/2}\} \tag{13}$$

uniformly as r and r_0 tend to infinity, subject to (12).

As a corollary we have the following theorem.

THEOREM 6. *Suppose that $f(z) = \sum a_n z^n$ is a transcendental entire function. Suppose also that $K > 1$, that $\alpha(t)$ is a comparison function, and that r_0 is normal with respect to K and α. Given a positive constant λ, define*

$$k = \operatorname{int}\left\{\frac{2\lambda+2}{|\alpha'(KN_0)|}\log\left(\frac{1}{|\alpha'(KN_0)|}\right)\right\}^{1/2}, \tag{14}$$

where $N_0 = N(r_0)$. *Then, for all z satisfying*

$$r_0 e^{-2/k} \le |z| \le r_0 e^{2/k}, \tag{15}$$

we have

$$f(z) = z^{N_0-k} P(z) + o\left\{\left(\frac{r}{r_0}\right)^{N_0} M(r_0)|\alpha'(KN_0)|^{\lambda}\right\}, \tag{16}$$

where $r = |z|$ *and*

$$P(z) = \sum_0^{2k} a_{n+N_0-k} z^n \tag{17}$$

uniformly as r and r_0 *tend to infinity, subject to* (15).

The next theorem, due essentially to Barry [1], is of the same kind as Theorem 5 but applies to functions that are small at certain points.

THEOREM 7 (cf. [10, Lemma 4]). *Suppose that* $f(z) = \sum a_n z^n$ *is a transcendental entire function. Suppose also that, for some* $R \ge 1$, $N(R) \le (\log R)^{p-1}$, *where* $1 \le p < 2$. *Given* $0 < \eta < 2 - p$, *we have*

$$\sum_{n \ne N} |a_n| r^n < 4\mu(r) \exp\{-(\log \sqrt{r})^{\eta}\} \tag{18}$$

for all r in $[1, R]$ *outside a set of logarithmic measure* $\le 5(\log R)^{p-1+\eta} + 2$.

Finally, we shall prove a result on the local behavior of entire functions near points at which the modulus is relatively large.

THEOREM 8 (cf. [10, Thm. 10]). *Suppose that* $f(z)$ *is a transcendental entire function. Suppose also that* $K > 1$, *that* $\alpha(t)$ *is a comparison function, and that* r_0 *is normal with respect to* K *and* α. *Given a positive constant* λ, *define*

$$k = \operatorname{int}\left\{\frac{2\lambda + 2}{|\alpha'(KN_0)|} \log\left(\frac{1}{|\alpha'(KN_0)|}\right)\right\}^{1/2}, \tag{19}$$

where $N_0 = N(r_0)$, *and let* η *be a number satisfying*

$$1 \ge \eta \ge |\alpha(KN_0)|^{\lambda}. \tag{20}$$

If z_0 *is such that* $|z_0| = r_0$ *and*

$$|f(z_0)| \ge \eta M(r_0), \tag{21}$$

then for all z of the form

$$z = z_0 e^{\tau} \quad \text{where } |\tau| \le \eta/(100k) \tag{22}$$

we have

$$\log \frac{f(z)}{f(z_0)} = (N_0 + \phi_1)\tau + \phi_2 \tau^2 + \delta(|\tau|), \tag{23}$$

where, if r_0 *is large,*

$$|\phi_j| \le 2(60k/\eta)^j \ (j = 1, 2) \quad \text{and} \quad |\delta(\tau)| \le 5(60k|\tau|/\eta)^3. \tag{24}$$

4. Proof of Theorems 5 and 6

We need a technical lemma, which Hayman uses implicitly.

LEMMA 9. *Given a real number q and a positive integer N, let*

$$S_{N,q}(t) = \sum_{n=N}^{\infty} n^q t^n. \tag{25}$$

Then, for any t satisfying $0 \le t < 1$,

$$S_{N,q}(t) \le C\frac{N^q t^N}{1-t}\{1 + (N(1-t))^{-k}\}, \tag{26}$$

where $k = \operatorname{int} q + 1$ and C depends only on q.

(Throughout the paper, C is used for a generic constant, not necessarily the same at each occurrence.)

To prove the lemma, suppose first that $q \le 0$. Then

$$S_{N,q}(t) = \sum_{n=N}^{\infty} n^q t^n \le N^q \sum_{n=N}^{\infty} t^n = \frac{N^q t^N}{1-t}, \tag{27}$$

so that (26) holds with $C = 1$. Otherwise $q > 0$, in which case

$$\begin{aligned}(1-t)S_{N,q}(t) &= N^q t^N + \sum_{n=N}^{\infty} n^q\{(1+1/n)^q - 1\}t^{n+1} \\ &= N^q t^N + q\sum_{n=N}^{\infty} n^{q-1}(1+\theta_n/n)^{q-1}t^{n+1} \\ &\le N^q t^N + q2^{|q-1|}\sum_{n=N}^{\infty} n^{q-1}t^n.\end{aligned}$$

Thus, with $C_q = q2^{|q-1|}$,

$$\begin{aligned}S_{N,q}(t) &\le \frac{N^q t^N}{1-t} + C_q\frac{S_{N,q-1}}{1-t} \\ &\le \frac{N^q t^N}{1-t} + C_q\frac{N^{q-1}t^N}{(1-t)^2} + \cdots + (C_q \ldots C_{q-k+1})\frac{S_{N,q-k}}{(1-t)^k} \\ &= \frac{N^q t^N}{1-t}\left\{1 + \frac{C_q}{N(1-t)} + \cdots + \frac{(C_q \ldots C_{q-k+1})S_{N,q-k}}{N^q t^N (1-t)^{k-1}}\right\},\end{aligned}$$

where $k = \operatorname{int} q + 1$. Using (27) to estimate $S_{N,q-k}$, we have

$$\begin{aligned}S_{N,q}(t) &\le \frac{N^q t^N}{1-t}\left\{1 + \frac{C_q}{N(1-t)} + \cdots + \frac{C_q \ldots C_{q-k+1}}{(N(1-t))^k}\right\} \\ &\le C\frac{N^q t^N}{1-t}\{1 + (N(1-t))^{-k}\},\end{aligned}$$

where C depends only on q. □

Turning now to the proof of Theorem 5, we consider the terms of the series in three blocks: those for which $n > KN_0$, those for which $n < K^{-1}N_0$, and those between. Here $N_0 = N(r_0)$, the central index at the normal value r_0.

Consider first the terms satisfying $n > KN_0$. From (6) we have

$$\begin{aligned}\frac{|a_n|r_0^n}{\mu(r_0)} &\le \left(\frac{\rho_{N_0}}{\rho_{KN_0}}\right)^{(1-K^{-1})n}\\ &= \exp\{n(1-K^{-1})(\alpha(KN_0)-\alpha(N_0))\}\\ &\le \exp\{nN_0K(1-K^{-1})^2\alpha'(KN_0)\}, \qquad (28)\end{aligned}$$

using (8). Thus, taking account of (10), (11), and (12),

$$\begin{aligned}\frac{|a_n|r^n}{\mu_0(r)} &= \frac{|a_n|r_0^n}{\mu(r_0)}\left(\frac{r}{r_0}\right)^{n-N_0}\\ &\le \exp\{nN_0K(1-K^{-1})^2\alpha'(KN_0)+(n-N_0)\log(r/r_0)\}\\ &\le \exp\{nN_0K(1-K^{-1})^2\alpha'(KN_0)+2nk^{-1}\}\\ &= \exp\{nN_0[K(1-K^{-1})^2\alpha'(KN_0)+o(|\alpha'(KN_0)|^{1/2}/N_0)]\}\\ &\le \exp\{\tfrac{1}{2}nN_0K(1-K^{-1})^2\alpha'(KN_0)\}, \qquad (29)\end{aligned}$$

provided that N_0 is large enough. Set $T = \exp\{\frac{1}{2}N_0K(1-K^{-1})^2\alpha'(KN_0)\}$. Given any (small) positive number ν,

$$|\alpha'(KN_0)| \ge \frac{\log N_0}{\nu N_0^2} \qquad (30)$$

for all large N_0, from (9), so $T \le N_0^{-K(1-K^{-1})^2/(2\nu N_0)}$ and, moreover,

$$1 - T \ge K(1-K^{-1})^2\frac{\log N_0}{4\nu N_0}$$

for all large N_0. From this together with (29) and the preceding lemma,

$$\begin{aligned}\sum_{n>KN_0} n^q|a_n|r^n &\le \mu_0(r)\sum_{n=N_0}^{\infty} n^qT^n\\ &= O\left(\mu_0(r)\frac{N_0^qT^{N_0}}{1-T}\right)\\ &= O\left(\frac{\mu_0(r)N_0^{q+1}N_0^{-K(1-K^{-1})^2/(2\nu)}}{\log N_0}\right)\\ &= o(\mu_0(r)N_0^{-\beta}) \qquad (31)\end{aligned}$$

for any positive β, since $\nu > 0$ is arbitrary.

Next, if $n < K^{-1}N_0$ then, from (5) and (8),

$$\begin{aligned}\frac{|a_n|r_0^n}{\mu(r_0)} &\le \frac{\alpha_n}{\alpha_N}\rho_N^{n-N}\\ &= \exp\left\{\int_n^{N_0}(t-n)\alpha'(t)\,dt\right\}\\ &\le \exp\{\tfrac{1}{2}(n-N_0)^2\alpha'(KN_0)\}\\ &\le \exp\{\tfrac{1}{2}N_0^2(1-K^{-1})^2\alpha'(KN_0)\};\end{aligned}$$

therefore, using (10), (11), and (12),

$$\begin{aligned}\frac{|a_n|r^n}{\mu_0(r)} &= \frac{|a_n|r_0^n}{\mu(r_0)}\left(\frac{r}{r_0}\right)^{n-N_0}\\ &\le \exp\{\tfrac{1}{2}N_0^2(1-K^{-1})^2\alpha'(KN_0)+(n-N_0)\log(r/r_0)\}\\ &\le \exp\{\tfrac{1}{2}N_0^2(1-K^{-1})^2\alpha'(KN_0)+2N_0k^{-1}\}\\ &= \exp\{\tfrac{1}{2}N_0^2(1-K^{-1})^2[\alpha'(KN_0)+o(|\alpha'(KN_0)|^{1/2}/N_0)]\}\\ &\le \exp\{\tfrac{1}{4}N_0^2(1-K^{-1})^2\alpha'(KN_0)\}\end{aligned}$$

if N_0 is large. From this and (30),

$$\begin{aligned}\sum_{n<K^{-1}N_0} n^q|a_n|r^n &\le \mu_0(r)N_0^{|q|+1}\exp\{\tfrac{1}{4}N_0^2(1-K^{-1})^2\alpha'(KN_0)\}\\ &\le \mu_0(r)N_0^{|q|+1}\exp\{-\tfrac{1}{4}(1-K^{-1})^2\nu^{-1}\log N_0\}\\ &= o(\mu_0(r)N_0^{-\beta}) \qquad (33)\end{aligned}$$

for any positive β, since $\nu > 0$ is arbitrary.

For the remaining terms, write $n = N_0 + p$, where $k \le p \le (K-1)N_0$ or $-(1-K^{-1})N_0 \le p \le -k$. As before,

$$\begin{aligned}\frac{n^q|a_n|r^n}{\mu_0(r)} &= n^q\frac{|a_n|r_0^n}{\mu(r_0)}\left(\frac{r}{r_0}\right)^{n-N_0}\\ &\le n^q\exp\{\tfrac{1}{2}\alpha'(N_0+|p|)p^2+2k^{-1}|p|\}\\ &\le K^qN_0^q\exp\{\tfrac{1}{2}\alpha'(KN_0)p^2+2k^{-1}|p|\}\\ &= K^qN_0^q\exp\{-bp^2+2k^{-1}|p|\}, \qquad (34)\end{aligned}$$

where $b = |\alpha'(KN_0)|/2$. The contribution of these terms to the sum is thus no more than

$$2K^q\mu_0(r)N_0^q\sum_{p=k}^{\infty}\exp(-bp^2+2k^{-1}p). \qquad (35)$$

Now $-bp^2+2k^{-1}p$ is decreasing for $p \ge k^{-1}b^{-1}$ and so certainly (when N_0 is large) for $p \ge k-1$, since $k^2b \sim \log(1/|\alpha(KN_0)|) \to \infty$ as $N_0 \to \infty$. Hence

$$\sum_{p=k}^{\infty} \exp(-bp^2 + 2k^{-1}p) \le \int_{k-1}^{\infty} \exp(-bx^2 + 2k^{-1}x)\,dx$$
$$= b^{-1/2} e^{k^{-2}b^{-1}} \int_{y_0}^{\infty} e^{-y^2}\,dy$$
$$\le \tfrac{1}{2} b^{-1/2} y_0^{-1} e^{(k^{-2}b^{-1} - y_0^2)}, \tag{36}$$

where $y_0 = (k-1)b^{1/2} - k^{-1}b^{-1/2}$, since

$$\int_{y_0}^{\infty} e^{-y^2}\,dy = \frac{e^{-y_0^2}}{2y_0} - \int_{y_0}^{\infty} \frac{e^{-y^2}}{4y^2}\,dy \le \frac{e^{-y_0^2}}{2y_0}.$$

Now $k^2 b \to \infty$ as $N_0 \to \infty$, so

$$\sum_{p=k}^{\infty} \exp(-bp^2 + 2k^{-1}p) = O\left(\frac{e^{-y_0^2}}{b^{1/2}y_0}\right). \tag{37}$$

Further, as is easily checked,

$$y_0 = (1 + o(1))\left\{\frac{1}{2}\gamma \log\left(\frac{1}{|\alpha'(KN_0)|}\right)\right\}^{1/2},$$

and thus

$$\frac{e^{-y_0^2}}{b^{1/2}y_0} = O\left\{\frac{|\alpha'(KN_0)|^{(1/2+o(1))\gamma}}{\{|\alpha'(KN_0)| \log(1/|\alpha'(KN_0)|)\}^{1/2}}\right\}$$
$$= o(|\alpha'(KN_0)|^{(1/2)(\gamma_1 - 1)}) \tag{38}$$

as $N_0 \to \infty$, for any $\gamma_1 < \gamma$. From this, (35), and (37), the sum of the terms in the third block is $o\{\mu_0(r) N_0^q |\alpha'(KN_0)|^{(1/2)(\gamma_1-1)}\}$. Further, $|\alpha'(KN_0)|^{(1/2)(\gamma_1-1)} \ge N_0^{-|\gamma_1 - 1|}$ for all large N_0, from (9). Theorem 5 follows, taking account of (31) and (33). □

To prove Theorem 6, apply Theorem 5 with $\gamma = 2\lambda + 2$, $\gamma_1 = 2\lambda + 1$, and $q = 0$. For z satisfying (15),

$$f(z) = \sum_{N_0-k}^{N_0+k} a_n z^n + o\{\mu_0(r)|\alpha'(KN_0)|^{\lambda}\}$$
$$= z^{N_0-k} P(z) + o\{\mu_0(r)|\alpha'(KN_0)|^{\lambda}\}, \tag{39}$$

where $P(z) = \sum_0^{2k} a_{n+N_0-k} z^n$. Thus

$$\frac{f(z)}{z^{N_0}} = \frac{P(z)}{z^k} + o\left(\frac{\mu_0(r)}{r^{N_0}}|\alpha'(KN_0)|^{\lambda}\right) = \frac{P(z)}{z^k} + o(|a_{N_0}||\alpha'(KN_0)|^{\lambda})$$
$$= \frac{P(z)}{z^k} + o\left(\frac{M(r_0)}{r_0^{N_0}}|\alpha'(KN_0)|^{\lambda}\right), \tag{40}$$

and the result follows. □

5. Proof of Theorem 7

If a nonnegative integer n is a value of the central index, let r_n be the smallest value of r at which $N(r) = n$. Otherwise let $r_n = r_N$, where N is the first value of the central index greater than n. (Thus, if $0 \le r < r_n$ then $N(r) < n$.)

Given η satisfying $0 < \eta < 2 - p$, define

$$k_n = \exp\{(\log^* r_n)^\eta\},$$

where $\log^* = \max\{1, \log\}$. For the purpose of the proof, a value r for which

$$r_n/k_n \le r \le k_n r_n$$

for some n will be called *exceptional*. It will be shown that the set of exceptional r in $[1, R]$ has logarithmic measure at most $5(\log R)^{p-1+\eta}$ for any $R \ge e$.

Suppose first that

$$\frac{r_{N_0}}{k_{N_0}} \le R \le \frac{r_{N_0+1}}{k_{N_0+1}},$$

where $N_0 = N(R)$. The logarithmic measure of the exceptional r in $[1, R]$ is then at most

$$\sum_{n=0}^{N_0} 2\log k_n \le 2(N_0+1)\log k_{N_0} \le 4(\log R)^{p-1}(\log^* r_{N_0})^\eta \le 4(\log R)^{p-1+\eta}.$$

Otherwise,

$$\frac{r_{N_0+1}}{k_{N_0+1}} < R < r_{N_0+1}$$

and, in addition to the set already identified, the interval $[r_{N_0+1}/k_{N_0+1}, R]$ is exceptional. Since $0 < \eta < 1$, so that $t\exp(-(\log^* t)^\eta)$ is increasing for $t \ge 0$, the additional exceptional interval lies in $[R\exp(-(\log R)^\eta), R]$ and has logarithmic measure at most $(\log R)^\eta$. Thus the exceptional set has logarithmic measure no more than $5(\log R)^{p-1+\eta}$.

Now suppose that r is a nonexceptional point in $[1, R]$, so that

$$r_N k_N < r < \frac{r_{N+1}}{k_{N+1}}$$

for some N. Necessarily, $N = N(r)$. Define r' and r'' by the equations

$$r = r'\exp\{(\log^* r')^\eta\}, \qquad r = r''\exp\{-(\log^* r'')^\eta\}.$$

Since $t\exp\{(\log^* t)^\eta\}$ and $t\exp\{-(\log^* t)^\eta\}$ are increasing for $t \ge 0$,

$$r_N < r' < r < r'' < r_{N+1}$$

and thus $N(r') = N(r'') = N$. For $n < N$,

$$|a_n| r'^n \le |a_N| r'^N,$$

so that

$$|a_n| r^n \le |a_N| r^N \Big(\frac{r}{r'}\Big)^{n-N} = \mu(r)\Big(\frac{r}{r'}\Big)^{n-N}$$

and therefore

$$\sum_{n<N} |a_n| r^n \leq \mu(r) \sum_{k=1}^{\infty} \left(\frac{r}{r'}\right)^{-k} = \frac{\mu(r)}{(r/r')-1}. \tag{41}$$

Similarly, for $n > N$,

$$|a_n| r''^n \leq |a_N| r''^N,$$

so that

$$|a_n| r^n \leq |a_N| r^N \left(\frac{r''}{r}\right)^{N-n} = \mu(r)\left(\frac{r''}{r}\right)^{N-n}$$

and therefore

$$\sum_{n>N} |a_n| r^n \leq \mu(r) \sum_{k=1}^{\infty} \left(\frac{r''}{r}\right)^{-k} = \frac{\mu(r)}{(r''/r)-1}. \tag{42}$$

Hence

$$\begin{aligned}\sum_{n\neq N} |a_n| r^n &\leq \mu(r)\left(\frac{1}{(r/r')-1} + \frac{1}{(r''/r)-1}\right) \\ &\leq \frac{2\mu(r)}{\exp((\log r')^{\eta})-1}.\end{aligned}$$

Now if $r \geq e^2$, then $r' \geq e$ and $\log r = \log r' + (\log r')^{\eta} \leq 2\log r'$, so

$$\begin{aligned}\sum_{n\neq N} |a_n| r^n &\leq \frac{2\mu(r)}{\exp((\log\sqrt{r})^{\eta})-1} \\ &\leq C\mu(r)\exp(-(\log\sqrt{r})^{\eta}),\end{aligned} \tag{43}$$

where $C = 2e/(e-1) < 4$. Thus (43) holds for all $r \in [1, R]$ outside a set of logarithmic measure at most $5(\log R)^{p-1+\eta} + 2$. □

6. Proof of Theorem 8

For all z such that $|z| = r_0$, by (16) it follows that

$$f(z) = z^{N_0-k}P(z) + o(M(r_0)|\alpha'(KN_0)|^{\lambda}), \tag{44}$$

so that

$$|P(z)| \leq (1+o(1))r_0^{k-N_0}M(r_0). \tag{45}$$

In particular, if $z = z_0$ then from (20) and (21) we have

$$|P(z_0)| = (1+o(1))r_0^{k-N_0}|f(z_0)| \geq (1+o(1))r_0^{k-N_0}\eta M(r_0). \tag{46}$$

Thus,

$$|P(z)| \leq \tfrac{3}{2} r_0^{k-N_0} M(r_0), \quad |z| = r_0, \tag{47}$$

and

$$|P(z_0)| \geq \tfrac{1}{2} r_0^{k-N_0} \eta M(r_0) \tag{48}$$

if r_0 is large. It follows [10, Lemma 8] that

$$\tfrac{1}{2}|P(z_0)| < |P(z)| < \tfrac{3}{2}|P(z_0)| \tag{49}$$

for $|z - z_0| < \eta r_0/(48k)$. For large r_0, this latter inequality is satisfied by $z = z_0 e^\tau$ if $|\tau| \le \eta/(60k)$, since $|1 - (z/z_0)| = |1 - e^\tau| = (1 + o(1))|\tau|$. Returning to (16), for these z we have

$$\begin{aligned} f(z) &= z^{N_0-k}\left\{P(z) + o\left(\left(\frac{r}{r_0}\right)^k r_0^{k-N_0} M(r_0)|\alpha'(KN_0)|^\lambda\right)\right\} \\ &= z^{N_0-k}\left\{P(z) + o\left(\left(\frac{r}{r_0}\right)^k P(z_0)\right)\right\} \\ &= (1 + o(1))z^{N_0-k}P(z), \end{aligned} \tag{50}$$

using (48), (49), (20), and (22). Now consider

$$\phi(\tau) = \log\frac{f(z_0 e^\tau)}{f(z_0)} - (N_0 - k)\tau \tag{51}$$

in the disk $|\tau| \le \tau_0 = \eta/(60k)$. From (50) we have

$$\operatorname{Re}\phi = \log|(1 + o(1))P(z)/P(z_0)|$$

and thus, from (49), $|\operatorname{Re}\phi| \le \log(5/2) < 1$ for all large r_0. With

$$\phi(\tau) = \sum_1^\infty \phi_n \tau^n \quad \text{for } |\tau| < \tau_0,$$

we deduce, following Hayman, that $|\phi_n| \le 2\tau_0^{-n}$ for $n \ge 1$. The cases $n = 1, 2$ establish the first half of (24). Further, if $|\tau| \le \frac{3}{5}\tau_0 = \eta/(100k)$ then

$$\left|\sum_3^\infty \phi_n \tau^n\right| \le 2\sum_3^\infty \left|\frac{\tau}{\tau_0}\right|^n \le 5\left(\frac{|\tau|}{\tau_0}\right)^3; \tag{52}$$

given the definition of $\phi(\tau)$, this proves the theorem. □

7. Derivatives of log $M(r)$

As Hayman has shown, the first two logarithmic derivatives of $\log M(r)$,

$$a(r) = r\frac{d}{dr}\log M(r) \quad \text{and} \quad b(r) = r\frac{d}{dr}a(r), \tag{53}$$

are closely related to the corresponding derivatives of $\log f(z)$ at points of maximum modulus. In fact [10, Lemma 6], except perhaps for isolated values of r,

$$z\frac{d}{dz}\log f(z) = a(r) \quad \text{and} \quad \left|\left(z\frac{d}{dz}\right)^2 \log f(z)\right| \le b(r), \tag{54}$$

where the expressions on the left are evaluated at a point at which $M(r)$ is attained. Comparing this with Theorem 8, we conclude that

$$N(r) + \phi_1 = a(r) \quad \text{and} \quad |\phi_2| \le \tfrac{1}{2}b(r) \tag{55}$$

for all but isolated normal values of r. It follows from (24) that

$$N(r) \sim a(r) \tag{56}$$

as $r \to \infty$ through all but certain isolated normal r. With the comparison function

$$\alpha(t) = \int_0^t \alpha'(\tau)\, d\tau,$$

where

$$\alpha'(\tau) = \begin{cases} -\frac{1}{\tau(\log \tau)^2}, & \tau \geq 2, \\ -\frac{1}{2(\log 2)^2}, & \tau \leq 2, \end{cases} \tag{57}$$

the exceptional values occupy a set of finite logarithmic measure, from (7). Thus (56) holds as $r \to \infty$ outside a set of finite logarithmic measure.

8. Inequalities for the Real Part

Given a normal value r and a point z, $|z| = r$, at which the maximum modulus is attained, write $f(z) = M(r)\exp(i\lambda)$, where $|\lambda| \leq \pi$. Suppose that r is not one of the isolated normal values at which (55) may fail. Applying Theorem 8 with $\eta = 1$, and with r_0, z_0, and $N_0 = N(r_0)$ replaced throughout by r, z, and $N = N(r)$, we conclude that

$$\begin{aligned} \log f(ze^{i\theta}) &= \log f(z) + (N + \phi_1)i\theta - \phi_2\theta^2 + \delta(|\theta|) \\ &= \log M(r) + (\lambda + \theta a(r))i - \phi_2\theta^2 + O(k^3|\theta|^3) \end{aligned}$$

for $\theta \leq 1/(100k)$. Set $\theta = -\lambda/a(r)$. This is permissible because then $|\theta| = O(N^{-1}) = o(k^{-1})$, from (19) and (9). We have

$$\log f(ze^{i\theta}) = \log M(r) - \phi_2 \frac{\lambda^2}{a(r)^2} + O(k^3 a(r)^{-3}). \tag{58}$$

The estimate (24) for ϕ_2 ensures that

$$-\phi_2 \frac{\lambda^2}{a(r)^2} + O(k^3 a(r)^{-3}) = o(1);$$

therefore, exponentiating (58), taking the real part, and using (55),

$$B(r) > M(r)\left\{1 - \frac{(1 + o(1))\pi^2 b(r)}{2a(r)^2} + O(k^3 a(r)^{-3})\right\}. \tag{59}$$

The same inequality holds with $-A(r)$ instead of $B(r)$, with the argument unchanged except for the choice of θ, which is $\theta = \pi - \lambda$ or $-\pi - \lambda$ depending on whether λ is positive or negative. Two cases of this result are useful in the sequel. With the comparison function (57), when $k = O(N^{1/2}(\log N)^{3/2})$ and the exceptional set has finite logarithmic measure, we have the following result.

THEOREM 10. *Suppose that $f(z)$ is a transcendental entire function and that $\delta > 0$. Then*

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + o(a(r)^{-(3/2)+\delta})\right\} \tag{60}$$

and

$$-A(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + o(a(r)^{-(3/2)+\delta})\right\} \tag{61}$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Choosing instead

$$\alpha(t) = -t^{1/(p-1)}$$

where $p > 2$, we have $k = O(N^{(p-2)/(2p-2)}(\log N)^{1/2})$ and the exceptional set E is such that logmeas $E_{(1,R)} = O(N(R)^{1/(p-1)})$, from (7). It is convenient to have this estimate in terms of $a(R)$. Toward this end note that, from the comment at the end of Section 7, given any large R we can find $R_0 < R$ such that $\log(R/R_0)$ is bounded and $a(R_0) \sim N(R_0)$ as $R \to \infty$. It follows that

$$\begin{aligned}\text{logmeas } E_{(1,R)} &\le \text{logmeas } E_{(1,R_0)} + O(1)\\ &= O(N(R_0)^{1/(p-1)}) = O(a(R_0)^{1/(p-1)}) = O(a(R)^{1/(p-1)}).\end{aligned}$$

Thus we have our next theorem.

THEOREM 11. *Suppose that $f(z)$ is a transcendental entire function, and that $p > 2$ and $\delta > 0$. Then*

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + o(a(r)^{-(3p/2(p-1))+\delta})\right\} \tag{62}$$

and

$$-A(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + o(a(r)^{-(3p/2(p-1))+\delta})\right\} \tag{63}$$

as $r \to \infty$ outside an exceptional set E such that

$$\text{logmeas } E_{(1,R)} = O(a(R)^{1/(p-1)})$$

as $R \to \infty$.

Stronger inequalities for $B(r)$ and $A(r)$ may be obtained using different methods if the central index is known to be small at certain points.

THEOREM 12. *Suppose that $f(z)$ is a transcendental entire function. Suppose that, for some $R \ge e^e$, $N(R) \le (\log R)^{p-1}$, where $1 \le p < 2$. Given $\varepsilon > 0$,*

$$B(r) > M(r)\left(1 - \frac{\varepsilon}{\log M(r)}\right) \tag{64}$$

and

$$-A(r) > M(r)\left(1 - \frac{\varepsilon}{\log M(r)}\right) \tag{65}$$

for all r in $[1, R]$ *outside a set of logarithmic measure no greater than*

$$5(\log R)^{p/2} + \log R/\log\log R + C,$$

where C depends only on ε *and* p.

Proof. From Theorem 7 with $\eta = \frac{1}{2}(2-p)$,

$$M(r) < \mu(r)\{1 + 4\exp(-(\log\sqrt{r})^{\eta})\} \tag{66}$$

for all r in $[1, R]$ outside a set E_R of logarithmic measure at most $5(\log R)^{p/2} + 2$. On the other hand, choosing θ so that, for $z = re^{i\theta}$, $a_N z^N$ is real and positive, we have, again from Theorem 7,

$$B(r) \geq a_N z^N - \sum_{n \neq N} |a_n z^n| > \mu(r)\{1 - 4\exp(-(\log\sqrt{r})^{\eta})\} \tag{67}$$

for $r \notin E_R$. Thus

$$M(r) - B(r) < 8\mu(r)\exp(-(\log\sqrt{r})^{\eta}) \leq 8M(r)\exp(-(\log\sqrt{r})^{\eta}),$$

so

$$B(r) > M(r)\{1 - 8\exp(-(\log\sqrt{r})^{\eta})\}$$

for $r \in [1, R] \setminus E_R$. Since $\log\mu(r) \leq N(r)\log r + O(1)$ and since $M(r) \leq 5\mu(r)$ for $r \in [1, R] \setminus E_R$, we have

$$\log M(r) \leq \log\mu(r) + O(1) \leq N(r)\log r + O(1) \leq (\log R)^p + O(1)$$

for $r \in [1, R] \setminus E_R$. Thus, given $\varepsilon > 0$ and $r \in [R^{1/\log\log R}, R] \setminus E_R$,

$$\begin{aligned}\frac{\log M(r)}{\exp((\log\sqrt{r})^{\eta})} &\leq \exp\{\log((\log R)^p + O(1)) - (\log\sqrt{r})^{\eta}\}\\ &\leq \exp\left\{2p\log\log R - \frac{(\log R)^{\eta}}{(2\log\log R)^{\eta}}\right\}\\ &< \varepsilon/8\end{aligned}$$

for $R > R_0 = R_0(\varepsilon, p)$. Hence

$$B(r) > M(r)\left\{1 - \frac{\varepsilon}{\log M(r)}\right\} \tag{68}$$

for all $r \in [R^{1/\log\log R}, R] \setminus E_R$, provided that $R > R_0$, and therefore for all $R > e^e$, outside an exceptional set of logarithmic measure at most

$$5(\log R)^{p/2} + \frac{\log R}{\log\log R} + \log R_0 + 2.$$

The result for $B(r)$ follows with $C = \log R_0 + 2$, and the case of $-A(r)$ is similar. □

9. Estimating the Error Term

The intention here is to prove the following lemma.

Lemma 13. *Suppose that $f(z)$ is a transcendental entire function. Given $\varepsilon > 0$,*

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} - \frac{\varepsilon}{\log M(r)}\right\} \tag{69}$$

and

$$-A(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} - \frac{\varepsilon}{\log M(r)}\right\} \tag{70}$$

as $r \to \infty$ outside a set of upper logarithmic density zero.

Proof. We consider only (69), since (70) is similar. From Theorem 10 with $\delta = 1/4$,

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + \varepsilon_1(r)\right\} \tag{71}$$

for all r outside a set of finite logarithmic measure C, where $\varepsilon_1(r) = o(a(r)^{-5/4})$. Given $\varepsilon > 0$, let $r' \geq 1$ be such that both $a(r') \geq 1$ and, for $r \geq r'$,

$$|\varepsilon_1(r)| < \varepsilon a(r)^{-5/4}.$$

Given any $R \geq r'$, consider the set of r in $[r', R]$ at which

$$a(r)^{-5/4} \geq \frac{1}{\log M(r)}. \tag{72}$$

There are two possibilities.

(i) The set of such r is empty. In this case,

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} - \frac{\varepsilon}{\log M(r)}\right\} \tag{73}$$

for all $r \leq R$ outside a set of logarithmic measure at most $C + \log r'$, and the procedure terminates.

(ii) Otherwise, the set of r in $[r', R]$ at which (72) holds is not empty. Let R' be an element of the set that is large enough that the logarithmic measure of the part of the set in $[R', R]$ is less than 1. Then (73) holds for $r \in (R', R]$ outside a set of logarithmic measure at most $C + 1$. Further, (72) holds at R', and since

$$\log M(R') \leq a(R') \log R' + C$$

for some constant C, we have

$$a(R') \leq (\log R' + C)^4. \tag{74}$$

Here we have used the fact that $a(R') \geq a(r') \geq 1$.

From Theorem 11 with $p = 7$ and $\delta = 1/20$, we conclude that

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} + \varepsilon_2(r)\right\} \tag{75}$$

for all r in $[1, R']$ outside a set of logarithmic measure

$$O(a(R')^{1/6}) = O((\log R')^{2/3}), \tag{76}$$

where $\varepsilon_2(r) = o(a(r)^{-17/10})$. Choose $r'' \geq r'$ such that

$$|\varepsilon_2(r)| < \varepsilon a(r)^{-17/10} \tag{77}$$

for all $r \geq r''$. Since Lemma 13 follows at once if R' remains bounded as $R \to \infty$, we may assume that $R' \geq r''$. Now consider the set of r in $[r'', R']$ at which

$$a(r)^{-17/10} \geq \frac{1}{\log M(r)}. \tag{78}$$

As before, there are two possibilities.

(iii) The set of such r is empty. In this case, from (75), (76), and (77), (73) holds for all $r \in [1, R']$ outside a set of logarithmic measure $O((\log R')^{2/3})$, and combining this with (ii) we conclude that (73) holds for all $r \in [1, R]$ outside a set of logarithmic measure $O((\log R)^{2/3})$. The procedure terminates.

(iv) Otherwise, the set of r in $[r'', R']$ at which (78) holds is not empty. Let R'' be an element of the set large enough that the logarithmic measure of the part of the set in $[R'', R']$ is less than 1. Then, from (75), (76), and (77), (73) holds for all $r \in (R'', R']$ outside a set of logarithmic measure $O((\log R')^{2/3})$, and combining this with (ii) we have (73) for all $r \in (R'', R]$ outside a set of logarithmic measure $O((\log R)^{2/3})$. Further, (78) holds at R'' and thus, arguing as before,

$$a(R'') \leq (\log R'' + C)^{10/7}. \tag{79}$$

From Theorem 11 with $p = 18/7$ and $\delta = 1/11$, we have

$$B(r) > M(r)\left\{1 - \frac{(1 + o(1))\pi^2 b(r)}{2a(r)^2} + \varepsilon_3(r)\right\} \tag{80}$$

for all r in $[1, R'')$ outside a set of logarithmic measure

$$O(a(R'')^{7/11}) = O((\log R'')^{10/11}), \tag{81}$$

where $\varepsilon_3(r) = o(a(r)^{-26/11})$. Choose $r''' \geq r''$ such that

$$|\varepsilon_3(r)| < \varepsilon a(r)^{-26/11} \tag{82}$$

for all $r > r'''$; again, we may assume that $R'' \geq r'''$. Consider the set of r in $[r''', R'']$ at which

$$a(r)^{-26/11} \geq \frac{1}{\log M(r)}. \tag{83}$$

As before, there are two possibilities.

(v) The set of such r is empty. In this case, from (80), (81), and (82), (73) holds for all $r \in [1, R'']$ outside a set of logarithmic measure $O((\log R'')^{10/11})$, and combining this with (iv) we conclude that (73) holds for all $r \in [1, R]$ outside a set of logarithmic measure $O((\log R)^{10/11})$. The procedure terminates.

(vi) Otherwise, the set of r in $[r''', R'']$ at which (83) holds is not empty. Let R''' be an element of the set large enough that the logarithmic measure of the part of the set in $[R''', R'']$ is less than 1. We have (73) for all $r \in (R''', R'']$ outside a set of logarithmic measure $O((\log R'')^{10/11})$, and combining this with (iv) we obtain (73) for all $r \in (R''', R]$ outside a set of logarithmic measure $O((\log R)^{10/11})$. Further, (83) holds at R''' and thus

$$a(R''') < (\log R''' + C)^{11/15}. \tag{84}$$

It is now possible to use Theorem 12 to halt what appears to be an interminable progression. For if R''' is large enough then, from (56),

$$N(R''') < (\log R''')^{4/5},$$

and thus from Theorem 12 with $p = 9/5$,

$$\begin{aligned} B(r) &> M(r)\left(1 - \frac{\varepsilon}{\log M(r)}\right) \\ &> M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} - \frac{\varepsilon}{\log M(r)}\right\} \end{aligned}$$

for all $r \in [1, R''']$ outside a set of logarithmic measure $o(\log R''')$. Hence (73) holds for all $r \in [1, R''']$ outside a set of logarithmic measure $o(\log R''')$. Combining this with (iv) we obtain (73) for all $r \in [1, R]$ outside a set of logarithmic measure $o(\log R)$.

Whatever the case, then, we have (73) for all $r \in [1, R]$ outside a set of logarithmic measure $o(\log R)$, and Lemma 13 is proved. □

10. Completing the Proof of Theorem 3

It remains to estimate $b(r)/a(r)^2$ in (69) and (70), which is done by means of the following growth lemma.

THEOREM 14 [7]. *Let $\Phi(r)$ be a positive, increasing, and convex function of r for $r \geq r_0$, and write*

$$p = \varliminf_{R\to\infty} \frac{\log \Phi(R)}{\log R} \quad \textit{and} \quad P = \varlimsup_{R\to\infty} \frac{\log \Phi(R)}{\log R} \tag{85}$$

so that $1 \leq p \leq P \leq \infty$. Suppose that $K > 0$ and that E is the set of r at which

$$\frac{\Phi(r)\Phi''(r)}{\Phi'(r)^2} \geq K. \tag{86}$$

Then:

(1) *if $K > 1$, $\overline{\text{dens}}\, E \leq 1/K$;*
(2) *if $p < \infty$ and $K > \sigma(p) = 1 - 1/p$, we have $\underline{\text{logdens}}\, E \leq \sigma(p)/K$;*
(3) *if $P < \infty$ and $K > \sigma(P)$, we have $\overline{\text{logdens}}\, E \leq \sigma(P)/K$.*

To prove Theorem 3, consider first the case $2 \leq P < \infty$. From Lemma 13 it follows that, given $R > 1$ and $\varepsilon_1 > 0$,

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2 b(r)}{2a(r)^2} - \frac{\varepsilon_1}{\log M(r)}\right\} \tag{87}$$

for all r outside a set of upper logarithmic density zero, and hence of upper log log density zero, from (3). Set $r = e^x$ and $\Phi(x) = \log M(e^x)$, so that

$$\Phi'(x) = a(e^x) \quad \text{and} \quad \Phi''(x) = b(e^x).$$

From part (3) of Theorem 14, given $\varepsilon > 0$ we have

$$\frac{\Phi(x)\Phi''(x)}{\Phi'(x)^2} = \frac{\log M(r) b(r)}{a(r)^2} \le \sigma(P) + \varepsilon \tag{88}$$

for all x outside a set E such that

$$\overline{\text{logdens}}\, E \le \frac{\sigma(P)}{\sigma(P)+\varepsilon}.$$

If $E' = \{r : \log r \in E\}$ then

$$\overline{\text{loglogdens}}\, E' = \overline{\text{logdens}}\, E \le \frac{\sigma(P)}{\sigma(P)+\varepsilon};$$

combining this with (87), we obtain

$$B(r) > M(r)\left\{1 - \frac{(1+o(1))\pi^2(\sigma(P)+\varepsilon)}{2\log M(r)} - \frac{\varepsilon_1}{\log M(r)}\right\} \tag{89}$$

for all r outside a set of upper log log density at most $\sigma(P)/(\sigma(P)+\varepsilon)$. Theorem 3 follows in this case since ε_1 is arbitrary.

Suppose next that $P = \infty$. The proof follows that for the case $2 \le P < \infty$, except that since $\sigma(P) + \varepsilon > 1$ we may use part (1) of Theorem 14. Then $\overline{\text{logdens}}\, E' = \overline{\text{dens}}\, E \le \sigma(P)/(\sigma(P)+\varepsilon)$ and the result follows.

Finally, if $P < 2$, we follow Hayman. Choose q so that $2 > q > P$. For all large r,

$$\log \mu(r) < \tfrac{1}{4}(\log r)^q,$$

so

$$N(r)\log r \le \int_r^{r^2} N(t)\frac{dt}{t} \le \log \mu(r^2) < \tfrac{1}{4}(\log r^2)^q < (\log r)^q$$

and therefore

$$N(r) \le (\log r)^{q-1}$$

for all large r. Part (iii) is then a consequence of Theorem 12. This proves the part of Theorem 3 concerning P, and the part concerning p is proved similarly. □

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