Seifert Manifolds with $\Gamma\backslash G/K$-Fiber

KYUNG BAI LEE & FRANK RAYMOND

Dedicated to Jehpill Kim of Seoul National University
teacher, colleague and friend

Introduction

Let G be a connected Lie group, and K a closed subgroup. Let W be a nice topological space. The Lie group G acts on the space $G \times W$ by $g(x, w) = (gx, w)$. Let $\text{TOP}_G(G \times W)$ be the group of self-homeomorphisms of $G \times W$ that are weakly G-equivariant (see below for an exact description). Consider the product space $G/K \times W$ of the space of left cosets $\{xK\}$ with W, and let $\text{TOP}_{G,K}(G/K \times W)$ be the group of self-homeomorphisms of $G/K \times W$ induced from weakly G-equivariant self-homeomorphisms of $G \times W$.

The aim of this paper is to study Seifert fiber spaces modeled on

$$(G/K \times W,\ \text{TOP}_{G,K}(G/K \times W)).$$

Such a space will have a double coset space $\Gamma\backslash G/K$ as a typical fiber. We shall pay special attention to the case where G is a semisimple Lie group in its adjoint form, and K is a maximal compact subgroup.

One of the important geometric problems that has motivated the development of Seifert fiberings has been the construction of closed aspherical manifolds realizing Poincaré duality groups Π of the form $1 \to \Gamma \to \Pi \to Q \to 1$, where Γ is a cocompact torsion-free lattice in a noncompact Lie group. In [CR2], [LR1], [KLR], and [LR2], closed aspherical manifolds were found for commutative, nilpotent, and solvable G provided Q could be made to act properly on some contractible manifold with compact quotient. In these cases, the Seifert construction enables one also to deduce interesting and relevant geometric information. The reason for this is that the Seifert construction, *which is a special embedding* (i.e., an injective homomorphism) *of the group* Π *into* $\text{TOP}_G(G \times W)$ *such that* Π *acts properly on* $G \times W$, preserves some of the properties of both G and W on $\Pi\backslash(G \times W)$. Furthermore, the action of Π on $G \times W$ "twists" the topology and geometry of G and W to create the orbit space $\Pi\backslash(G \times W)$ in the same way that the group structures of Γ and Q "twist" to create the group Π. In other words, this algebraic twisting of Π makes the geometric twisting of the "bundle with singularities"

$$\Gamma\backslash G \to \Pi\backslash(G \times W) \to Q\backslash W,$$

where the homogeneous space $\Gamma\backslash G$ is a typical fiber.

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In the references cited, G is a simply connected Lie group diffeomorphic to a Euclidean space. Hence it seems advantageous to enlarge the concept of Seifert constructions to include Seifert fiber spaces modeled on $G/K \times W$ as well, where G is an arbitrary Lie group and K is a closed subgroup. Then a Seifert construction would yield a "bundle with singularities", $\Gamma\backslash G/K \to \Pi\backslash G/K \times W \to Q\backslash W$, where the double coset space is a typical fiber. In particular, if G is an arbitrary noncompact Lie group with a finite number of connected components and K is a maximal compact subgroup of G, then G/K is diffeomorphic to a Euclidean space. An earlier paper of Raymond and Wigner [RW] dealt with the construction of closed aspherical manifolds $M = K(\Pi, 1)$, where Γ was a lattice in a noncompact semisimple Lie group G in adjoint form. The method, which was rather ad hoc, yielded the desired manifolds. The groups Γ and G had to be replaced by the isomorphic groups $\mathrm{Inn}(\Gamma)$ and $\mathrm{Inn}(G)$ respectively. However, it has not been clear just how that construction fitted into the general theory of Seifert constructions. Also, the geometry of the spaces obtained could not be easily explained in terms of geometries of G and W. In this paper, we explain how one goes about creating a general theory for Seifert constructions on $G/K \times W$, where G is a connected Lie group and K is a closed subgroup. As a consequence, the main result of [RW] is recaptured in Theorem 4.6. An advantage of the present approach over the earlier one is Corollary 4.7, where it is easily shown that the constructed manifold $M(\Pi)$ inherits the product geometry from G/K and W.

Section 1 describes the general set-up culminating in a complete description of the universal uniformizing group $\mathrm{TOP}_{G,K}(G/K \times W)$ into which discrete groups Π need to be mapped to create Seifert fiberings modeled on $G/K \times W$. Section 2 calculates $\mathrm{TOP}_{G,K}(G/K \times W)$ when $N_G(K) = K$ and $\mathrm{Aut}^0(G, K) = 1$. The main result is Theorem 2.2, which is crucial for Sections 4 and 5.

Section 3 is a technical section that will allow us to assert uniqueness of our construction when G is specialized as in Sections 4 and 5. It is, however, phrased in the general context of classifying all the homomorphisms of one short exact sequence into another short exact sequence.

Section 4 then solves the embedding and uniqueness problem for $1 \to \Gamma \to \Pi \to Q \to 1$ into $\mathrm{TOP}_{G,K}(G/K \times W)$ when (G, K) is a Riemannian symmetric pair of noncompact type.

Section 5 is concerned with solvable G. If G is solvable, not all the nice features of the theory of Seifert constructions that work so well in the nilpotent or semisimple cases remain valid. With the aid of the earlier sections, we show that under certain conditions the usual embedding and uniqueness theorems are valid if we enlarge G to G' by extending it by a compact abelian group K. Then, using the pair (G', K) and the technique of the earlier sections, we obtain the desired solution. This explains the geometry of the infra-solvmanifolds constructed by Auslander and Johnson [AJ] as Seifert manifolds.

Applications and specific examples illustrating how the theory works in practice are given.

1. General

Let W be a completely regular space admitting covering space theory. Therefore, W is locally path-connected, semilocally-1-connected, and path-connected.

We denote by $\mathrm{M}(W, G)$ the group of all continuous maps from W into G with multiplication

$$(\lambda \cdot \nu)(w) = \lambda(w) \cdot \nu(w).$$

$\mathrm{TOP}(W)$ denotes the group of all homeomorphisms of W, and $\mathrm{Aut}(G)$ denotes the Lie group of all continuous automorphisms of G. Then $\mathrm{Aut}(G) \times \mathrm{TOP}(W)$ acts on $\mathrm{M}(W, G)$ via

$$^{(\alpha,h)}\lambda = \alpha \circ \lambda \circ h^{-1} : W \xrightarrow{h^{-1}} W \xrightarrow{\lambda} G \xrightarrow{\alpha} G$$

for $(\alpha, h) \in \mathrm{Aut}(G) \times \mathrm{TOP}(W)$ and $\lambda \in \mathrm{M}(W, G)$. The group $\mathrm{TOP}_G(G \times W)$ of *weakly G-equivariant self-homeomorphisms* of $G \times W$ is defined as follows: A homeomorphism f of $G \times W$ onto itself belongs to $\mathrm{TOP}_G(G \times W)$ if and only if there exists a continuous automorphism α_f of G such that

$$f(a \cdot x, w) = \alpha_f(a) f(x, w)$$

for all $a \in G$ and $(x, w) \in G \times W$.

LEMMA 1.1 [LR1]. $\mathrm{TOP}_G(G \times W) = \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W))$.

The group law is

$$(\lambda_1, \alpha_1, h_1) \cdot (\lambda_2, \alpha_2, h_2) = (\lambda_1 \cdot \alpha_1 \circ \lambda_2 \circ h_1^{-1}, \alpha_1 \circ \alpha_2, h_1 \circ h_2),$$

and the action of $\mathrm{TOP}_G(G \times W)$ on $G \times W$ is given by

$$(\lambda, \alpha, h) \cdot (x, w) = (\alpha(x) \cdot \lambda(hw)^{-1}, hw).$$

Then $\mathrm{M}(W, G) \rtimes \mathrm{Aut}(G)$ is the group of all weakly G-equivariant self-homeomorphisms of $G \times W$ that move only along the fibers.

For $a \in G$, the constant map $W \to G$ sending W to a is denoted by $r(a)$. Clearly,

$$r(a) = (a, 1, 1) \in \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W)).$$

This is a right translation by a^{-1} on the first factor of $G \times W$ so that $r(a)(x, w) = (x \cdot a^{-1}, w)$, and the subgroup of all such right translations is denoted by $r(G) \subset \mathrm{M}(W, G)$. Let $l(G)$ denote the group of left translations on the first factor so that $l(a)(x, w) = (a \cdot x, w)$. Then elements of $l(G)$ are of the form

$$l(a) = (a^{-1}, \mu(a), 1) \in \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W));$$

$\mu(a) \in \mathrm{Inn}(G)$ is conjugation by a. Note that $l(G)$ is normal in $\mathrm{TOP}_G(G \times W)$. In fact, $\mathrm{TOP}_G(G \times W)$ is the largest subgroup of $\mathrm{TOP}(G \times W)$ in which $l(G)$ is normal.

NOTATION.

$$\begin{aligned}
N_G(K) &= \text{normalizer of } K \text{ in } G,\ K \text{ a closed subgroup of } G,\\
C_G(K) &= \text{centralizer of } K \text{ in } G,\\
\mathrm{Aut}(G,K) &= \{\alpha \in \mathrm{Aut}(G) : \alpha|_K \in \mathrm{Aut}(K)\},\\
\mathrm{Inn}(G,K) &= \mathrm{Inn}(G) \cap \mathrm{Aut}(G,K),\\
\mathrm{Out}(G,K) &= \mathrm{Aut}(G,K)/\mathrm{Inn}(G,K),\\
\mathrm{Aut}^0(G,K) &= \{\alpha \in \mathrm{Aut}(G) : x^{-1}\alpha(x) \in K \text{ for all } x \in G\},\\
\mu(a) &= \text{conjugation by } a;\text{ so, } \mu(a)(x) = axa^{-1} \text{ for } x \in G.
\end{aligned}$$

Let $\mathrm{TOP}_{G,K}(G \times W)$ be the subgroup of $\mathrm{TOP}_G(G \times W)$ consisting of elements that induce maps on $G/K \times W$ (i.e., mapping left K-cosets to left K-cosets); and $\mathrm{TOP}_{G,K}(G/K \times W)$ the image of $\mathrm{TOP}_{G,K}(G \times W)$ in $\mathrm{TOP}(G/K \times W)$. Therefore, $\mathrm{TOP}_{G,K}(G/K \times W)$ is the group of homeomorphisms on $G/K \times W$ that are induced from the weakly G-equivariant homeomorphisms on $G \times W$:

$$\begin{array}{ccccc}
\mathrm{TOP}_{G,K}(G \times W) & \subset & \mathrm{TOP}_G(G \times W) & \subset & \mathrm{TOP}(G \times W)\\
\downarrow & & & & \\
\mathrm{TOP}_{G,K}(G/K \times W) & \subset & \mathrm{TOP}(G/K \times W). & &
\end{array}$$

We need to study these groups, $\mathrm{TOP}_{G,K}(G \times W)$ and $\mathrm{TOP}_{G,K}(G/K \times W)$, in detail.

PROPOSITION 1.2. $\mathrm{TOP}_{G,K}(G \times W) = l(G)\cdot[\mathrm{M}(W, N_G(K)) \rtimes \mathrm{Aut}(G,K)] \rtimes \mathrm{TOP}(W)$.

Proof. We need to prove that $f \in \mathrm{TOP}_G(G \times W)$ belongs to $\mathrm{TOP}_{G,K}(G \times W)$ if and only if f is of the form $l(a)\cdot(\lambda, \alpha, h) = (\lambda a^{-1}, \mu(a)\alpha, h)$, where $a \in G$ and

(1) $\alpha \in \mathrm{Aut}(G,K)$, and
(2) $\lambda \in \mathrm{M}(W, N_G(K))$.

Suppose

$$f = (\lambda_1, \alpha_1, h) \in \mathrm{TOP}_{G,K}(G \times W).$$

Let $(\lambda_1, \alpha_1, h)(xK, w) = (x'K, h(w))$ for some $x' \in G$. Then

$$x'^{-1}\alpha_1(x)\alpha_1(K)\lambda_1(hw)^{-1} = K. \tag{1.2.1}$$

In particular, we must have

$$x'^{-1}\alpha_1(x)\lambda_1(hw)^{-1} = x'^{-1}\alpha_1(x)\alpha_1(1)\lambda_1(hw)^{-1} \in K. \tag{1.2.2}$$

The two equalities (1.2.1) and (1.2.2) yield

$$\alpha_1(K) = \lambda_1(hw)^{-1}K\lambda_1(hw).$$

Notice that the left-hand side is independent of w. Fix $w_0 \in W$, and let $a^{-1} = \lambda_1(hw_0)$, $\lambda = \lambda_1 a$, and $\alpha = \mu(a^{-1}) \circ \alpha_1$. Since $\alpha(K) = a^{-1}\alpha_1(K)a = K$, we have

$$\alpha \in \mathrm{Aut}(G, K).$$

Also $\lambda(hw)\cdot K\cdot\lambda(hw)^{-1} = \lambda(hw)\cdot\alpha(K)\cdot\lambda(hw)^{-1} = \lambda_1(hw)\cdot\alpha_1(K)\cdot\lambda_1(hw)^{-1} = K$ for all $w \in W$ shows that $\lambda(w) \in N_G(K)$ for all $w \in W$. Thus,

$$\lambda \in \mathrm{M}(W, N_G(K)).$$

Consequently, $f = (\lambda a^{-1}, \mu(a)\alpha, h) = l(a)\cdot(\lambda, \alpha, h)$, where $a \in G$, $\alpha \in \mathrm{Aut}(G, K)$, and $\lambda \in \mathrm{M}(W, N_G(K))$.

Conversely, let $\alpha \in \mathrm{Aut}(G, K)$ and $\lambda \in \mathrm{M}(W, N_G(K))$. Then it is easy to see that $(\lambda, \alpha, h)(xK, w) = (\alpha(x)\lambda(hw)^{-1}K, hw)$ so that (λ, α, h) maps K-cosets to K-cosets. It is clear that $l(G)$ preserves K-cosets also. This completes the proof. □

LEMMA 1.3 (Ineffective part of $\mathrm{TOP}_{G,K}(G \times W)$). *An element*

$$(\lambda a^{-1}, \mu(a)\alpha, h) \in \mathrm{TOP}_{G,K}(G \times W)$$

acts trivially on $G/K \times W$ *if and only if*

(1) $h = \mathrm{id}$ *on* W,
(2) $\lambda a^{-1} \in \mathrm{M}(W, K)$, *and*
(3) $x^{-1}\cdot(a\alpha(x)a^{-1}) \in K$ *for all* $x \in G$.

Consequently, the kernel of $\mathrm{TOP}_{G,K}(G \times W) \to \mathrm{TOP}_{G,K}(G/K \times W)$ *is exactly* $\mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K)$.

Proof. Suppose $(\lambda a^{-1}, \mu(a)\alpha, h) \in \mathrm{TOP}_{G,K}(G \times W)$ acts trivially on $G/K \times W$. Two points $(\lambda a^{-1}, \mu(a)\alpha, h)(x, w) = (a\alpha(x)\lambda(hw)^{-1}, hw)$ and (x, w) represent the same point in $G/K \times W$ if and only if $hw = w$ and $x^{-1}a\alpha(x)\lambda(w)^{-1} \in K$. These should hold for all $x \in G$ and all $w \in W$. For $x = 1$, the latter reduces to $a\lambda(w)^{-1} \in K$ so that $a\lambda^{-1} \in \mathrm{M}(W, K)$. Now, $x^{-1}a\alpha(x)a^{-1}(a\lambda(w)^{-1}) = x^{-1}a\alpha(x)\lambda(w)^{-1} \in K$ and $a\lambda(w)^{-1} \in K$ yield $x^{-1}(\mu(a)\alpha)(x) = x^{-1}a\alpha(x)a^{-1} \in K$ for all $x \in G$, so that $\mu(a)\alpha \in \mathrm{Aut}^0(G, K)$. Thus $(\lambda a^{-1}, \mu(a)\alpha, 1) \in \mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K)$. Conversely, for $(\lambda_1, \alpha_1) \in \mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K)$, we have $(\lambda_1, \alpha_1)(xK, w) = (xK, w)$. □

COROLLARY 1.4 (Ineffective part of $[l(G)\cdot\mathrm{M}(W, N_G(K))] \rtimes \mathrm{Inn}(G, K)$). *The part of the kernel of* $\mathrm{TOP}_{G,K}(G \times W) \to \mathrm{TOP}_{G,K}(G/K \times W)$ *in* $\mathrm{M}(W, G) \rtimes \mathrm{Inn}(G)$ *is exactly* $\mathrm{M}(W, K) \rtimes \mathrm{Inn}^0(G, K)$.

COROLLARY 1.5.

$$\mathrm{TOP}_{G,K}(G/K \times W) = \frac{l(G)\cdot[\mathrm{M}(W, N_G(K)) \rtimes \mathrm{Aut}(G, K)]}{\mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K)} \rtimes \mathrm{TOP}(W).$$

Also, since $[l(G)\cdot\mathrm{M}(W, N_G(K))] \rtimes \mathrm{Inn}(G, K) = l(G)\cdot\mathrm{M}(W, N_G(K))$, because $\mu(a) = l(a)r(a)$,

$$1 \to l(G)\cdot\mathrm{M}(W, N_G(K)) \to \mathrm{TOP}_{G,K}(G \times W) \to \mathrm{Out}(G, K) \times \mathrm{TOP}(W) \to 1$$

is exact. Hence we have the following.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathrm{M}(W,K)\rtimes \mathrm{Inn}^0(G,K) & \rightarrow & \mathrm{M}(W,K)\times \mathrm{Aut}^0(G,K) & \rightarrow & \mathrm{Out}^0(G,K) & \rightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & l(G)\cdot \mathrm{M}(W,N_G(K)) & \rightarrow & \mathrm{TOP}_{G,K}(G\times W) & \rightarrow & \mathrm{Out}(G,K)\times \mathrm{TOP}(W) & \rightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \frac{l(G)\cdot \mathrm{M}(W,N_G(K))}{\mathrm{M}(W,K)\rtimes \mathrm{Inn}^0(G,K)} & \rightarrow & \mathrm{TOP}_{G,K}(G/K\times W) & \rightarrow & \overline{\mathrm{Out}(G,K)}\times \mathrm{TOP}(W) & \rightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

COROLLARY 1.6. *There exists a commuting "9-diagram" (with exact rows and columns) as shown on page 442.*

The group $\mathrm{TOP}_{G,K}(G/K \times W)$ is the group of homeomorphisms on $G/K \times W$ that are induced from the weakly (left) G-equivariant homeomorphisms on $G \times W$. If we were to require the stronger condition that $\mathrm{TOP}_{G,K}(G/K \times W)$ consist of the homeomorphisms of $G/K \times W$ induced from weakly (left) $G \times K$-equivariant homeomorphisms (with K acting by $kx = xk^{-1}$), we would then need to replace $N_G(K)$ by $C_G(K)$ in the formulation of Corollaries 1.4, 1.5, and 1.6. In Section 3, where G is semisimple in adjoint form, we have $N_G(K) = K$ but $C_G(K)$ is trivial. On the other hand, in Section 4, where $G = S \rtimes K$ with suitable conditions, we have $N_G(K) = C_G(K) = K$ and so there will be no difference in this case.

PROPOSITION 1.7. *Suppose H is a closed subgroup of K, and is normal in G. Let $G/H = \bar{G}$ and $K/H = \bar{K}$. Then* $\mathrm{TOP}_{G,K}(G/K \times W) = \mathrm{TOP}_{\bar{G},\bar{K}}(\bar{G}/\bar{K} \times W)$.

Proof. Since H is normal in G, $[G, H] \subset H \subset K$. This implies that $\mu(H) \subset \mathrm{Aut}^0(G, K)$. Therefore,

$$l(H) = \{(h^{-1}, \mu(h)) : h \in H\} \subset \mathrm{M}(W, H) \rtimes \mu(H) \subset \mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K).$$

Also, $N_G(K)/H = N_{\bar{G}}(\bar{K})$. Finally, $\mathrm{Aut}(G, K)/(\mathrm{Aut}(H) \cap \mathrm{Aut}(G, K)) = \mathrm{Aut}(\bar{G}, \bar{K})$, so that

$$\frac{\mathrm{Aut}(G, K)}{\mathrm{Aut}^0(G, K)} = \frac{\mathrm{Aut}(G, K)/(\mathrm{Aut}(H) \cap \mathrm{Aut}(G, K))}{\mathrm{Aut}^0(G, K)/(\mathrm{Aut}(H) \cap \mathrm{Aut}^0(G, K))} = \frac{\mathrm{Aut}(\bar{G}, \bar{K})}{\mathrm{Aut}^0(\bar{G}, \bar{K})}.$$

Consequently, we get $\mathrm{TOP}_{G,K}(G/K \times W) = \mathrm{TOP}_{\bar{G},\bar{K}}(\bar{G}/\bar{K} \times W)$. □

In particular, suppose K itself is normal in G (i.e., $H = K$ in Proposition 1.7). With $\bar{G} = G/K$ (and $\bar{K} = K/K = 1$), we have

$$\mathrm{TOP}_{G,K}(G/K \times W) = \mathrm{M}(W, \bar{G}) \rtimes (\mathrm{Aut}(\bar{G}) \rtimes \mathrm{TOP}(W))$$

which is exactly the same as $\mathrm{TOP}_{\bar{G}}(\bar{G} \times W)$.

Let H be a subgroup of G, and let $\alpha \in \mathrm{Aut}(G)$. Even though $\alpha(H) \subset H$, $\alpha|_H : H \to H$ may not be an automorphism of H in general. For example, let $G = \mathrm{GL}(3, \mathbb{R})$ and let H be the subgroup generated by $x = I + e_{12}$, where I is the 3×3 identity matrix and e_{ij} is the matrix whose (i, j)-entry is 1, and 0 elsewhere. Then H is a closed subgroup of G. Consider $a = e_{11} + (1/p)e_{22} + e_{33} \in G$, where p is an integer greater than 1. Let α be the automorphism of G, which is conjugation by a. Then $\alpha(x) = x^p$. Even though α maps H into H itself, it does not induce an automorphism of H. Even though $aHa^{-1} \subset H$, $a \notin N_G(H)$. The following lemma provides a sufficient condition, which will be used in the subsequent sections.

LEMMA 1.8. *Let H be a closed subgroup of G with finitely many connected components, and let $\alpha \in \mathrm{Aut}(G)$ so that $\alpha(H) \subset H$. Then $\alpha(H) = H$.*

Proof. Let H_0 be the connected component of the identity of H. Then H_0 is closed in G. Since α is a global homeomorphism, $\alpha(H_0)$ is closed in G and, hence, closed in H_0. Because $\alpha(H_0)$ is a connected manifold having the same dimension as H_0 and is embedded in H_0 as a closed subset, invariance of domain implies that $\alpha(H_0) = H_0$.

Now, H_0 is normal in H, and α induces a homomorphism $\bar{\alpha}$ of H/H_0 into itself. It is enough to show that $\bar{\alpha}$ is onto; it is, because H/H_0 is a finite group. □

If we wish to find all Seifert fiberings over the space $Q\backslash W$ with typical fiber $\Gamma\backslash G/K$, we must do the following:

(A) Find a proper action of Q on W, that is, find a representation $\rho\colon Q \to \mathrm{TOP}(W)$ such that Q acts properly on W. This ensures that $Q\backslash W$ is Hausdorff and inherits some of the geometry of W.

(B) For each group extension

$$1 \to \Gamma \to \Pi \to Q \to 1,$$

find all homomorphisms $\theta\colon \Pi \to \mathrm{TOP}_{G,K}(G/K \times W)$ such that $\theta|_\Gamma\colon \Gamma \to l(G)$ restricts to an injective homomorphism onto a lattice of $G = l(G)$ and such that the diagram

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\varphi\times\rho} & & \\ 1 & \to & \frac{l(G)\cdot \mathrm{M}(W,N_G(K))}{\mathrm{M}(W,K)\rtimes \mathrm{Inn}^0(G,K)} & \to & \mathrm{TOP}_{G,K}(G/K \times W) & \to & \overline{\mathrm{Out}(G,K)} \times \mathrm{TOP}(W) & \to & 1 \end{array}$$

is commutative. Since Π acts on Γ by conjugation, the action of $\theta(\Pi)$ extends to an action on the kernel of $\mathrm{TOP}_{G,K}(G/K \times W) \to \overline{\mathrm{Out}(G,K)} \times \mathrm{TOP}(W)$. Induced will be the homomorphism $\varphi\colon Q \to \overline{\mathrm{Out}(G,K)}$. Because $\theta(\Gamma)$ is a lattice and Q acts properly on W, it can be seen that Π acts properly on $G/K \times W$ via $\theta(\Pi)$.

We call the homomorphism θ a *Seifert construction.* If θ is injective, we call θ an *embedding* (into $\mathrm{TOP}_{G,K}(G/K \times W)$). The space $X = \theta(\Pi)\backslash(G/K \times W)$ is a *Seifert fiber space* and the induced mapping $\theta(\Pi)\backslash(G/K \times W) \to Q\backslash W$ is called a *Seifert fibering* (or Seifert bundle) *with typical fiber* $\Gamma\backslash G/K$ *over the base* $Q\backslash W$.

We also say that the fibering is *modeled on* $G/K \times W$. If $G/K \times W$ is a manifold and $\theta(\Pi)$ acts freely, we call X a *Seifert manifold.* Sometimes we abuse the technical meaning of "orbifold" and call X an orbifold. The constructions are done *smoothly* if we replace $\mathrm{TOP}(W)$ by diffeomorphisms of W when W is a smooth manifold.

For a fixed θ_0, a conjugation of θ_0 by an element of $\mathrm{TOP}_{G,K}(G/K \times W)$ is called an *automorphism* of the Seifert construction θ_0. Running through all conjugacy classes (conjugating by elements of $\mathrm{TOP}_{G,K}(G/K \times W)$ and varying i and ρ) yields all the constructions modeled on $G/K \times W$ for the group Π. The group $\mathrm{TOP}_{G,K}(G/K \times W)$ is called the *universal uniformizing group.* A construction θ into a subgroup of $\mathrm{TOP}_{G,K}(G/K \times W)$ is called a *reduction* of the universal uniformizing group. Because $\mathrm{TOP}_{G,K}(G/K \times W)$ contains many geometrically interesting subgroups, a reduction to one of these groups will induce the interesting geometric structure on X. In practice, we shall begin with a fixed $\rho\colon Q \to \mathrm{TOP}(W)$ and a fixed $i\colon \Gamma \to l(G)$. Then we find all θs so that θ induces i and ρ.

If $K = \{1\} \subset G$, then $\mathrm{TOP}_{G,K}(G/K \times W)$ becomes $\mathrm{TOP}_G(G \times W) = \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W))$ (cf. [LL1] or [LL2]).

2. When $N_G(K) = K$ and $\mathrm{Aut}^0(G, K) = 1$

Most of the Lie groups treated in Sections 4 and 5 satisfy the two conditions of this section's title. Consequently, as we shall see, the kernel of $\mathrm{TOP}_{G,K}(G/K \times W) \to \mathrm{TOP}(W)$ becomes a Lie group. This has the tendency to make Seifert constructions, if they exist, more rigid. On the other hand, being able to work with kernels that are Lie groups (instead of much larger kernels) enables us to prove the strong existence and uniqueness theorems for the Seifert constructions investigated in Sections 4 and 5.

Throughout this section, G is a connected Lie group and K is a closed subgroup of G. The main result is Theorem 2.2. We begin with a sufficient condition for $\mathrm{Aut}^0(G, K) = 1$.

LEMMA 2.1. *Let G be a connected Lie group and K a closed subgroup of G. Suppose the largest normal subgroup of G contained in K is trivial. Then* $\mathrm{Aut}^0(G, K)$ *is trivial.*

Proof. Suppose $\alpha \in \mathrm{Aut}^0(G, K)$. By the definition of $\mathrm{Aut}^0(G, K)$, $x^{-1}\alpha(x) \in K$ for every $x \in G$. Let

$$K' = \text{ the subgroup of } K \text{ generated by } \{x^{-1}\alpha(x) : x \in G\}.$$

For any $b \in G$,

$$b^{-1} \cdot (x^{-1}\alpha(x)) \cdot b = (xb)^{-1}\alpha(xb) \cdot (b^{-1}\alpha(b))^{-1},$$

which is an element of K'. Thus K' is a subgroup of K that is normal in G. However, the largest normal subgroup of G contained in K is trivial. Therefore K' is trivial. This implies $\alpha = \mathrm{id}$. Consequently, $\mathrm{Aut}^0(G, K)$ is a trivial group. □

For simplicity, we fix some notation:

$$\mathrm{Aff}(G, K) = l(G) \rtimes \mathrm{Aut}(G, K).$$

Since $r(K)$ is normal in $\mathrm{Aff}(G, K)$, we define

$$\overline{\mathrm{Aff}}(G, K) = \frac{l(G) \rtimes \mathrm{Aut}(G, K)}{r(K)},$$

$$\overline{\mathrm{Aff}}_0(G, K) = \frac{l(G) \rtimes \mathrm{Inn}(G, K)}{r(K)}.$$

Then $\overline{\mathrm{Aff}}_0(G, K)$ is the connected component of $\overline{\mathrm{Aff}}(G, K)$ that contains the identity.

THEOREM 2.2. *Let G be a connected Lie group and K a closed subgroup of G. Suppose*

(1) $N_G(K) = K$, *and*
(2) $\mathrm{Aut}^0(G, K)$ *is trivial.*

Then, $\mathrm{TOP}_{G,K}(G/K \times W) = \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W)$.

Proof. From Corollary 1.5, we have

$$\begin{aligned}\mathrm{TOP}_{G,K}(G/K \times W) &= \frac{l(G) \cdot [\mathrm{M}(W, N_G(K)) \rtimes \mathrm{Aut}(G, K)]}{\mathrm{M}(W, K) \rtimes \mathrm{Aut}^0(G, K)} \rtimes \mathrm{TOP}(W) \\ &= \frac{l(G) \cdot [\mathrm{M}(W, K) \rtimes \mathrm{Aut}(G, K)]}{\mathrm{M}(W, K)} \rtimes \mathrm{TOP}(W)\end{aligned}$$

from the two conditions given in the statement. Now the factors $\mathrm{M}(W, K)$ drop out. However, notice that $[l(G) \rtimes \mathrm{Aut}(G, K)] \cap \mathrm{M}(W, K) = r(K)$. Therefore,

$$\mathrm{TOP}_{G,K}(G/K \times W) = \frac{l(G) \rtimes \mathrm{Aut}(G, K)}{r(K)} \times \mathrm{TOP}(W).$$

This is a direct product rather than a semidirect product, since $(1, 1, h) \in \mathrm{TOP}(W)$ commutes with $(a, \alpha, 1) \in \mathrm{Aff}(G, K)$ because a is a constant map. Of course, the group $l(G) \rtimes \mathrm{Aut}(G, K)$ acts on G/K by $(a, \alpha) \cdot xK = a\alpha(x)K$. □

PROPOSITION 2.3. *Let G be a connected Lie group and K a closed subgroup of G. Suppose that* $N_G(K) = K$ *and that every closed subgroup of G isomorphic to K is a conjugate of K. Then there exists an isomorphism* $\Psi\colon \overline{\mathrm{Aff}}(G, K) \to \mathrm{Aut}(G)$ *making the square*

$$\begin{array}{ccc} G & \xrightarrow{\mu} & \mathrm{Aut}(G) \\ \cong\big\downarrow l & & \cong\big\uparrow \Psi \\ l(G) & \longrightarrow & \overline{\mathrm{Aff}}(G, K) \end{array}$$

commutative.

Proof. Since G is normal in $G \rtimes \mathrm{Aut}(G, K)$, conjugation by elements $(a, \alpha) \in G \rtimes \mathrm{Aut}(G, K)$ on $(x, 1) \in G$,

$$(a, \alpha)(x, 1)(a, \alpha)^{-1} = (a \cdot \alpha(x) \cdot a^{-1}, 1),$$

yields a homomorphism $\Psi\colon G \rtimes \mathrm{Aut}(G, K) \to \mathrm{Aut}(G)$ given by

$$\Psi(a, \alpha) = \mu(a)\alpha.$$

For $k \in K$, $(k, \mu(k^{-1})) \in G \rtimes \mathrm{Aut}(G, K)$ and $\Psi(k, \mu(k^{-1})) = 1$. Conversely, suppose $\Psi(a, \alpha) = 1$. Then $\alpha = \mu(a^{-1})$ so that $\mu(a^{-1}) \in \mathrm{Aut}(G, K)$. Therefore, $a \in N_G(K)$. However, $N_G(K) = K$ so that $a \in K$. We have shown that the kernel of Ψ is exactly

$$K \cong \{\, (k, \mu(k^{-1})) : k \in K \,\}.$$

To show Ψ is surjective, let $\beta \in \mathrm{Aut}(G)$. Since $\beta(K)$ is isomorphic to K, there exists $a \in G$ for which $\beta(K) = aKa^{-1}$. Then $\mu(a)^{-1}\beta \in \mathrm{Aut}(G, K)$. This shows that $(a, \mu(a)^{-1}\beta) \in G \rtimes \mathrm{Aut}(G, K)$ maps to α by Ψ.

Since K is closed and Ψ is continuous and surjective, Ψ induces an isomorphism of groups

$$\frac{G \rtimes \mathrm{Aut}(G, K)}{K} \longrightarrow \mathrm{Aut}(G)$$

which is a diffeomorphism, so they are isomorphic as Lie groups.

The group $G \rtimes \mathrm{Aut}(G, K)$ acts on G by $(a, \alpha) \cdot x = a\alpha(x)$. In other words, $G \rtimes \mathrm{Aut}(G, K)$ is naturally identified with $l(G) \rtimes \mathrm{Aut}(G, K)$. Under this identification, the kernel of Ψ is exactly $r(K)$ since $(k, \mu(k^{-1}))x = k \cdot k^{-1}xk = xk$. □

COROLLARY 2.4. *With the same conditions as in Proposition 2.3, the following diagram is commutative:*

$$\begin{array}{ccccccccc}
1 & \rightarrow & \mathrm{Inn}(G) & \rightarrow & \mathrm{Aut}(G) & \rightarrow & \mathrm{Out}(G) & \rightarrow & 1 \\
 & & {\scriptstyle l}\downarrow\cong & & {\scriptstyle \Psi^{-1}}\downarrow\cong & & \downarrow= & & \\
1 & \rightarrow & \overline{\mathrm{Aff}}_0(G, K) & \rightarrow & \overline{\mathrm{Aff}}(G, K) & \rightarrow & \mathrm{Out}(G) & \rightarrow & 1.
\end{array}$$

3. Classification up to Conjugation

This section will be used later to show that the Seifert constructions with typical fiber a locally symmetric space of noncompact type are necessarily unique. The general question of classification of Seifert constructions reduces to the classification of mappings of one short exact sequence into another. We address this general problem now.

Consider a commutative diagram of group homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
1 & \rightarrow & \Delta & \rightarrow & \Pi & \rightarrow & Q & \rightarrow & 1 \\
 & & \downarrow i & & \downarrow \theta & & \downarrow \rho & & \\
1 & \rightarrow & \mathfrak{A} & \rightarrow & \mathcal{U} & \rightarrow & \mathcal{T} & \rightarrow & 1,
\end{array} \tag{3.0.1}$$

where i is injective. With the homomorphisms i and ρ fixed, how many θs are there, keeping the diagram commutative? In the following, for brevity we identify $a \in \Delta$ with $i(a) \in \mathfrak{A}$. We fix such a homomorphism θ_0 once and for all and use the action of Π on $\mathcal{U}$ obtained from $\Pi \xrightarrow{\theta_0} \mathcal{U} \xrightarrow{\mu} \mathrm{Aut}(\mathcal{U})$. Therefore, for $\alpha \in \Pi$,

$$^{\alpha}u = \theta_0(\alpha) \cdot u \cdot \theta_0(\alpha)^{-1}$$

for any $u \in \mathcal{U}$.

Let $\theta\colon \Pi \to \mathcal{U}$ be such a homomorphism fitting the diagram. This θ will be expressed via the fixed θ_0. Then θ must be of the form

$$\theta(\alpha) = \lambda(\alpha)\theta_0(\alpha)$$

for some map $\lambda\colon \Pi \to \mathfrak{A}$. It is easy to verify that λ satisfies

$$\begin{aligned}\lambda(\alpha\beta) &= \lambda(\alpha)\cdot\theta_0(\alpha)\lambda(\beta)\theta_0(\alpha)^{-1}\\ &= \lambda(\alpha)\cdot{}^{\alpha}\lambda(\beta).\end{aligned} \tag{3.0.2}$$

Since $\theta|_\Delta = i = \theta_0|_\Delta$, we have $\lambda(a) = 1$ for all $a \in \Delta$. For any $a \in \Delta$ and $\alpha \in \Pi$, $\lambda(\alpha a) = \lambda(\alpha)$. Moreover, $\lambda(\alpha) = \lambda(\alpha\cdot(\alpha^{-1}a\alpha)) = \lambda(a\alpha) = \lambda(a)\cdot\theta_0(a)\lambda(\alpha)\theta_0(a)^{-1} = 1\cdot i(a)\lambda(\alpha)i(a)^{-1} = i(a)\lambda(\alpha)i(a)^{-1}$. This shows that λ has values in the centralizer $C_{\mathfrak{A}}(\Delta)$. Furthermore, (3.0.2) shows that $\theta_0(\alpha)\lambda(\beta)\theta_0(\alpha)^{-1} \in C_{\mathfrak{A}}(\Delta)$. Therefore,

$$\begin{aligned}\theta_0(\alpha)\lambda(\beta)\theta_0(\alpha)^{-1} &= i(a)\theta_0(\alpha)\lambda(\beta)\theta_0(\alpha)^{-1}i(a)^{-1}\\ &= \theta_0(a\alpha)\lambda(\beta)\theta_0(a\alpha)^{-1}.\end{aligned} \tag{3.0.3}$$

By (3.0.3), the actions of Π induced by $\mu\circ\theta_0$ on $C_{\mathfrak{A}}(\Delta)$ factor through Q so that the following diagram is commutative:

$$\begin{array}{ccc}\Pi & \xrightarrow{\mu\circ\theta_0} & \mu(\theta_0(\Pi)) \quad \subset \mathrm{Aut}(\mathfrak{A})\\ {\scriptstyle\text{projection}}\downarrow & & \downarrow{\scriptstyle\text{restriction}}\\ Q & \longrightarrow & \mathrm{Aut}(C_{\mathfrak{A}}(\Delta)).\end{array}$$

With the homomorphism $Q \to \mathrm{Aut}(C_{\mathfrak{A}}(\Delta))$, we may consider $Z^1(Q;\, C_{\mathfrak{A}}(\Delta))$ and $H^1(Q;\, C_{\mathfrak{A}}(\Delta))$. The equality (3.0.2) shows that $\lambda \in Z^1(Q;\, C_{\mathfrak{A}}(\Delta))$. Therefore, $Z^1(Q;\, C_{\mathfrak{A}}(\Delta))$ represents the set of all homomorphisms θ fitting the commutative diagram (3.0.1). In fact, $\lambda \leftrightarrow \theta = \lambda\cdot\theta_0$ gives the one–one correspondence

$$\begin{aligned}Z^1(Q; C_{\mathfrak{A}}(\Delta)) &= \{\lambda\colon Q \to C_{\mathfrak{A}}(\Delta) \mid \lambda(\alpha\beta) = \lambda(\alpha)\cdot{}^{\alpha}\lambda(\beta)\}\\ &\overset{\cong}{\leftrightarrow} \{\theta\colon \Pi \to \mathcal{U} \text{ inducing } i \text{ and } \rho \text{ on } \Delta \text{ and on } Q\}.\end{aligned}$$

The group $C_{\mathfrak{A}}(\Delta)$ acts on $Z^1(Q;\, C_{\mathfrak{A}}(\Delta))$ as follows: Let $c \in C_{\mathfrak{A}}(\Delta)$ and $\lambda \in Z^1(Q;\, C_{\mathfrak{A}}(\Delta))$. Then ${}^c\lambda$ is given by

$$\begin{aligned}({}^c\lambda)(\alpha) &= c\cdot\lambda\cdot{}^{\alpha}c^{-1}\\ &= c\cdot\lambda\cdot\theta_0(\alpha)\cdot c^{-1}\cdot\theta_0(\alpha)^{-1}.\end{aligned}$$

It is easy to see that $({}^c\lambda)(\alpha) \in C_{\mathfrak{A}}(\Delta)$, and that ${}^c\lambda$ satisfies the cocycle condition. After the abelian case, we denote the orbit spaces of these actions by

$$H^1(Q;\, C_{\mathfrak{A}}(\Delta)) \equiv Z^1(Q;\, C_{\mathfrak{A}}(\Delta))/C_{\mathfrak{A}}(\Delta).$$

Proofs of the following two statements can be found in [LLR].

LEMMA 3.1. *$\theta_1 = \lambda_1\cdot\theta_0$ and $\theta_2 = \lambda_2\cdot\theta_0$ are conjugate by an element of $\mathfrak{A}$ if and only if λ_1 and λ_2 belong to the same orbit of the $C_{\mathfrak{A}}(\Delta)$-action.*

THEOREM 3.2. *The set of all homomorphisms θ fitting the commutative diagram* (3.0.1) *with the fixed i and ρ are classified, up to conjugation by elements of $\mathfrak{A}$, by $H^1(Q;\, C_{\mathfrak{A}}(\Delta))$.*

Theorem 3.2 will be used in Theorem 4.6 to prove uniqueness of the Seifert constructions modeled on $G/K \times W$, where G is semisimple and K is a maximal compact subgroup, and in Theorem 5.6 for a special type of solvable pair (G, K). These will be done by applying Theorem 3.2 to the diagram from Section 1:

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi\times\rho} & & \\ 1 & \to & \frac{l(G)\cdot \mathrm{M}(W, N_G(K))}{\mathrm{M}(W,K)\rtimes \mathrm{Inn}^0(G,K)} & \to & \mathrm{TOP}_{G,K}(G/K \times W) & \to & \overline{\mathrm{Out}(G,K)} \times \mathrm{TOP}(W) & \to & 1. \end{array}$$

Conjugation of θ_0 by any element $u \in \mathrm{TOP}_{G,K}(G/K \times W)$ gives rise to a new homomorphism $\theta\colon \Pi \to \mathrm{TOP}_{G,K}(G/K \times W)$. This induces a homeomorphism of $\theta_0(\Pi)\backslash(G \times W)$ to $\theta(\Pi)\backslash(G \times W)$ that respects the Seifert structure. The conjugation may change ρ or the embedding of Γ into $l(G)$. This is called a *Seifert automorphism*. If conjugation by u induces the identity on $l(G)$ and $\rho(Q)$, we say u induces a *strict* Seifert automorphism between Seifert fiber spaces $\theta_0(\Pi)\backslash(G\times W)$ and $\theta(\Pi)\backslash(G\times W)$. If $u \in [l(G)\cdot\mathrm{M}(W, N_G(K))]/[\mathrm{M}(W, K) \rtimes \mathrm{Inn}^0(G, K)]$, then the Seifert automorphism is called a strict Seifert automorphism *moving only along the fibers*. Therefore, $H^1(Q; C_{\mathfrak{A}}(\Gamma))$, where $\mathfrak{A} = [l(G)\cdot\mathrm{M}(W, N_G(K))]/[\mathrm{M}(W, K) \rtimes \mathrm{Inn}^0(G, K)]$, classifies all strict Seifert automorphism moving only along the fibers. For more details, readers are referred to [LLR].

4. Symmetric Spaces of Noncompact Type

A *symmetric space* is a triple (G, K, σ) consisting of a connected Lie group G, a closed subgroup K of G, and an involutive automorphism σ of G such that $(G^\sigma)_0 \subset K \subset G^\sigma$, where G^σ is the fixed-point set of σ. (G, K, σ) is (almost) *effective* if the largest normal subgroup N of G contained in K is trivial (discrete). If (G, K, σ) is a symmetric space then $(G/N, K/N, \sigma^*)$ is an effective symmetric space, where σ^* is the automorphism of G/N induced from σ. If, in addition, the group $\mathrm{Ad}_G(K)$ is compact then (G, K, σ) is said to be a *Riemannian symmetric space.*

Throughout this section, (G, K, σ) will be an effective Riemannian symmetric space of noncompact type. Therefore G is a connected, semisimple Lie group in its adjoint form with no compact normal factors, and K is a closed maximal compact subgroup of G. We collect some facts for such groups.

LEMMA 4.1. *Let G be a connected, centerless, semisimple Lie group without any normal compact factors. Let K be a maximal compact subgroup of G. Then*:

(1) *$N_G(K) = K$, and K is connected*;
(2) *every closed subgroup of G isomorphic to K is a conjugate of K*; *and*
(3) $\mathrm{Out}(G)$ *is finite.*

For (1), see [Hel, p. 275, A3(i)]. For (2), see [Hel; VI Thm. 2.1]. (3) is well known. Now Theorem 2.2 characterizes our universal uniformizing group completely.

COROLLARY 4.2. *Let G be a connected, centerless, semisimple Lie group without any normal compact factors. Let K be a maximal compact subgroup of G. Then*

$$\mathrm{TOP}_{G,K}(G/K \times W) = \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W).$$

LEMMA 4.3. *Let $\bar{G} = \mathrm{Aut}(G)$ and let $\bar{K}$ be its maximal compact subgroup containing $\mu(K) = \mathrm{Ad}_G(K)$. Then $\bar{K} = \mathrm{Aut}(G, K)$ and $\bar{K}/K \cong \mathrm{Out}(G)$.*

Proof. Since $\mathrm{Out}(G)$ is finite, $\bar{K}/K$ is discrete. Hence $\bar{K}_0 = \mu(K) \subset \bar{G}$ is the connected component of $\bar{K}$ containing the identity element. Thus, $\mu(K)$ is normal in $\bar{K}$. Let $\alpha \in \bar{K}$ be any element. We claim that $\alpha(K) = K$. Pick any $k \in K$. We shall show $\alpha(k) \in K$. Since $\mu(K)$ is normal in $\bar{K}$, $\alpha \cdot \mu(k) \cdot \alpha^{-1} = \mu(k')$ for some $k' \in K$. Note that these are equal as elements of $\bar{K}$ (and hence as elements of $\bar{G} = \mathrm{Aut}(G)$). Since $\alpha \cdot \mu(k) \cdot \alpha^{-1} = \mu(\alpha(k))$, we have

$$\mu(k') = \mu(\alpha(k)).$$

That is, conjugations by k' and $\alpha(k)$ produce the same automorphisms of G. Consequently, $\alpha(k) \cdot k'^{-1} \in \mathcal{Z}(G)$, the center of G, which is trivial. We have shown that $\alpha(k) \cdot k'^{-1} = 1$, so $\alpha(k) = k' \in K$. Hence $\alpha(K) \subset K$. Since K is compact, one can apply Lemma 1.8 to get $\alpha(K) \subset K$ implies that α induces an automorphism of K so that $\alpha \in \mathrm{Aut}(G, K)$. Thus $\bar{K} \subset \mathrm{Aut}(G, K)$.

Since $N_G(K) = K$, we have $\mathrm{Inn}(G, K) = \mathrm{Ad}_G(K) \cong K$. Thus, $\mathrm{Aut}(G, K)/\mathrm{Inn}(G, K) \subset \mathrm{Out}(G)$, a finite group. However, all maximal compact subgroups of G are conjugate to each other. Therefore, for every $\beta \in \mathrm{Aut}(G)$ there exists $a \in G$ for which $\beta(K) = aKa^{-1}$. Then $\mu(a)^{-1}\beta \in \mathrm{Aut}(G, K)$. This implies that $\mathrm{Aut}(G, K)/\mathrm{Inn}(G, K) \to \mathrm{Out}(G)$ is surjective. Thus we have a short exact sequence

$$1 \to K \to \mathrm{Aut}(G, K) \to \mathrm{Out}(G) \to 1.$$

Since K and $\mathrm{Out}(G)$ are compact, so is $\mathrm{Aut}(G, K)$. By maximality of $\bar{K}$, we have $\bar{K} = \mathrm{Aut}(G, K)$. □

REMARK 4.4. The action of $\mathrm{Aut}(G)$ on G/K can also be interpreted as follows: By Lemma 4.3, the conjugation map induces an identification

$$\bar{\mu}: G/K \xrightarrow{\approx} \bar{G}/\bar{K}.$$

In fact, $\bar{\mu}$ is bijective because $N_G(K) = K$ and $\bar{K} = \mathrm{Aut}(G, K)$. We claim that: *With respect to the action of* $\mathrm{Aut}(G)$ *on G/K via Ψ^{-1} and the action of* $\mathrm{Aut}(G) = \bar{G}$ *on $\bar{G}/\bar{K}$ as left multiplications, the diffeomorphism $\bar{\mu}$ is* $\mathrm{Aut}(G)$*-equivariant.* In other words, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Aut}(G) \times G/K & \longrightarrow & G/K \\ {\scriptstyle \mathrm{id} \times \bar{\mu}}\downarrow & & {\scriptstyle \bar{\mu}}\downarrow{\scriptstyle \approx} \\ \mathrm{Aut}(G) \times \bar{G}/\bar{K} & \longrightarrow & \bar{G}/\bar{K}. \end{array}$$

Let $\beta \in \mathrm{Aut}(G)$, and let $(a,\alpha) \in l(G) \rtimes \mathrm{Aut}(G,K)$ be a representative of $\Psi^{-1}(\beta)$. Then $\alpha = \mu(a^{-1})\beta \in \mathrm{Aut}(G,K)$ and

$$\begin{aligned}\beta \cdot xK &= (a,\alpha)xK \\ &= a\alpha(x)K.\end{aligned}$$

On the other hand,

$$\beta \cdot \mu(x)\bar{K} = \beta\mu(x)\bar{K}.$$

To see that $a\alpha(x)K \in G/K$ corresponds to $\beta\mu(x)\bar{K} \in \bar{G}/\bar{K}$ via $\bar{\mu}$, it is enough to have

$$\mu(a\alpha(x))^{-1}\beta\mu(x) \in \bar{K}.$$

However, a calculation shows that $\mu(a\alpha(x))^{-1}\beta\mu(x) = \alpha \in \mathrm{Aut}(G,K) = \bar{K}$. Pictorially,

$$\begin{array}{ccc}(\beta, xK) & \longrightarrow & a\alpha(x)K \\ {\scriptstyle \mathrm{id}\times\bar{\mu}}\big\downarrow & & {\scriptstyle \bar{\mu}}\big\downarrow{\scriptstyle\approx} \\ (\beta, \mu(x)\bar{K}) & \longrightarrow & \beta\mu(x)\bar{K}.\end{array}$$

This verifies the commutativity of the above square.

PROPOSITION 4.5. *With the G-invariant Riemannian metric on G/K induced by the Killing–Cartan form of* $\mathfrak{g}$, $\overline{\mathrm{Aff}}(G,K) = \mathrm{Isom}(G/K)$.

Proof. Recall the Killing–Cartan form is defined by

$$B(X,Y) = \mathrm{Trace}(\mathrm{ad}\,X \cdot \mathrm{ad}\,Y).$$

Let $\alpha \in \mathrm{Aut}(G,K)$ and $\alpha_* = d\alpha \in \mathrm{Aut}(\mathfrak{g})$. Then $\mathrm{ad}(\alpha_* X) = \alpha_* \circ \mathrm{ad}\,X \circ \alpha_*^{-1}$. Applying $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ twice, we get

$$B(\alpha_* X, \alpha_* Y) = B(X,Y).$$

Therefore, α_* leaves the quadratic form B on $\mathfrak{g}$ invariant. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the orthogonal decomposition, where $\mathfrak{k}$ is the Lie algebra of K. Since α_* maps $\mathfrak{k}$ onto itself and B is invariant under α_*, α_* leaves the orthogonal complement $\mathfrak{p}$ invariant. Thus, α_* maps $\mathfrak{p}$ to itself and preserves the quadratic form B on $\mathfrak{p}$. The metric on G/K is just a scalar multiple of the restriction of the Killing–Cartan form B on $\mathfrak{p}$. We have shown that α_* is an isometry on G/K. Since $l(G) \subset \mathrm{Isom}(G/K)$, clearly we have

$$l(G) \rtimes \mathrm{Aut}(G,K) \to \mathrm{Isom}(G/K).$$

We show this homomorphism to be surjective. It is well known that $l(G) = \mathrm{Isom}_0(G/K)$. Suppose $\bar{f} \in \mathrm{Isom}(G/K)$. By transitivity of the action of $l(G)$ on G/K, we may assume that $\bar{f}$ fixes the point $K = eK \in G/K$. Since $l(G)$ is normal in $\mathrm{Isom}(G/K)$, conjugation by $\bar{f}$ defines an automorphism of $l(G)$. Let $f = l^{-1} \circ \mu(\bar{f}) \circ l \in \mathrm{Aut}(G)$. Then

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ {\scriptstyle l}\downarrow & & {\scriptstyle l}\downarrow \\ l(G) & \xrightarrow{\mu(\bar{f})} & l(G) \end{array}$$

commutes and

$$l(f(a)) = \bar{f} \circ l(a) \circ \bar{f}^{-1}.$$

With $\bar{f}(eK) = eK$, an easy calculation shows that

$$\bar{f}(aK) = \bar{f} \circ l(a)(eK) = l(f(a)) \circ \bar{f}(eK) = l(f(a))(eK) = f(a)K$$

for all $a \in G$. Thus, for $k \in K$, $f(k)K = \bar{f}(kK) = \bar{f}(K) = K$ so that $f(k) \in K$. We have shown that the automorphism f of G maps K to itself. This proves the surjectivity of $l(G) \rtimes \operatorname{Aut}(G, K) \to \operatorname{Isom}(G/K)$. Since the kernel of this homomorphism is $r(K)$, we have completed the proof. □

THEOREM 4.6 (cf. [RW, Thm. 2]). *Let G be a connected, centerless, semisimple Lie group without any normal compact factors or 3-dimensional factors. Let K be a maximal compact subgroup of G. Let Γ be a lattice of G, let $\rho\colon Q \to \mathrm{TOP}(W)$ be a proper action of a discrete group Q (i.e., a properly discontinuous action), and let $1 \to \Gamma \to \Pi \to Q \to 1$ be an exact sequence. Then there exists a homomorphism $\theta\colon \Pi \to \mathrm{TOP}_{G,K}(G/K \times W) = \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W)$ so that the diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle \theta}\downarrow & & {\scriptstyle \varphi\times\rho}\downarrow & & \\ 1 & \to & \overline{\mathrm{Aff}}_0(G, K) & \to & \mathrm{TOP}_{G,K}(G/K \times W) & \to & \mathrm{Out}(G) \times \mathrm{TOP}(W) & \to & 1 \end{array}$$

is commutative, yielding a Seifert fiber space with typical fiber the double coset space $\Gamma\backslash G/K$. Such a homomorphism θ with fixed i and $\varphi \times \rho$ is unique. The action is free if and only if the preimage of each stabilizer Q_w in Π is torsion-free.

Proof. (Existence) Since Γ is normal in Π, there is a natural homomorphism $\mu\colon \Pi \to \operatorname{Aut}(\Gamma)$. Under the conditions on G stated, Mostow's rigidity theorem ensures that the pair (Γ, G) has the UAEP (unique automorphism extension property). The UAEP gives rise to a homomorphism $\operatorname{Aut}(\Gamma) \to \operatorname{Aut}(G)$. Consequently, we have a homomorphism $\Pi \to \operatorname{Aut}(G)$ so that the following diagram with exact rows is commutative:

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Inn}(G) & \to & \mathrm{Aut}(G) & \to & \mathrm{Out}(G) & \to & 1. \end{array}$$

Composing $\Pi \to \operatorname{Aut}(G)$ with $\Psi^{-1}\colon \operatorname{Aut}(G) \to \overline{\mathrm{Aff}}(G, K)$ in Corollary 2.4, we get a homomorphism $\Pi \to \operatorname{Aut}(G) \to \overline{\mathrm{Aff}}(G, K)$ under which Γ is mapped into

$\overline{\mathrm{Aff}}_0(G, K)$. This, together with the action $\Pi \to Q \to \mathrm{TOP}(W)$, gives rise to a homomorphism $\Pi \to \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W) = \mathrm{TOP}_{G,K}(G/K \times W)$.

(Uniqueness) We apply Theorem 3.2 to the commuting diagram in the statement of the theorem (with $\mathfrak{A} = \overline{\mathrm{Aff}}_0(G, K)$ and $\Delta = \Gamma$). To this end, we need to calculate the centralizer of Γ in $\overline{\mathrm{Aff}}_0(G, K) = (l(G) \rtimes \mathrm{Inn}(G, K))/r(K)$. An element $(a, \mu(b)) \in l(G) \rtimes \mathrm{Inn}(G, K)$ represents an element in the centralizer if and only if

$$[(a, \mu(b)), (x, 1)] = (abxb^{-1}a^{-1}x^{-1}, 1) = (\mu(ab)(x) \cdot x^{-1}, 1) \in r(K)$$

for every $x \in \Gamma$. Recall that elements of $r(K)$ in $l(G) \rtimes \mathrm{Aut}(G, K)$ are of the form $(k^{-1}, \mu(k))$ with $k \in K$. Therefore, it happens if and only if $\mu(ab)(x) \cdot x^{-1} = 1$ for every $x \in \Gamma$. By the UAEP, this should happen for every $x \in G$. Then $\mu(ab) \in \mathrm{Inn}^0(G, K)$. But $\mathrm{Inn}^0(G, K)$ is trivial by Lemma 2.1. Therefore, $\mu(ab) = 1$ so that $ab = 1$. Since $\mu(b) \in \mathrm{Inn}(G, K)$, $(a, \mu(b)) = (b^{-1}, \mu(b)) \in r(K)$, which represents the identity element of $\overline{\mathrm{Aff}}_0(G, K)$. We have shown $C_{\overline{\mathrm{Aff}}_0(G,K)}(\Gamma)$ is trivial so that $Z^1(Q; C_{\overline{\mathrm{Aff}}_0(G,K)}(\Gamma)) = 0$. (Note that we do not need conjugation for the uniqueness here.) Finally, since G/K is diffeomorphic to $\mathbb{R}^n$, the action of Π is free if and only if the preimage of each stabilizer Q_w in Π is torsion-free. □

In [LLR] we discussed a situation where G is semisimple and K is trivial. When G is in adjoint form, then $\mathrm{Aut}(G) = G \rtimes \mathrm{Out}(G)$, and the main Lemma 2.2 in [LLR] describes a necessary and sufficient condition for an extension Π to be mapped into $\mathrm{TOP}_G(G \times W)$. In the applications there, G was of compact type and the lattice Γ was assumed to go into $l(G) \times r(G)$. Section 5 of [LLR] discusses uniqueness for that situation.

When W is a Riemannian manifold, the space $G/K \times W$ acquires the natural product metric. Then,

$$\overline{\mathrm{Aff}}(G, K) \times \mathrm{Isom}(W) = \mathrm{Isom}(G/K) \times \mathrm{Isom}(W) \subset \mathrm{Isom}(G/K \times W).$$

Corollary 4.7. *Suppose W is a Riemannian manifold, and Q acts on W as isometries (i.e., ρ maps Q into* $\mathrm{Isom}(W)$). *Then the construction yields a representation*

$$\Pi \to \mathrm{Isom}(G/K) \times \mathrm{Isom}(W) \subset \mathrm{Isom}(G/K \times W),$$

yielding a Riemannian orbifold $\Pi\backslash(G/K \times W)$. □

The space $\Pi\backslash(G \times W)$ has a Seifert fiber structure

$$\Gamma\backslash G/K \to \Pi\backslash(G/K \times W) \to Q\backslash W,$$

where $\Gamma\backslash G/K$, the typical fiber, is a Riemannian symmetric space. Singular fibers are finite quotients of the typical fiber, where the finite actions are via isometries of G/K.

Here is a more precise account. Let Q_0 be the kernel of $\phi\colon Q \to \mathrm{Out}(\Gamma)$. Then the preimage of Q_0 in Π splits as a direct product $\Gamma \times Q_0$ because Γ has trivial center. Since $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \mathrm{Out}(G)$, Q_0 maps trivially into $\overline{\mathrm{Aff}}(G, K)$.

Consequently, the normal subgroup $\Gamma \times Q_0$ of Π acts on $G/K \times W$ in such a way that Γ acts only on the G/K-factor as left translations and Q_0 acts only on the W-factor via ρ, yielding $(\Gamma\backslash G/K) \times (Q_0\backslash W)$. Because $\mathrm{Out}(\Gamma)$ is finite, Q/Q_0 is finite. The finite quotient group $F = \Pi/(\Gamma \times Q_0)$ acts diagonally on $(\Gamma\backslash G/K) \times (Q_0\backslash W)$:

$$\begin{array}{ccc}
G/K \times W & \xrightarrow{Q_0\backslash} & G/K \times Q_0\backslash W \\
\Gamma\backslash\downarrow & & \Gamma\backslash\downarrow \\
\Gamma\backslash G/K \times W & \xrightarrow{Q_0\backslash} & (\Gamma\backslash G/K) \times (Q_0\backslash W) \\
 & & F\backslash\downarrow \\
 & & (\Gamma\backslash G/K) \times_F (Q_0\backslash W) \xrightarrow{=} \Pi\backslash(G/K \times W).
\end{array}$$

EXAMPLES 4.8. Let $G = \mathrm{SO}_0(1,3)$ and $W = \mathbb{R}^n$. Let Γ be a (resp. torsion-free) lattice of G, and let $Q \subset E(n)$ be a crystallographic group. Then $K = \mathrm{SO}(3)$ and $\mathrm{SO}_0(1,3)/\mathrm{SO}(3) = \mathbf{H}^3$, the 3-dimensional hyperbolic space. For any extension Π of Γ by Q, there exists a Seifert fibering (resp. an aspherical manifold)

$$\Gamma\backslash\mathbf{H}^3 \to \Pi\backslash(\mathbf{H}^3 \times \mathbb{R}^n) \to Q\backslash\mathbb{R}^n$$

with typical fiber the hyperbolic spaceform $\Gamma\backslash\mathbf{H}^3$ and base orbifold $Q\backslash\mathbb{R}^n$.

REMARK 4.9. Seifert constructions for $G = \mathbb{R}^n$ are more numerous and more twisted than constructions with G semisimple or G/K as above. For example, for Q take the Fuchsian group whose orbit space $Q\backslash\mathbf{H}^2$ is the 2-sphere with multiplicities 2, 3, and 7. For Seifert fiberings modeled on $\mathbb{R}^3 \times \mathbf{H}^2$ and $\Gamma = \mathbb{Z}^3$, we obtain an infinite number of different fiberings parameterized by $\mathbb{Z}^3$ when $Q \to \mathrm{Aut}(\mathbb{Z}^3)$ is trivial. If we take just those constructions that yield $K(\Pi, 1)$s, we obtain an infinite number of distinct 5-manifolds that fiber over the 2-torus with finite abelian structure group; see [CR3, 2.2]. The infinite number of distinct extensions follows from $H^2(Q;\mathbb{Z}) = \mathbb{Z}$. These aspherical 5-manifolds all exhibit $\mathbb{R}^2 \times \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ geometry. (From [CR3, 2.2] we can view Π as having a finite indexed normal subgroup $\mathbb{Z}^2 \times$(the fundamental group of a closed Seifert 3-manifold M) with a cyclic quotient. The aspherical 5-manifold is then "diagonally" covered by $T^2 \times M$.) In other words, the universal uniformizing group can be reduced to $\mathrm{Isom}(\mathbb{R}^2) \times \mathrm{Isom}_0(\widetilde{\mathrm{PSL}(2,\mathbb{R})}) = E(2) \times (\mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}(2,\mathbb{R})})$.

On the other hand, take $G/K \times \mathbf{H}^2$, where $G = \mathrm{SO}_0(1,3) \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}_0(\mathbf{H}^3)$ and $K = \mathrm{SO}(3)$. For any lattice $\Gamma \subset G$ and homomorphism $\phi\colon Q \to \mathrm{Aut}(\Gamma)$, there exists just one extension $1 \to \Gamma \to \Pi \to Q \to 1$, because the center of Γ is trivial. Furthermore, Π contains a finite indexed normal subgroup $\Gamma \times Q_0$, where Q_0 is the kernel of $Q \to \mathrm{Out}(\Gamma)$ ($\mathrm{Out}(\Gamma)$ is finite). The Seifert construction $M(\Pi) = \theta(\Pi)\backslash(G/K \times \mathbf{H}^2)$ is regularly covered (possibly branched) by the product space $(\Gamma\backslash G/K) \times (Q_0\backslash W)$, where Q/Q_0 acts diagonally and isometrically as in Theorem 4.6. This means that $M(\Pi)$ has $(\mathbf{H}^3 \times \mathbf{H}^2)$-geometry as an orbifold. If $Q \to \mathrm{Out}(\Gamma)$ is trivial, then Π cannot be torsion-free even if Γ is torsion-free. Consequently, $M(\Pi)$ is not aspherical (the underlying space of this orbifold is $\Gamma\backslash\mathbf{H}^3 \times S^2$).

If we want $M(\Pi)$ to be a closed aspherical 5-manifold, then Π must be torsion-free. This means that Π_w in $1 \to \Gamma \to \Pi_w \to Q_w \to 1$ must be torsion-free for each $Q_w \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_7$. Moreover, the image of Q in $\mathrm{Out}(\Gamma)$, which is isomorphic to Q/Q_0, can have no abelian quotient other than 1 because Q is perfect. In addition, Γ must be normally contained in other torsion-free lattices in $\mathrm{PSL}(2,\mathbb{C})$ with quotients $\mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_7$. This is of course difficult to achieve in general. (We may easily find Γ so that $\mathrm{Out}(\Gamma)$ does not contain each of $Q_w \cong \mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_7$ as subgroups.) The point here is that the possible group extensions Π of Γ by Q and the structure of Π severely restrict the possible Seifert constructions that yield torsion-free Π, and hence, aspherical manifolds when compared with extensions of $\mathbb{Z}^3$ by Q. One caveat, though, is that there are far more nonisomorphic lattices in $\mathrm{SO}_0(1,3)$ than in $\mathbb{R}^3$, which leads to a rich supply of Seifert fiberings despite the finiteness of each $\mathrm{Out}(\Gamma)$.

EXAMPLES 4.10. In this example, let us choose $\Gamma = (\mathbb{Z} * \mathbb{Z}) \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{Z} * \mathbb{Z}$ by $a \mapsto ab$, $b \mapsto a$. This group embeds in $\mathrm{SO}_0(1,3)$ as a noncocompact lattice. In fact, $\Gamma' = (\mathbb{Z} * \mathbb{Z}) \rtimes 2\mathbb{Z}$ is a subgroup of index 2 and $\Gamma'\backslash\mathbf{H}^3$ is the well-known complement of the (hyperbolic) figure-eight knot. $M = \Gamma\backslash\mathbf{H}^3$ is a nonorientable finite-volume hyperbolic manifold doubly covered by the complement of the figure-eight knot. Now, $\mathrm{Out}(\Gamma)$ is precisely $\mathbb{Z}_2$ [CR4, 6.8 & 7.2]. The generator of $\mathbb{Z}_2$ lifts to a hyperbolic involution on M and has a circle of fixed points.

Let $\phi\colon Q \to \mathrm{Out}(\Gamma) = \mathbb{Z}_2$ be a homomorphism and Q_0 be the kernel. Let $1 \to \Gamma \to \Pi \to Q \to 1$ be the extension induced by ϕ.

Case 1 ($Q = Q_0$): Then the Seifert construction is the embedding of $\Pi = \Gamma \times Q$ into $(\mathrm{SO}_0(1,3) \rtimes \mathbb{Z}_2) \times \mathrm{TOP}(W)$ and the action of Π on $\mathbf{H}^3 \times W$ is just the product action. The action will be free if and only if the Q action on W is free.

Case 2 (the index $[Q; Q_0]$ is 2): The Seifert construction again leads to an embedding in the above group. The action of Π on $\mathbf{H}^3 \times W$ factors through $\mathbf{H}^3 \times W \to \Gamma\backslash\mathbf{H}^3 \times Q_0\backslash W \to (\Gamma\backslash\mathbf{H}^3) \times_{\mathbb{Z}_2} (Q_0\backslash W)$. Clearly this action is free if and only if the action of Q on W is free. In the free case, $\Pi\backslash(\mathbf{H}^3 \times W)$ is a $\Gamma\backslash\mathbf{H}^3$ bundle over $Q\backslash W$ with structure group $\mathbb{Z}_2$. Since these are the only possible group extensions, there are no other Seifert constructions with this lattice.

REMARK 4.11. Let us now explain more carefully the connection with [RW]. Let Q_0 be the kernel of $Q \to \mathrm{Out}(\Gamma)$. Then $\Gamma \times Q_0$ is a normal subgroup of Π with quotient isomorphic to the finite group Q/Q_0. The exact sequence $1 \to \Gamma \to \Pi/Q_0 \to Q/Q_0 \to 1$ injects into $1 \to \mathrm{Inn}(\Gamma) \to \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma) \to 1$. Since every automorphism of $i(\Gamma) \subset G = l(G)$ extends uniquely to an automorphism of G, Π/Q_0 is mapped into $\mathrm{Aut}(G) = \bar{G}$ carrying Γ to $\mu(i(\Gamma)) = \bar{\Gamma}$ in $\mathrm{Inn}(\Gamma) \subset \mathrm{Inn}(G)$. The action, in [RW], of Π on $\bar{G}/\bar{K} \times W$ is given by the composite

$$\bar{G}/\bar{K}\times W \xrightarrow{Q_0\backslash} \bar{G}/\bar{K}\times(Q_0\backslash W) \xrightarrow{\bar{\Gamma}\backslash} (\bar{\Gamma}\backslash\bar{G}/\bar{K})\times(Q_0\backslash W) \xrightarrow{Q/Q_0} \Pi\backslash(\bar{G}/\bar{K}\times W).$$

Specifically, the action of $\beta \in \Pi$ on $\bar{\alpha}\bar{K} \times w$ is given by $\bar{\alpha}\bar{K} \times w \mapsto \bar{\beta}\bar{\alpha}\bar{K} \times w'$, where $\bar{\beta}$ is the automorphism of G induced by conjugation by $\beta \in \Pi$, and $w' = \rho(j(\beta))(w)$, where $j\colon \Pi \to Q$.

Remark 4.4 tells us that the action of $\theta(\Pi) \subset \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W)$ on $G/K \times W$ is equivalent to the action of Π on $\bar{G}/\bar{K} \times W$ via the isomorphism Ψ^{-1}, and the diffeomorphism $G/K \to \bar{G}/\bar{K}$.

If $\rho\colon Q \to \mathrm{TOP}(W)$, where W is a contractible manifold, and if Π is torsion free, then the space $M(\Pi) = \theta(\Pi)\backslash(G/K \times W)$ is the $K(\Pi, 1)$-manifold constructed in [RW] as mentioned in the Introduction, and $M(\Pi)$ will be smooth if ρ is smooth.

Mostow's rigidity theorem does not apply to $G = \mathrm{PSL}(2, \mathbb{R})$. However, by changing the embedding of Γ, one can still embed the group Π into $\mathrm{TOP}_{G,K}(G \times W)$ provided that the image of the abstract kernel in $\mathrm{Out}(\Gamma)$ of the given extension is finite.

THEOREM 4.12. *Let $G = \mathrm{PSL}(2, \mathbb{R})$, and let $K = S^1 \subset \mathrm{PSL}(2, \mathbb{R})$ be a maximal compact subgroup. Let Γ be a lattice of G. Let $\rho\colon Q \to \mathrm{TOP}(W)$ be a properly discontinuous action, and let $1 \to \Gamma \to \Pi \to Q \to 1$ be an exact sequence. Assume that the abstract kernel $\varphi\colon Q \to \mathrm{Out}(\Gamma)$ associated with this extension has finite image. Then there exists a homomorphism $\theta\colon \Pi \to \mathrm{TOP}_{G,K}(G/K \times W)$ so that the diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow i & & \downarrow \theta & & \downarrow \varphi\times\rho & & \\ 1 & \to & \mathrm{PSL}(2,\mathbb{R}) & \to & \mathrm{TOP}_{G,K}(G/K \times W) & \to & \mathrm{Out}(G)\times\mathrm{TOP}(W) & \to & 1 \end{array}$$

is commutative (where $i\colon \Gamma \to l(G)$ may be different from the original $\Gamma \subset l(G)$) This yields a Seifert fiber space with the surface orbifold $\Gamma\backslash G/K = \Gamma\backslash \mathbf{H}^2$ as typical fiber. The action is free if and only if the preimage of each stabilizer Q_w in Π is torsion-free and, in particular, Γ is a surface group.

Proof. First we need to calculate $\mathrm{TOP}_{G,K}(G/K \times W)$. Since $N_G(K) = K$, the general case still applies. We have

$$\mathrm{TOP}_{G,K}(G/K \times W) = \frac{l(G) \rtimes \mathrm{Aut}(G, K)}{r(K) \rtimes \mathrm{Aut}^0(G, K)} \times \mathrm{TOP}(W).$$

Let Q_0 be the kernel of $\varphi\colon Q \to \mathrm{Out}(\Gamma)$. Then $\bar{Q} = Q/Q_0$ is finite. Consider the extension

$$1 \to \Gamma \to \bar{\Pi} \to \bar{Q} \to 1,$$

where $\bar{\Pi} = \Pi/Q_0$. By Nielsen's theorem, as completed by S. Kerkhoff, there exists a homomorphism $\bar{\Pi} \to \mathrm{PSL}(2, \mathbb{R}) \rtimes \mathbb{Z}_2$ realizing this extension as a group action. This, together with the action $Q \to \mathrm{TOP}(W)$, gives rise to a desired homomorphism θ. If the abstract kernel of Q into $\mathrm{Out}(\Gamma)$ is not finite, then we cannot apply Nielsen's theorem and a Seifert construction is not possible by this method. □

Theorem 4.6 still holds if G contains 3-dimensional factors (i.e., $\mathrm{PSL}(2,\mathbb{R})$-factors), provided that the projection of Γ to each of these factors is dense, because the lattice will still satisfy the UAEP condition [Mos; Pra]. The other extreme case will be generalization of Theorem 4.12. Suppose $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \times \cdots \times \mathrm{PSL}(2,\mathbb{R})$, and assume Γ is a lattice in G such that *none* of the images of Γ by the projection onto each factor is dense. Then Γ lies in a group of the form $\Delta = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_s$ (simply take the images of projections). The argument of Theorem 4.12 goes through, and the statement holds true in this more general setting.

EXAMPLE 4.13. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a compact surface group of genus 9, and let $\hat{Q} = \mathbb{Z}^2 \subset E(2)$. A finite group $F = \mathbb{Z}_2$ acts on the surface as a covering transformation yielding a surface of genus 5. It also acts on $\hat{Q}$ by sending the generators $t_1 \mapsto t_1^{-1}$ and $t_2 \mapsto t_2^{-1}$ so that it has four fixed points on the 2-torus. Let Π be an extension of $\Gamma \times \mathbb{Z}^2$ by $F = \mathbb{Z}_2$ so that $1 \to \Gamma \times \mathbb{Z}^2 \to \Pi \to \mathbb{Z}_2 \to 1$ is exact. We view Π as $1 \to \Gamma \to \Pi \to Q \to 1$, where $Q = \hat{Q} \rtimes \mathbb{Z}_2$. Then Π acts freely on $(\mathrm{PSL}(2,\mathbb{R})/S^1) \times \mathbb{R}^2 = \mathbf{H} \times \mathbb{R}^2$. The resulting space $\Pi\backslash(\mathbf{H} \times \mathbb{R}^2)$ is an aspherical Seifert manifold over a flat orbifold (topologically the 2-sphere) with typical fiber the surface of genus 9. There are four singular fibers, all of which are surfaces of genus 5.

5. Solvmanifolds

A solvmanifold X is a space on which a solvable Lie group acts transitively. This is equivalent to saying $X = G/H$ where G is a solvable Lie group and H is a closed subgroup. An *infra-solvmanifold* is a quotient space of a solvable Lie group G by a closed subgroup H' of $G \rtimes \mathrm{Aut}(G)$ which is finitely covered by a solvmanifold. Therefore, $H' \cap G$ must have finite index in H'.

We shall work with a special kind of solvable Lie group: The split Lie hull of a predivisible group (definitions to follow). This is not very restrictive, because every poly{cyclic or finite} group contains a characteristic predivisible group. The following definition can be found in [AJ].

DEFINITION 5.1. A torsion-free group Γ is called a *predivisible group* if it fits the short exact sequence $1 \to \Delta \to \Gamma \to \mathbb{Z}^k \to 1$ and satisfies the following conditions.

(1) Δ is nilpotent.
(2) For $\gamma \in \Gamma$, let $\mu(\gamma)$ be the automorphism of $\Delta_{\mathbb{R}}$ (the Malcev completion of Δ). Then, for each eigenvalue θ of $\mu(\gamma)$,

$$\theta|\theta|^{-1} = \cos 2\pi\rho + i \sin 2\pi\rho$$

with $\rho = 0$ or irrational.

For a predivisible group Γ, there exists a connected solvable Lie group $G = S \rtimes K$, called the *split Lie hull of* Γ, satisfying:

(P1) Γ is a lattice of S;
(P2) (Γ, G) has the UAEP;
(P3) S is a closed normal subgroup of G; and
(P4) K is a maximal compact subgroup of G which is a torus.

LEMMA 5.2. *With G as above, let N be the largest normal subgroup of G contained in K. Then N is fully invariant in G.*

Proof. Since S is normal in G, $[S, N] \subset [S, G] \subset S$. Similarly, since N is normal in G, $[S, N] \subset N$. Consequently, $[S, N] \subset S \cap N = \{1\}$. Since K is abelian, this implies that $N \subset \mathcal{Z}(G)$. In fact, $N = \mathcal{Z}(G) \cap K$.

We claim that N is fully invariant. Since N is closed, it is compact. Therefore, either N is finite or the set of elements of finite order is dense in N. Let $f\colon G \to G$ be an automorphism. Assume $f(N) \not\subset N$. If $f(n) \in N$ for every element n of finite order, then $f(N) \subset N$. Therefore, there exists $n \in N$ of finite order, say of order p, such that $f(1, n) = (a, \alpha)$ with $a \neq 1$. Since $(a, \alpha) \in \mathcal{Z}(G)$, $\alpha = \mu(a^{-1})$ so that $\alpha(a) = a$. Thus,

$$(1,1) = f((1,n)^p) = (f(1,n))^p = (a,\alpha)^p = (a^p, \alpha^p).$$

However, since S is torsion-free, $a^p = 1$ is not possible. This proves $f(1, n)$ is of the form $(1, \alpha)$, which implies $f(N) \subset K$. However, $\mathcal{Z}(G)$ is fully invariant so that a central element maps to a central element. Thus $\alpha \in N$ again. Since N is compact, one can apply Lemma 1.8 so that $\alpha(N) \subset N$ implies that α induces an automorphism of N. We have proved that N is fully invariant. □

When we divide out G by this group N, properties (P1)–(P4) are preserved (with K replaced by K/N). We lose nothing by dividing out by N, since G/K is diffeomorphic to $(G/N)/(K/N)$ and the universal uniformizing groups are the same; see Proposition 1.7. Therefore, we may assume that N is trivial from the beginning. So we add one more property to the list:

(P5) the largest normal subgroup of G contained in K is trivial.

An *MW* (Mostow–Wang) *group* Γ is one occurring in an exact sequence $1 \to \Delta \to \Gamma \to \mathbb{Z}^s \to 1$, where Δ is a torsion-free, finitely generated nilpotent group. It is known that an MW group contains a characteristic predivisible polycyclic group of finite index. *From now on, Γ is a lattice of $G = S \rtimes K$ satisfying conditions* (P1)–(P5).

LEMMA 5.3. *Every compact subgroup of G isomorphic to K is conjugate to K.*

Proof. Let $K' \subset G$ be a torus. It acts on the coset space $G/K \cong S$ smoothly, as left translations. It is well known that a torus action on a Euclidean space has a fixed point, say pK. Now $K' \cdot pK = pK$ implies $p^{-1}K'p \subset K$. If $K' \cong K$, then clearly $p^{-1}K'p = K$. In fact, the statement is true in more generality: Let G be a Lie group with a finite number of components and let K be a maximal compact subgroup. Then every compact subgroup of G can be conjugated into K. □

Let

$$S^K = \{ s \in S : [k, s] = 1 \text{ for all } k \in K \}$$

be the fixed point set of the K-action on S. Since $[(1, k), (a, \alpha)] = (k(a) \cdot a^{-1}, 1)$ and K is abelian, we have the following lemma.

LEMMA 5.4. $N_G(K) = C_G(K) = S^K \times K$. □

PROPOSITION 5.5. *Suppose S^K is trivial. Then*:

(1) *there exists an isomorphism* $\Psi: \overline{\text{Aff}}(G, K) \to \text{Aut}(G)$; *and*
(2) $\text{TOP}_{G,K}(G/K \times W) = \overline{\text{Aff}}(G, K) \times \text{TOP}(W)$, *where* $\overline{\text{Aff}}(G, K) = (l(G) \rtimes \text{Aut}(G, K)/r(K)$.

Proof. Lemma 5.4 implies $N_G(K) = K$ in our case. (1) Lemma 2.3 applies because of Lemma 5.3. (2) Lemma 2.1 together with the condition (P5) implies that $\text{Aut}^0(G, K)$ is trivial. Now one applies Theorem 2.2. □

THEOREM 5.6. *Let Γ be a predivisible group, and let $G = S \rtimes K$ be a solvable Lie group satisfying* (P1)–(P5). *Also assume that S^K is trivial. Let $\rho: Q \to \text{TOP}(W)$ be a properly discontinuous action, and let $1 \to \Gamma \to \Pi \to Q \to 1$ be an exact sequence. Then there exists a homomorphism $\theta: \Pi \to \text{TOP}_{G,K}(G/K \times W) = \overline{\text{Aff}}(G, K) \times \text{TOP}(W)$ so that the diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi\times\rho} & & \\ 1 & \to & \overline{\text{Aff}}(G, K) & \to & \text{TOP}_{G,K}(G/K \times W) & \to & \text{Out}(G) \times \text{TOP}(W) & \to & 1 \end{array}$$

is commutative, yielding a Seifert fiber space with typical fiber the double coset space $\Gamma\backslash G/K$, a solvmanifold. Such a homomorphism θ with fixed i and $\varphi \times \rho$ is unique. The action is free if and only if the preimage of each stabilizer Q_w in Π is torsion-free.

Proof. (Existence) Since Γ is normal in Π, there is a natural homomorphism μ : $\Pi \to \text{Aut}(\Gamma)$. The UAEP, by (P2), gives rise to a homomorphism $\text{Aut}(\Gamma) \to \text{Aut}(G)$. Consequently, we have a homomorphism $\Pi \to \text{Aut}(G)$. Composing $\Pi \to \text{Aut}(G)$ with $\Psi^{-1} : \text{Aut}(G) \to \overline{\text{Aff}}(G, K)$ in Proposition 5.5, we get a homomorphism $\Pi \to \text{Aut}(G) \to \overline{\text{Aff}}(G, K)$. Under this homomorphism, Γ is mapped into $\overline{\text{Aff}_0}(G, K)$. This, together with the action $\Pi \to Q \to \text{TOP}(W)$, gives rise to a homomorphism $\Pi \to \overline{\text{Aff}}(G, K) \times \text{TOP}(W) = \text{TOP}_{G,K}(G/K \times W)$.

(Uniqueness) Same as the proof of Theorem 4.6. □

In the previous theorem we assumed that S^K is trivial. When S^K is nontrivial, the universal uniformizing group is pretty big and is not so easy to handle. However, when Q is finite (as in [AJ]), a Seifert construction can be made.

THEOREM 5.7 (when Q is finite; cf. [AJ]). *Let Γ be a predivisible group, and let $G = S \rtimes K$ be a solvable Lie group satisfying* (P1)–(P5). *Let $1 \to \Gamma \to \Pi \to Q \to 1$ be an exact sequence, with Q finite. Then there exists a homomorphism $\theta : \Pi \to \overline{\mathrm{Aff}}(G, K)$ so that the diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \to & \overline{\mathrm{Aff}}_0(G, K) & \to & \overline{\mathrm{Aff}}(G, K) & \to & \mathrm{Out}(G) & \to & 1 \end{array}$$

is commutative. Such a homomorphism θ with fixed i and φ is unique. The action is free if and only if Π is torsion-free, in which case $\theta(\Pi)\backslash G/K$ will be an infra-solvmanifold.

Proof. We choose W to be a point. Since Q is finite, the trivial action of Q on W is proper. By Corollary 1.5, the universal uniformizing group $\mathrm{TOP}_{G,K}(G/K \times W)$ is then

$$\mathrm{TOP}_{G,K}(G/K) = \frac{l(G) \cdot [r(N_G(K)) \rtimes \mathrm{Aut}(G, K)]}{r(K) \rtimes \mathrm{Aut}^0(G, K)}.$$

We can still apply Lemma 2.1 to yield $\mathrm{Aut}^0(G, K) = 1$. Thus,

$$\mathrm{TOP}_{G,K}(G/K) = \frac{l(G) \cdot [r(N_G(K)) \rtimes \mathrm{Aut}(G, K)]}{r(K)}.$$

Since $r(N_G(K)) \subset l(G) \rtimes \mathrm{Aut}(G, K)$, this is equal to $(l(G) \rtimes \mathrm{Aut}(G, K))/r(K) = \overline{\mathrm{Aff}}(G, K)$.

(Existence) We shall first map Π into $\mathrm{Aff}(G, K) = l(G) \rtimes \mathrm{Aut}(G, K)$. The UAEP, by (P2), gives rise to an extension $1 \to G \to \Pi \cdot G \to Q \to 1$. However, it is known that every finite extension of G splits (see [Aus, p. 251]). Therefore the group $\Pi \cdot G \cong G \rtimes Q = (S \rtimes K) \rtimes Q$. Let K' be a maximal compact subgroup of $(S \rtimes K) \rtimes Q$ containing K. Then, clearly, $(S \rtimes K) \rtimes Q = S \rtimes K'$, where $K' \cong K \rtimes Q$. In other words, $\Pi \cdot G \cong G \rtimes Q \cong S \rtimes (K \rtimes Q)$. Then the conjugation map sends Q into $\mathrm{Aut}(G, K)$. Consequently, we have mapped $\Pi \cdot G$ into $l(G) \rtimes \mathrm{Aut}(G, K)$ via

$$\Pi \to \Pi \cdot G \to G \rtimes Q \to S \rtimes (K \rtimes Q) \to l(G) \rtimes \mathrm{Aut}(G, K).$$

This, together with the projection $l(G) \rtimes \mathrm{Aut}(G, K) \to \overline{\mathrm{Aff}}(G, K)$, gives a desired homomorphism $\Pi \to \overline{\mathrm{Aff}}(G, K)$.

(Uniqueness) Again, we apply Theorem 3.2 to the commuting diagram in the statement of the theorem (with $\mathfrak{A} = \overline{\mathrm{Aff}}_0(G, K)$ and $\Delta = \Gamma$). We need to calculate the centralizer of Γ in $\overline{\mathrm{Aff}}_0(G, K) = (l(G) \rtimes \mathrm{Inn}(G, K))/r(K)$. An element $(a, \mu(b)) \in l(G) \rtimes \mathrm{Inn}(G, K)$ represents an element in the centralizer if and only if

$$[(a, \mu(b)), (x, 1)] = (abxb^{-1}ax^{-1}, 1) = (\mu(ab)(x) \cdot x^{-1}, 1) \in r(K)$$

for every $x \in \Gamma$. Recall that elements of $r(K)$ in $l(G) \rtimes \mathrm{Aut}(G, K)$ are of the form $(k^{-1}, \mu(k))$ with $k \in K$. Therefore, it happens if and only if $\mu(ab)(x) \cdot x^{-1} \in$

$\mathcal{Z}(G) \cap K$. However, $\mathcal{Z}(G) \cap K$ is trivial so that $\mu(ab)(x) \cdot x^{-1} = 1$ for every $x \in \Gamma$. By the UAEP, this should happen for every $x \in G$. Then $\mu(ab) \in \mathrm{Inn}^0(G, K)$. But $\mathrm{Inn}^0(G, K)$ is trivial by Lemma 2.1. Therefore, $\mu(ab) = 1$, so that $ab \in \mathcal{Z}(G)$. Let $a = zb^{-1}$ for some $z \in \mathcal{Z}(G)$. Since $\mu(b) \in \mathrm{Inn}(G, K)$,

$$(a, \mu(b)) = (zb^{-1}, \mu(b)) = (z, 1)(b^{-1}, \mu(b)),$$

which represents the element $(z, 1) \in l(\mathcal{Z}(G))$ of $\overline{\mathrm{Aff}}_0(G, K)$. We have shown $C_{\overline{\mathrm{Aff}}_0(G,K)}(\Gamma) = l(\mathcal{Z}(G))$. Since Q is a finite group and $\mathcal{Z}(G)$ is isomorphic to $\mathbb{R}^k$ for some k,

$$H^1(Q; C_{\overline{\mathrm{Aff}}_0(G,K)}(\Gamma)) = H^1(Q; \mathbb{R}^k) = 0.$$

By Theorem 3.2, such a homomorphism θ with fixed i and φ is unique, up to conjugation by elements of $\overline{\mathrm{Aff}}(G, K)$. Finally, since G/K is diffeomorphic to $\mathbb{R}^n$, the action is free if and only if Π is torsion-free. ❑

The Seifert construction in Theorem 5.7 for Π, when Q is finite, gives us detailed knowledge of the geometric structure of the spaces constructed earlier by Auslander and Johnson [AJ]. For example, if G has a left invariant metric that is also right K-invariant, then the resulting space G/K will inherit the metric.

More generally, we shall take the case where $\varphi: Q \to \mathrm{Out}(G)$ has a finite image.

COROLLARY 5.8 ($\mathcal{Z}(\Gamma)$ trivial and $\varphi: Q \to \mathrm{Out}(G)$ has a finite image). *Let Γ be a predivisible group* without center, *and let $G = S \rtimes K$ be a solvable Lie group satisfying* (P1)–(P5). *Let $\rho: Q \to \mathrm{TOP}(W)$ be a properly discontinuous action, and let $1 \to \Gamma \to \Pi \to Q \to 1$ be an exact sequence. Assume that the abstract kernel $\varphi: Q \to \mathrm{Out}(G)$ associated to this extension has a finite image. Then there exists a homomorphism $\theta: \Pi \to \mathrm{TOP}_{G,K}(G/K \times W) = \overline{\mathrm{Aff}}(G, K) \times \mathrm{TOP}(W)$ so that the diagram with exact rows*

$$\begin{array}{ccccccccc}
1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi\times\rho} & & \\
1 & \to & \overline{\mathrm{Aff}}(G, K) & \to & \mathrm{TOP}_{G,K}(G/K \times W) & \to & \mathrm{Out}(G) \times \mathrm{TOP}(W) & \to & 1
\end{array}$$

is commutative, yielding a Seifert fiber space with typical fiber the solvmanifold $\Gamma\backslash G/K$. Such a homomorphism θ with fixed i and $\varphi \times \rho$ is unique. The action is free if and only if the preimage of each stabilizer Q_w in Π is torsion-free.

Proof. Let Q_0 be the kernel of $\varphi: Q \to \mathrm{Out}(G)$. Then, since Γ is centerless, Q_0 lifts to a normal subgroup of Π. Now consider the exact sequence $1 \to \Gamma \to \Pi/Q_0 \to Q/Q_0 \to 1$. Since Q/Q_0 is finite, Theorem 5.7 applies to obtain a homomorphism $\Pi/Q_0 \to \overline{\mathrm{Aff}}(G, K)$. Now the two homomorphisms $\Pi \to \Pi/Q_0 \to \overline{\mathrm{Aff}}(G, K)$ and $\Pi \to Q \to \mathrm{TOP}(W)$ give the desired homomorphism. □

The structure of the space $\theta(\Pi)\backslash(G/K \times W)$ is similar to the symmetric space case. That is, it has a Seifert fiber structure

$$\Gamma\backslash G/K \to \theta(\Pi)\backslash(G/K \times W) \to Q\backslash W,$$

where the typical fiber $\Gamma\backslash G/K$ is a solvmanifold. Singular fibers are finite quotients of the typical fiber, where the finite actions are via elements of $\overline{\mathrm{Aff}}(G, K)$. Let Q_0 be the kernel of $\phi\colon Q \to \mathrm{Out}(\Gamma)$. Then the preimage of Q_0 in Π splits as a direct product $\Gamma \times Q_0$ because Γ has trivial center. Since $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \mathrm{Out}(G)$, Q_0 maps trivially into $\overline{\mathrm{Aff}}(G, K)$. Consequently, the normal subgroup $\Gamma \times Q_0$ of Π acts on $G/K \times W$ in such a way that Γ acts only on the G/K-factor as left translations and Q_0 acts only on the W-factor via ρ, yielding $(\Gamma\backslash G/K) \times (Q\backslash W)$. Because $\varphi\colon Q \to \mathrm{Out}(\Gamma)$ has a finite image, Q/Q_0 is finite. The finite quotient group $F = \Pi/(\Gamma \times Q_0)$ acts diagonally on $(\Gamma\backslash G/K) \times (Q\backslash W)$.

Recalling that F acts on a space X, let $\tilde{F}$ be the group of all lifts of elements of F to the universal covering $\tilde{X}$ of X. Then $\tilde{F}$ acts on $\tilde{X}$ and normalizes the covering transformations $\Pi = \pi_1(X)$. The exact sequence

$$1 \to \Pi \to \tilde{F} \to F \to 1$$

is called the *lifting sequence* for F, and $\tilde{F}$ is effective if and only if F is effective. In particular, if X is a closed aspherical manifold then F is effective if and only if the centralizer $C_{\tilde{F}}(\Pi)$ is torsion-free [LR3].

As mentioned at the beginning of this section, for a torsion-free poly{cyclic or finite} group Π we can always find a (characteristic) predivisible subgroup Γ of finite index in Π. Let Q be the finite quotient Π/Γ, and choose $W =$ point. Then the Seifert construction of Theorem 5.7 produces an embedding $\theta(\Pi) \subset \overline{\mathrm{Aff}}(G, K)$ and the Seifert manifold $M(\Pi) = \theta(\Pi)\backslash G/K$ is a closed smooth $K(\Pi, 1)$ manifold. Suppose now $\psi\colon F \to \mathrm{Out}(\Pi) = \pi_0\mathcal{E}(M(\Pi))$ is a homomorphism of a finite group F into the homotopy classes of self-homotopy equivalences of $M(\Pi)$. We then have the following.

COROLLARY 5.9 (Geometric realization of group actions from homotopy data). *F acts on $M(\Pi)$ if and only if there exists an extension*

$$1 \to \Pi \to E \to F \to 1$$

realizing the abstract kernel ψ. Moreover, the action can be chosen to be smooth, induced from smooth Seifert automorphisms contained in $\overline{\mathrm{Aff}}(G, K)$. The action of F is effective if and only if $C_E(\Pi)$ is torsion-free.

Proof. In order to have an action, we must have a lifting sequence and hence an extension, $1 \to \Pi \to E \to F \to 1$, that realizes the abstract kernel ψ. Since Γ is characteristic in Π, it is normal in E and and $1 \to Q = \Pi/\Gamma \to E/\Gamma \to F \to 1$ is exact. Because of the commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & \Gamma & \to & \Pi & \to & Q & \to & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
1 & \to & \Gamma & \to & E & \to & E/\Gamma & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & \xrightarrow{=} & F, & &
\end{array}$$

we can find a Seifert construction $\theta' : E \to \overline{\mathrm{Aff}}(G, K)$ that extends $\theta : \Pi \to \overline{\mathrm{Aff}}(G, K)$. Therefore the group F acts on $M(\Pi)$ smoothly as diffeomorphisms preserving the Seifert structure. The action of F, as mentioned above, is effective if and only if $C_E(\Pi)$ is torsion-free. In any case, we have a lift $\tilde{\psi}$,

$$\begin{array}{ccc}
F & \xrightarrow{\tilde{\psi}} & \mathrm{Diff}(M(\Pi)) \\
\psi\downarrow & & j\downarrow \\
\mathrm{Out}(\Pi) & \xrightarrow{=} & \mathcal{E}(M(\Pi)),
\end{array}$$

where j sends a self-diffeomorphism to its homotopy class. In case there exists one extension realizing the abstract kernel ψ, then for each element of $H^2(F, \mathcal{Z}(\Pi))$ there is a congruence class of extensions E realizing the abstract kernel ψ. Each of these extensions gives rise to a (not necessarily effective) action of F on $M(\Pi)$. □

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K. B. Lee
Department of Mathematics
University of Oklahoma
Norman, OK 73019

kblee@math.uoknor.edu

F. Raymond
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

Frank.Raymond@math.lsa.umich.edu