Banach Spaces of Hypercyclic Vectors

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1. Introduction

Beauzamy [Be1; Be2; Be3] constructed examples of linear operators on Hilbert space having dense, invariant linear manifolds all of whose nonzero elements are hypercyclic. Recently, Bourdon [Bo] has shown that any hypercyclic operator on Hilbert space has a dense invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors. Special cases of Bourdon's result were proved by Godefroy and Shapiro [GoS]. Interest in constructing linear spaces of hypercyclic vectors arises from the invariant subset problem.

Let us denote by $H(\mathbb{C})$ the space of the entire functions with the topology of uniform convergence on compact subsets. A Euclidean translation $\varphi(z) = z + a$ ($a \neq 0$) defines a continuous linear operator C_φ on $H(\mathbb{C})$ that assigns to each function $f \in H(\mathbb{C})$ the function $C_\varphi(f) = f \circ \varphi$. If we apply Theorem 1.2 in [BM] to the sequence $\{z + na\}$ we obtain that there is an infinite closed vector space consisting, except for zero, of hypercyclic vectors for the operator C_φ. A similar result can be obtained from Theorem 1.2 in [BM] for the space of the holomorphic functions on the unit disk with the Euclidean translation replaced by a non-Euclidean one. The space of entire functions is a Fréchet space, so this result only complements partially the result of Bourdon. The proof of Theorem 1.2 in [BM] did not appear to adapt easily to other situations. The result suggests many questions. J. H. Shapiro has proposed to the author in private communication the following problems: What are the analogs of this result for:

(a) sequences of powers of a single bounded operator on a Banach space?

(b) sequences of powers of a composition operator on Hardy spaces?

It seems that nothing is known about any of these problems, and research into them may lead to interesting relations between function theory and operator theory. The main result of this paper isolates sufficient conditions for an operator to have an infinite-dimensional Banach subspace of hypercyclic vectors. Of course, results of this kind are primarily of interest in regard to Hilbert spaces. Section 3 contains an application of the theorem to composition operators on Hardy spaces.

In this section we also prove that there are operators that do have hypercyclic vectors, but not an infinite-dimensional Banach space of them—namely, certain scalar multiples of the backward shift defined on the sequence spaces l^p ($1 \leq p < \infty$) and c_0.

2. Main Result

In all that follows, B will be a separable Banach space and T a bounded linear operator on B. We recall that a vector z in a Banach space is hypercyclic for the bounded linear operator $T: B \to B$ if the orbit $\{T^n z\}_{n \geq 1}$ is dense in B. A bounded linear operator T on a Banach space is hypercyclic if it has a hypercyclic vector. Separability of the underlying space is clearly a necessary condition for hypercyclicity. There is a sufficient condition for an operator T on a Banach space B to have hypercyclic vectors. This condition has come to be called the *hypercyclicity criterion*. It reads as follows (see [GS, Thm. 2.2]).

THEOREM 2.1. *Suppose T is a bounded linear operator on a separable Banach space B. Suppose there is a dense subset $X \subset B$ and a right inverse S for T ($TS =$ identity on X) such that $\|T^n x\| \to 0$ and $\|S^n x\| \to 0$ for every $x \in X$. Then T has hypercyclic vectors.*

The first result of this type was discovered by Kitai [Ki] in her 1982 dissertation, which was never published. This result was rediscovered by Gethner and Shapiro [GS] with a simpler proof, who used it to unify the proofs of theorems of Birkhoff [Bi], McLane [Mc], Rolewicz [Ro], Seidel and Walsh [SW], and many others. Since then, the theorem has been used to discover hypercyclic behavior in subsequent studies of hypercyclicity [BS1; BS2; GoS; CS; He; HW; Sh]. We have stated Theorem 2.1 here because its hypotheses are part of the hypotheses of the following theorem.

THEOREM 2.2. *Let B be a separable Banach space, and let $T: B \to B$ be a bounded linear operator that satisfies the following conditions.*

(a) *There is a dense subset X of B and a right inverse S (possibly discontinuous) for T ($TS =$ identity on X) such that $\|T^n x\| \to 0$ and $\|S^n x\| \to 0$ for every $x \in X$.*

(b) *There is an infinite-dimensional Banach subspace $B_0 \subset B$ such that $\|T^n e\| \to 0$ for every $e \in B_0$.*

Then there is an infinite-dimensional Banach space B_1 such that each $z \in B_1 \setminus \{0\}$ is hypercyclic for T.

Proof. By a theorem of Mazur (see e.g. [Di, p. 39]), we can find a basic sequence $\{e_m\}_{m \geq 1} \subset B_0$ that we may suppose to be normalized. Let $K > 0$ be the basis constant for this basic sequence and let $\{\varepsilon_m\}_{m \geq 1}$ be a decreasing sequence of positive numbers such that $\sum_{m=1}^{\infty} \varepsilon_m < 1/2K$. Finally, we also consider a denumerable subset $\{x_n\}_{n \geq 1} \subset X$ that is dense in B.

By setting $i(m, n) = (m + n - 1)(m + n)/2 - n + 1$, the positive integers are distributed as the entries of an infinite matrix.

$\dfrac{m}{n}$	1	2	3	...	m	...
1	1	3	6		$i(m, 1)$	
2	2	5			$\vdots$	
3	4				$\vdots$	
$\vdots$					$\vdots$	
n	$i(1, n)$	....			$i(m, n)$	...
$\vdots$					$\vdots$	

The following comments on $i(m, n)$ will be useful later. First, observe that the function $i(m, n)$ is strictly increasing with respect to both m and n. Second, if an element $i(k, l)$ is on the diagonal that goes from $i(1, n)$ to $i(n, 1)$, then $k + l = n + 1$. In fact, the elements of this diagonal are $i(j, n+1-j)$, $j = 1, \ldots, n$, which form a strictly increasing finite sequence of positive integers. Finally, observe that $i(k, n)$ belongs to the diagonal that goes from $i(1, n + k - 1)$ to $i(n + k - 1, 1)$.

The idea of the proof is to contruct a basic sequence $\{z_m\}_{m \geq 1}$ of hypercyclic vectors that is a perturbation of the sequence $\{e_m\}_{m \geq 1}$. Once this has been done, it will be proved that the hypercylicity of the elements of the sequence $\{z_m\}_{m \geq 1}$ is inherited by the elements of its closed linear span. Our proof is based on the following crucial claim, whose proof is postponed.

CLAIM. *There exist infinitely many strictly increasing sequences of positive integers* $\{r(i(m, n))\}_{n \geq 1}$ *($m \geq 1$), and for each of them there exists a corresponding vector z_m such that the following conditions hold.*

(i) *Each z_m is a hypercyclic vector for T. In fact, we have*

$$\|T^{r(i(m,n))} z_m - x_n\| < \varepsilon_m / 2^n \quad (n \geq 1).$$

(ii) *For $k \neq m$ we have*

$$\|T^{r(i(k,n))}(z_m - e_m)\| < \varepsilon_m / 2^n \quad (n \geq 1).$$

(iii) *The sequence $\{z_m\}_{m \geq 1}$ is close to $\{e_m\}_{m \geq 1}$; that is, $\|z_m - e_m\| < \varepsilon_m$ for every $m \geq 1$.*

Now we prove that $\{z_m\}_{m \geq 1}$ is a basic sequence. Let $\{e_m^*\}_{m \geq 1}$ be the coefficient functionals corresponding to the basic sequence $\{e_m\}_{m \geq 1}$. Because the sequence is normalized, we know that $\|e_m^*\| \leq 2K$ for every $m \geq 1$. So, using (iii), we find that

$$\sum_{m=1}^{\infty} \|e_m^*\| \|e_m - z_m\| < 2K \sum_{m=1}^{\infty} \varepsilon_m < 1.$$

Since $\{e_m\}_{m \geq 1}$ is a basic sequence, we have that $\{z_m\}_{m \geq 1}$ is a basic sequence equivalent to $\{e_m\}_{m \geq 1}$ (see [Di, Thm. 9, p. 46]). This means that the closed linear span generated by $\{e_m\}_{m \geq 1}$ is isomorphic to the closed linear span generated by $\{z_m\}_{m \geq 1}$. The Banach space $\mathcal{B}_1$ we are looking for will be the latter. Clearly, $\mathcal{B}_1$ is an infinite-dimensional Banach space. Hence we need only prove that each

vector $z \in \mathcal{B}_1$ is hypercyclic for T. Taking account of the fact that $\{z_m\}_{m \geq 1}$ is a basic sequence equivalent to $\{e_m\}_{m \geq 1}$, we find that each element $z \in \mathcal{B}_1$ has a unique representation as a series $z = \sum_{m=1}^{\infty} \alpha_m z_m$. Let us see that each vector $z = \sum_{m=1}^{\infty} \alpha_m z_m$ that is not the null vector is a hypercyclic vector for T. Since $\sum_{m=1}^{\infty} \alpha_m z_m$ is not the null vector, there is an $\alpha_k \neq 0$. Since every nonzero scalar multiple of a hypercyclic vector is again hypercyclic, we may suppose that $\alpha_k = 1$. We consider the sequence $\{T^{r(i(k,n))}\}_{n \geq 1}$ and compute

$$\left\| T^{r(i(k,n))} \sum_{m=1}^{\infty} \alpha_m z_m - x_n \right\|. \tag{1}$$

By the triangle inequality, (1) is less than

$$\left\| T^{r(i(k,n))} z_k - x_n \right\| + \left\| T^{r(i(k,n))} \sum_{m \neq k} \alpha_m e_m \right\|$$

$$+ \sum_{m \neq k} |\alpha_m| \| T^{r(i(k,n))} (z_m - e_m) \|. \tag{2}$$

We set $z_0 = \sum_{m \neq k} \alpha_m e_m$. Clearly, z_0 belongs to $\mathcal{B}_0$. By applying (i) to the first term in (2) and (ii) to the third term, we find that (2) is less than

$$\frac{\varepsilon_k}{2^n} + \sum_{m \neq k} |\alpha_m| \frac{\varepsilon_m}{2^n} + \| T^{r(i(k,n))} z_0 \| = \frac{1}{2^n} \sum_{m=1}^{\infty} |\alpha_m| \varepsilon_m + \| T^{r(i(k,n))} z_0 \|$$

$$\leq \frac{2K \|z\|}{2^n} \sum_{m=1}^{\infty} \varepsilon_m + \| T^{r(i(k,n))} z_0 \|$$

$$< \frac{\|z\|}{2^n} + \| T^{r(i(k,n))} z_0 \|. \tag{3}$$

Since $z_0 \in \mathcal{B}_0$ and $\{r(i(k,n))\}_{n \geq 1}$ is a strictly increasing sequence of positive integers, we may apply hypothesis (b) to obtain that (3) tends to 0 when n tends to ∞. Therefore, we have

$$\lim_{n \to \infty} \left\| T^{r(i(k,n))} \sum_{m=1}^{\infty} \alpha_m z_m - x_n \right\| = 0.$$

Since the sequence $\{T^{r(i(k,n))} z\}_{n \geq 1}$ is near enough to $\{x_n\}_{n \geq 1}$ and we can extract from the latter a subsequence converging to any $x' \in \mathcal{B}$, we have that we can extract from the former a subsequence converging to any $x' \in \mathcal{B}$; this proves the hypercyclicity of z. Therefore, the proof of the theorem will be concluded once we have proved the claim.

Proof of the Claim. The proof of (i)–(iii) will be by induction on $i = i(m, n)$. In order to save some notation we set $r(0) = 0$, $T^0 = $ identity on $\mathcal{B}$, and $x_{m,1} = e_m$ for every $m \geq 1$. In addition, we will sometimes denote $r(i(m, n))$ by $r(i)$. For $i(1, 1) = 1$, by applying hypotheses (a) and (b) we may find a positive integer $r(i(1, 1))$ such that

$$\|T^{r(i(1,1))}x_{1,1}\| < \frac{\varepsilon_1}{2^{i(1,1)+1}} \quad \text{and} \quad \|S^{r(i(1,1))}x_1\| < \frac{\varepsilon_1}{2^{i(1,1)+1}}.$$

We define $z_{1,1} = x_{1,1} + S^{r(i(1,1))}x_1$. Now suppose that we have already chosen $z_{m',n'}$ and $x_{m',n'}$ for all $i(m',n') < i(m,n)$. Then $z_{m,n}$ and $x_{m,n}$ are chosen as follows.

Case 1. If $n = 1$, then by hypothesis again we may find a positive integer $r(i(m,1))$ such that

$$\|T^{r(i(m,1))}x_{m',1}\| < \frac{\varepsilon_m}{2^{i(m,1)+1}} \quad \text{for } 1 \le m' \le m, \tag{4}$$

$$\|T^{r(i(m,1))}x_{m',n'}\| < \frac{\varepsilon_m}{2^{i(m,1)+2}}$$

$$\text{for } i(1,m) \le i(m',n') \le i(m,1), \tag{5}$$

$$\|T^{r(i(m,1))-r(i(m',n'))}x_{n'}\| < \frac{\varepsilon_m}{2^{i(m,n)+2}}$$

$$\text{for } i(1,m) \le i(m',n') < i(m,1), \tag{6}$$

and

$$\|S^{r(i(m,1))-r(j)}x_1\| < \frac{\varepsilon_m}{2^{i(m,1)+1}} \quad \text{for } 0 \le j < i(m,1). \tag{7}$$

Hence, we define $z_{m,1} = x_{m,1} + S^{r(i(m,1))}x_1$.

Case 2. If $n > 1$, then we have already constructed $z_{m,n-1}$. Since X is a dense subset and $\{T^{r(j)}; 0 \le j < i(m,n)\}$ is equicontinuous, we may find an element $x_{m,n} \in X$ such that

$$\|T^{r(j)}(x_{m,n} - z_{m,n-1})\| < \frac{\varepsilon_m}{2^{i(m,n)+1}} \quad \text{for } 0 \le j < i(m,n). \tag{8}$$

By applying the hypothesis again, we find a positive integer $r(i(m,n))$ such that

$$\|T^{r(i(m,n))}x_{m',1}\| < \frac{\varepsilon_{m+n-1}}{2^{n+1}} \quad \text{for } 1 \le m' < m+n-1, \tag{9}$$

$$\|T^{r(i(m,n))}x_{m',n'}\| < \frac{\varepsilon_m}{2^{i(m,n)+2}}$$

$$\text{for } i(m,n-1) < i(m',n') \le i(m,n), \tag{10}$$

$$\|T^{r(i(m,n))-r(i(m',n'))}x_{n'}\| < \frac{\varepsilon_m}{2^{i(m,n)+2}}$$

$$\text{for } i(m,n-1) < i(m',n') < i(m,n), \tag{11}$$

and

$$\|S^{r(i(m,n))-r(j)}x_n\| < \frac{\varepsilon_m}{2^{i(m,n)+1}} \quad \text{for } 0 \le j < i(m,n). \tag{12}$$

Again, we define $z_{m,n} = x_{m,n} + S^{r(i(m,n))}x_n$. From (10), (11), and the triangle inequality, and recalling that $TS = $ identity on X, for $n > 1$ we have

$$\|T^{r(i(m,n))}z_{m',n'}\| = \|T^{r(i(m,n))}x_{m',n'} + T^{r(i(m,n))-r(i(m',n'))}x_{n'}\|$$

$$\le \|T^{r(i(m,n))}x_{m',n'}\| + \|T^{r(i(m,n))-r(i(m',n'))}x_{n'}\|$$

$$< \frac{\varepsilon_m}{2^{i(m,n)+1}} \quad \text{for } i(m,n-1) < i(m',n') < i(m,n). \tag{13}$$

Similarly, from (5) and (6), for $n = 1$ we have

$$\|T^{r(i(m,1))}z_{m',n'}\| < \frac{\varepsilon_m}{2^{i(m,n)+1}} \quad \text{for} \quad i(1,m) \leq i(m',n') < i(m,1). \tag{14}$$

From (8), (12), and the triangle inequality, we have the following estimation:

$$\|T^{r(j)}(z_{m,n} - z_{m,n-1})\| = \|T^{r(j)}(x_{m,n} - z_{m,n-1}) + S^{r(i(m,n))-r(j)}x_n\|$$

$$\leq \|T^{r(j)}(x_{m,n} - z_{m,n-1})\| + \|S^{r(i(m,n))-r(j)}x_n\|$$

$$< \frac{\varepsilon_m}{2^{i(m,n)}} \quad \text{for} \quad 0 \leq j < i(m,n). \tag{15}$$

By using (15) (for $r(0) = 0$), we find that for each $m \geq 1$ the sequence $\{z_{m,n}\}_{n \geq 1}$ is convergent to a vector $z_m \in \mathcal{B}$. These vectors can be written, for any $n \geq 1$, as

$$z_m = z_{m,n} + \sum_{k=n}^{\infty}(z_{m,k+1} - z_{m,k}). \tag{16}$$

Hence, by using (16), (7) (for $r(0) = 0$), and (15) (for $r(0) = 0$), we have

$$\|z_m - e_m\| = \left\|z_{m,1} - e_m + \sum_{k=1}^{\infty}(z_{m,k+1} - z_{m,k})\right\|$$

$$\leq \|S^{r(i(m,1))}x_1\| + \sum_{k=1}^{\infty}\|z_{m,k+1} - z_{m,k}\|$$

$$< \frac{\varepsilon_m}{2^{i(m,1)+1}} + \sum_{k=1}^{\infty}\frac{\varepsilon_m}{2^{i(m,k+1)}}$$

$$< \frac{\varepsilon_m}{2^{i(m,1)}}$$

$$< \varepsilon_m$$

for every $m \geq 1$; thus we have (iii).

Now, let us see that z_m is hypercyclic for T. Recalling that $TS =$ identity on X, we have the equality below. The second inequality is obtained from (15) ($i(m,n) < i(m, j+1)$ for every $j \geq n$) and (10) (or (5), if $n = 1$):

$$\|T^{r(i(m,n))}z_m - x_n\| = \left\|T^{r(i(m,n))}x_{m,n} + \sum_{k=n}^{\infty}T^{r(i(m,n))}(z_{m,k+1} - z_{m,k})\right\|$$

$$\leq \|T^{r(i(m,n))}x_{m,n}\| + \sum_{k=n}^{\infty}\|T^{r(i(m,n))}(z_{m,k+1} - z_{m,k})\|$$

$$< \frac{\varepsilon_m}{2^{i(m,n)+1}} + \sum_{k=n}^{\infty}\frac{\varepsilon_m}{2^{i(m,k+1)}}$$

$$< \frac{\varepsilon_m}{2^{i(m,n)}}$$

$$< \frac{\varepsilon_m}{2^n}.$$

Therefore, for every $m \geq 1$, we have

$$\lim_{n \to \infty} \|T^{r(i(m,n))} z_m - x_n\| = 0.$$

For every $m \geq 1$, $T^{r(i(m,n))} z_m$ is near enough to $\{x_n\}_{n \geq 1}$, so we may conclude as before that z_m is hypercyclic for T for every $m \geq 1$. This is (i).

It remains to prove (ii). Toward this end, we fix any two positive integers $k \geq 1$ and $n \geq 1$. If $m \geq 1$ ($m \neq k$) is given, then either $m \geq n + k - 1$ or $1 \leq m < n + k - 1$. If $m \geq n + k - 1$ and $m \neq k$, then $i(m, j) \geq i(m, 1) > i(k, n)$ for every $j \geq 1$, so we have

$$\|T^{r(i(k,n))}(z_m - e_m)\| = \left\| S^{r(i(m,1)) - r(i(k,n))} x_1 + \sum_{j=1}^{\infty} T^{r(i(k,n))}(z_{m,j+1} - z_{m,j}) \right\|$$

$$\leq \|S^{r(i(m,1)) - r(i(k,n))} x_1\| + \sum_{j=1}^{\infty} \|T^{r(i(k,n))}(z_{m,j+1} - z_{m,j})\|$$

$$< \frac{\varepsilon_m}{2^{i(m,1)+1}} + \sum_{j=1}^{\infty} \frac{\varepsilon_m}{2^{i(m,j+1)}}$$

$$< \frac{\varepsilon_m}{2^{i(m,1)}}$$

$$< \frac{\varepsilon_m}{2^n}.$$

We have used (7) and (15) in the second inequality above.

If $1 \leq m < n + k - 1$, then the number defined as

$$s = \begin{cases} n + k - m & \text{if } 1 \leq m < k, \\ n + k - m - 1 & \text{if } k < m < n + k - 1, \end{cases} \tag{17}$$

is the unique positive integer such that $i(m, s) < i(k, n) < i(m, s + 1)$. Since s satisfies $i(k, n - 1) < i(m, s) < i(k, n)$ if $n > 1$ (or $i(1, k) \leq i(m, s) < i(k, 1)$ if $n = 1$), we may apply (13) (or (14)) and (15) ($i(m, j + 1) > i(k, n)$ for every $j \geq s$) in the second inequality below. Thus we have

$$\|T^{r(i(k,n))} z_m\| = \left\| T^{r(i(k,n))} z_{m,s} + \sum_{j=s}^{\infty} T^{r(i(k,n))}(z_{m,j+1} - z_{m,j}) \right\|$$

$$\leq \|T^{r(i(k,n))} z_{m,s}\| + \sum_{j=s}^{\infty} \|T^{r(i(k,n))}(z_{m,j+1} - z_{m,j})\|$$

$$< \frac{\varepsilon_m}{2^{i(m,s)+1}} + \sum_{j=s}^{\infty} \frac{\varepsilon_m}{2^{i(m,j+1)}}$$

$$< \frac{\varepsilon_m}{2^{i(m,s)}}$$

$$< \frac{\varepsilon_m}{2^{n+1}}.$$

In the last inequality we have applied the fact that if $n > 3$ then, for s defined as in (17), we have $i(m, s) \geq i(k + 1, n - 2) \geq i(2, n - 2) > n + 1$. It is easy to see that this inequality is true for $n = 1, 2$ and for $n = 3$ it is also true, except if $k = 1$ and $m = 2$ (but this is unimportant). Therefore, by using (9) or (4),

$$\|T^{r(i(k,n))}(z_m - e_m)\| \leq \|T^{r(i(k,n))}z_m\| + \|T^{r(i(k,n))}e_m\| < \frac{\varepsilon_m}{2^{n+1}} + \frac{\varepsilon_{k+n-1}}{2^{n+1}} < \frac{\varepsilon_m}{2^n}.$$

So, in any case we have proved (ii). Hence (i)–(iii) are fulfilled and the proof of the claim is completed. Therefore, the theorem is proved.   □

In a similar way as in [GS], we have the following remarks.

REMARK 1.   One can generalize the concept of hypercyclicity by using sequences of operators more general than the positive powers of a fixed operator. The resulting concept is generally called *universality*. In fact, the proof of Theorem 2.2 also works for this more general situation. More precisely, suppose $\{T_n\}_{\geq 1}$ is a sequence of continuous linear operators for which $T_n \to 0$ pointwise in a dense subset X of $\mathcal{B}$ such that, for each n, the operator T_n has a right inverse S_n and $S_n \to 0$ pointwise on X. Suppose also there is an infinite-dimensional Banach space $\mathcal{B}_0$ such that $T_n \to 0$ on $\mathcal{B}_0$. Then there is an infinite-dimensional Banach space $\mathcal{B}_1$ such that each of its elements, except for zero, is universal; that is, $\{ T_n z : n \geq 1 \}$ is dense for every $z \in \mathcal{B}_1 \setminus \{0\}$.

REMARK 2.   Theorem 2.2 and Remark 1 continue to hold if we replace condition (a) by the following two conditions.

(a1)   There is a dense subset $X \subset \mathcal{B}$ such that $\|T^n x\| \to 0$ for every $x \in X$.

(a2)   There is a dense subset $Y \subset \mathcal{B}$ and a map $S: Y \to Y$ (possibly discontinuous) such that $TS = $ identity on Y and $\|S^n y\| \to 0$ for every $y \in Y$.

The proof would involve much more notation. However, the applications in the next section do not require this generality.

3. Composition Operators and Backward Shifts

Throughout this section, $\mathbb{D}$ will stand for the open unit disk of the complex plane. We denote by $L^p(\partial \mathbb{D})$ $(1 \leq p < \infty)$ the Banach space of the complex functions on the boundary of the unit disk $\partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}$ for which the norm

$$\|f\|_p = \left(\int_{\partial \mathbb{D}} |f(e^{i\theta})|^p \, d\mu \right)^{1/p}$$

is finite. Here, $d\mu = \frac{1}{2\pi} d\theta$ is the normalized Lebesgue measure. The Hardy space $H^p(\mathbb{D})$ will be the space of the analytic functions on $\mathbb{D}$ whose boundary values are in $L^p(\partial \mathbb{D})$. The Hardy space will be regarded as a closed subspace of the latter.

If $\varphi: \mathbb{D} \to \mathbb{D}$ is an automorphism of the unit disk, then the Littlewood subordination theorem (see [Zh, Thm. 10.4.2 p. 220]) asserts that the corresponding composition operator C_φ which assigns to each function $f \in H^p(\mathbb{D})$ the function $C_\varphi(f) = f \circ \varphi$ takes $H^p(\mathbb{D})$ boundedly into itself. We denote by φ_0 the identity on $\mathbb{D}$ by $\varphi_n = \varphi \circ \varphi_{n-1}$ the subsequent iterates of φ; finally, their corresponding inverses $(\varphi_n)^{-1} = \varphi_n^{-1}$ will be denoted by φ_{-n}. The connection between the iterates and composition operators is the equation $C_{\varphi_n} = C_\varphi^n$.

Recall that a non-Euclidean rotation is an automorphism of the unit disk $\mathbb{D}$ that has a fixed point in $\mathbb{D}$. It is known that if φ is not a non-Euclidean rotation then there

is a hypercyclic vector for the corresponding composition operator (see [BS2] or [Sh]). We will prove that in fact there is an infinite-dimensional Banach space of them. More precisely, we will prove the following theorem.

THEOREM 3.1. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an automorphism of the unit disk that is not a non-Euclidean rotation. Then there is an infinite-dimensional Banach subspace $\mathcal{B}_1$ such that each $f \in \mathcal{B}_1 \setminus \{0\}$ is hypercyclic for C_φ.*

We will show this theorem by seeing that the hypotheses of Theorem 2.2 are satisfied. It is known that composition operators under the hypothesis of Theorem 3.1 satisfy the hypercyclicity criterion (condition (a) of Theorem 2.2) and can be found in [BS2] or [Sh]. In fact, this is what proves that these composition operators have hypercyclic vectors. We reproduce the proof here, not only for the sake of completeness but also because it sheds light on where to look for the space $\mathcal{B}_0$, which is required by condition (b) of Theorem 2.2.

To verify (a) of Theorem 2.2, we follow the lines of [Sh, pp. 110–111]. An automorphism of the unit disk that is not a non-Euclidean rotation either fixes a point on the boundary of the unit disk (in which case it is called a non-Euclidean *limit rotation*) or it fixes two points on the boundary of the unit disk (in which case it is called a non-Euclidean *translation*). If it is a non-Euclidean translation then there is a unique attractive fixed point $\alpha \in \partial\mathbb{D}$, known as the Denjoy–Wolff point. The other fixed point $\beta \in \partial\mathbb{D}$ is the attractive fixed point of the inverse of φ^{-1} (see [Bu] for details). For every $\xi \in \partial\mathbb{D} \setminus \{\beta\}$ we have $\varphi_n(\xi) \to \alpha$, and for every $\xi \in \partial\mathbb{D} \setminus \{\alpha\}$ we have $\varphi_{-n}(\xi) \to \beta$. Let X denote the set of functions which are continuous on the closed unit disc, analytic on the interior, and which vanish at α and β. We claim that $C_\varphi^n \to 0$ on X. If $f \in X$, then $f(\varphi_n) \to f(\alpha) = 0$. An application of the Lebesgue bounded convergence theorem yields the desired result:

$$\|C_\varphi^n(f)\|_p^p = \int_{\partial\mathbb{D}} |f(\varphi_n(e^{i\theta}))|^p \, d\mu \to 0 \quad (n \to \infty).$$

Now C_φ is invertible in $H^p(\partial\mathbb{D})$: its inverse is $C_{\varphi^{-1}}$. The same argument gives that $C_{\varphi^{-1}}^n \to 0$ on X. It remains to see that X is a dense subset of $H^p(\mathbb{D})$. This follows from the fact that X contains the subspace generated by $z^n(z - \alpha)(z - \beta)$ and Beurling's approximation theorem (see [Du, Cor. 1, p. 114]). The case where φ has just one fixed point α on $\partial\mathbb{D}$ is easier. Take X to be the set of continuous functions on the closed unit disk, analytic in the interior, and which vanish at α. In this case $\varphi_n(\xi) \to \alpha$ as well as $\varphi_{-n}(\alpha) \to \alpha$ for every $\xi \in \partial\mathbb{D}$, and again the same arguments apply.

It remains to see that composition operators under the hypothesis of Theorem 3.1 satisfy condition (b) of Theorem 2.2. This will be a consequence of Lemma 3.2 and Lemma 3.3.

LEMMA 3.2. *Let $f \in H^p(\mathbb{D})$ be a function that is continuous at a point $\alpha \in \partial\mathbb{D}$ with $f(\alpha) = 0$, and let φ be an automorphism of the unit disk such that its Denjoy–Wolff point is α. Then $\lim_{n\to+\infty} \|f(\varphi_n)\|_p = 0$.*

Proof. Without loss of generality, we may suppose that $\alpha = 1$. Since φ is an automorphism that fixes a point on the unit circle, φ is either a non-Euclidean limit rotation or a non-Euclidean translation. First, we consider the case where φ is a limit rotation. To obtain a clear image of the action of the iterates $\{\varphi_n\}$ on the unit circle, we write $w = \varphi(z)$ as follows:

$$\frac{1}{w-1} = \frac{1}{z-1} + bi, \quad b \neq 0 \text{ and real.} \tag{1}$$

That is, the limit rotation may be viewed as the sequence of transformations starting with an inversion taking $1 \mapsto \infty$ and $|z| < 1$ into a half plane $\Re z < -\frac{1}{2}$. Then comes a translation through b parallel to the imaginary axis, and finally an inversion taking back the half plane to the unit disk with $\infty \mapsto 1$. From now on, we assume that $b > 0$ (the case $b < 0$ can be handled analogously).

Equation (1) allows us compute $\varphi_n(z)$ for every integer $n \in \mathbb{Z}$:

$$\varphi_n(z) = \frac{(1 + nbi)z - nbi}{nbiz + 1 - nbi}.$$

Now, for each integer $k \in \mathbb{Z}$, we consider the intervals on the boundary of $\Re z < -\frac{1}{2}$ defined by $I_k = \{z : z = -\frac{1}{2} + ibt \text{ with } t \in [-\infty, k)\}$ and $J_k = \{z : z = -\frac{1}{2} + ibt \text{ with } t \in [k, \infty)\}$; let $\{U_k\}_{k \in \mathbb{Z}}$ and $\{V_k\}_{k \in \mathbb{Z}}$ denote their preimages on the unit circle by the linear transformation $1/(w-1) \mapsto z$. The final (initial) point of U_k (V_k) is

$$w_k = \frac{4k^2b^2 - 1}{1 + 4k^2b^2} - \frac{4kb}{1 + 4k^2b^2}i,$$

which tends to 1 in the counterclockwise sense. Observe that that U_k and V_k are semineighborhoods of the point 1 for every $k \in \mathbb{Z}$ (see Figure 1).
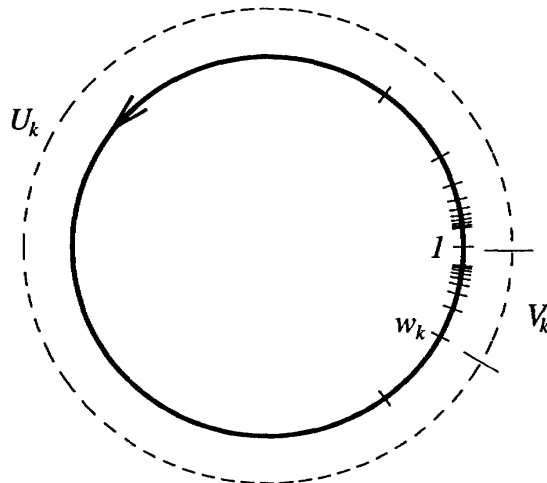

Figure 1. U_k and V_k

By means of equation (1) it is easy to see the following properties:

(a) $U_k \cup V_k = \partial \mathbb{D}$ for every $k \in \mathbb{Z}$;

(b) $\varphi_n(U_k) = U_{k+n}$ and $\varphi_n(V_k) = V_{k+n}$ for every $k, n \in \mathbb{Z}$.

Now, we start properly to prove the lemma. Since f is continuous at the point 1, for all $\varepsilon > 0$ there is a V_k such that $|f(e^{i\theta})|^p < \frac{\varepsilon}{2}$ for every $e^{i\theta} \in V_k$. From now on, this k will be fixed. Since $\varphi_n(V_{k-n}) = V_k$ we have $|f(\varphi_n(e^{i\theta}))|^p < \frac{\varepsilon}{2}$ for every $e^{i\theta} \in V_{k-n}$. Therefore, we have

$$\int_{V_{k-n}} |f(\varphi_n(e^{i\theta}))|^p \, d\mu < \frac{\varepsilon}{2}\mu(V_{k-n}) < \frac{\varepsilon}{2}. \tag{2}$$

So, by property (a) we need only prove

$$\int_{U_{k-n}} |f(\varphi_n(e^{i\theta}))|^p \, d\mu < \frac{\varepsilon}{2} \tag{3}$$

for n large enough. The change of variables $e^{it} = \varphi_n(e^{i\theta})$ gives

$$\int_{U_{k-n}} |f(\varphi_n(e^{i\theta}))|^p \, d\mu(\theta) = \int_{\varphi_n(U_{k-n})} |f(e^{it})|^p |\varphi'_{-n}(e^{it})| \, d\mu(t)$$

$$= \int_{U_k} |f(e^{it})|^p |\varphi'_{-n}(e^{it})| \, d\mu(t). \tag{4}$$

An elementary computation shows that, for each $n \geq 1$, the function

$$|\varphi'_{-n}(e^{it})| = \frac{1}{|-nbie^{it} + 1 + nbi|^2} = \frac{1}{1 + n^2b^2(1 - \cos t) + 2nb \sin t}$$

attains its maximum and its minimum at the points

$$e^{it_0} = \frac{nb}{\sqrt{1 + n^2b^2}} - \frac{i}{\sqrt{1 + n^2b^2}} \quad \text{and} \quad e^{t_1} = -\frac{nb}{\sqrt{1 + n^2b^2}} + \frac{i}{\sqrt{1 + n^2b^2}},$$

which tend to 1 and -1 (respectively) as $n \to +\infty$. Hence, $|\varphi'_{-n}(e^{it})|$ is strictly decreasing on the unit circle interval (e^{it_0}, e^{it_1}) in the counterclockwise sense for every $n \geq 1$ (see Figure 2).
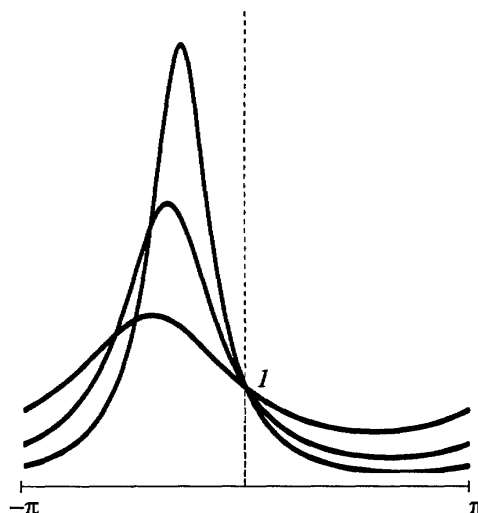

Figure 2. Graph of $|\varphi'_{-n}|$ for some positive values of n

Since $|\varphi'_{-n}(1)| = 1$, we may find an integer j_0 such that $|\varphi'_{-n}(e^{it})| < 1$ for every $t \in U_j$, for every integer $j \le j_0$, and for every $n \ge 1$. On the other hand, since $\mu(U_j)$ tends to 0 as j tends to $-\infty$, we may find an integer $l \le j_0$ such that

$$\int_{U_l} |f(e^{it})|^p \, d\mu(t) < \frac{\varepsilon}{4}.$$

We also observe that $|\varphi'_{-n}(e^{it})|$ tends to 0 uniformly on the interval $U_k \setminus U_l$ as $n \to +\infty$. We can therefore find a positive integer n_0 such that $|\varphi'_{-n}(e^{it})| < \varepsilon/4\|f\|^p$ for every $n \ge n_0$ and for every $t \in U_k \setminus U_l$. So, for n large enough, we have

$$\int_{U_k} |f(e^{it})|^p |\varphi'_{-n}(e^{it})| \, d\mu = \int_{U_l} |f(e^{it})|^p |\varphi'_{-n}(e^{it})| \, d\mu$$

$$+ \int_{U_k \setminus U_l} |f(e^{it})|^p |\varphi'_{-n}(e^{it})| \, d\mu$$

$$< \int_{U_l} |f(e^{it})|^p \, d\mu + \frac{\varepsilon}{4\|f\|^p} \int_{U_k \setminus U_l} |f(e^{it})|^p \, d\mu$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4\|f\|^p} \int_{\partial \mathbb{D}} |f(e^{it})|^p \, d\mu$$

$$= \frac{\varepsilon}{2}.$$

With this, (3) is proved.

From a geometrical viewpoint, the properties of the sequence $\{|\varphi'_{-n}|\}$ are due to the fact that any compact interval on the unit circle that does not contain 1 is contracted by φ_{-n} for large n. A similar observation can be made in the case below.

The case in which φ is a non-Euclidean translation is easier. In addition, we suppose that the other fixed point of φ is -1. Thus we may write $w = \varphi(z)$ as

$$\frac{w-1}{w+1} = \lambda \frac{z-1}{z+1}, \quad 0 < \lambda < 1. \tag{5}$$

This has the effect of mapping $|z| < 1$ onto a half plane $\Re z < 0$ with $1 \mapsto 0$ and $-1 \mapsto \infty$, shrinking by a factor λ, and then mapping back to the unit disk with $0 \mapsto 1$ and $\infty \mapsto -1$. Equation (5) allows us to compute $\varphi_n(z)$ for every integer $n \in \mathbb{Z}$, giving

$$\varphi_n(z) = \frac{(1 + \lambda^n)z + 1 - \lambda^n}{(1 - \lambda^n)z + 1 + \lambda^n}.$$

Now, for each $k \in \mathbb{Z}$, we consider the subsets on the boundary of $\Re z < 0$ defined by $I_k = \{z : z = t\lambda^k i, |t| \ge 1\}$ and $J_k = \{z : z = t\lambda^k i, |t| < 1\}$; let $\{U_k\}_{k \in \mathbb{Z}}$ and $\{V_k\}_{k \in \mathbb{Z}}$ denote their preimages on the unit circle by the linear transformation $(w - 1)/(w + 1) \mapsto z$. Observe that that U_k and V_k are neighborhoods of the points -1 and 1 (respectively) for every $k \in \mathbb{Z}$ (see Figure 3).
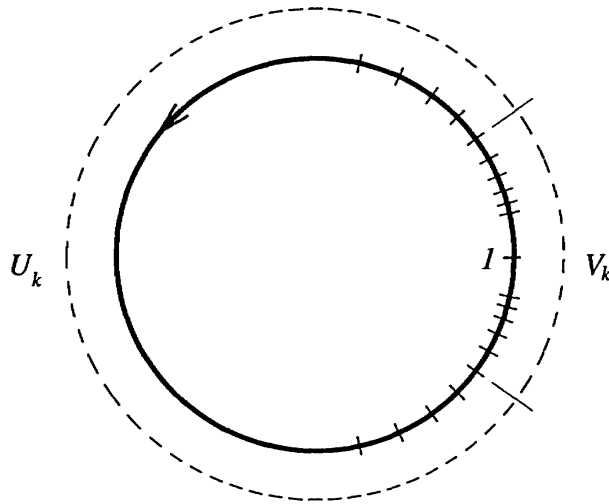
Figure 3. U_k and V_k

By means of equation (5), it is easy to see the properties (a) and (b) above. Analogously as before we may find an integer $k \in \mathbb{Z}$ such that (2) is satisfied, so we need only prove (3) for n large enough. The same change of variables $e^{it} = \varphi_n(e^{i\theta})$ gives (4). Again, an elementary computation shows that, for each $n \geq 1$, the function

$$|\varphi'_{-n}(e^{it})| = \frac{4\lambda^n}{|(\lambda^n - 1)e^{it} + \lambda^n + 1|^2} = \frac{2\lambda^n}{\lambda^{2n} + 1 + (\lambda^{2n} - 1)\cos t}$$

attains its maximum and its minimum at the points 1 and -1, whose values are λ^{-n} and λ^n, respectively (see Figure 4). This time the situation is easier because $|\varphi_{-n}(e^{it})|$ tends uniformly to 0 on U_k, and we may again conclude the assertion of the lemma. $\square$
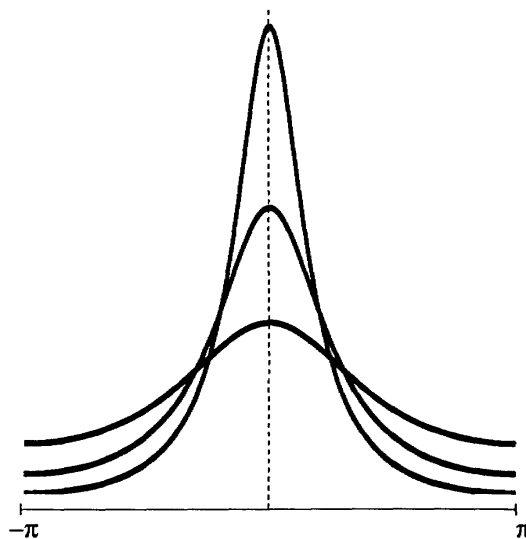

Figure 4.   Graph of $|\varphi'_{-n}|$ for some positive values of n

To prove the following lemma we need the de la Vallée Poussin kernel (see [Ka, pp. 9–15] for details). We summarize here those properties that will be needed. The de la Vallée Poussin kernel is defined as $V_n(t) = 2K_{2n+1} - K_n$, where

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}$$

is Fejer's kernel. Hence V_n is a trigonometrical polynomial of degree $2n + 1$. For each $n \geq 0$, the de la Vallée Poussin kernel defines a bounded linear operator on $H^p(\mathbb{D})$ (and on $L^p(\partial\mathbb{D})$) which is also represented by V_n. This operator assigns to each function $f \in H^p(\mathbb{D})$ the function $V_n(f) = V_n \star f \in H^p(\mathbb{D})$, that is, the convolution of V_n and f. The norm of this operator is $\|V_n\| \leq \|V_n\|_1 \leq 2\|K_{2n+1}\|_1 + \|K_n\|_1 = 3$. The following properties are easily verified:

(i) $V_n \star P = P$ if P is a trigonometrical polynomial of degree less or equal than $n + 1$;

(ii) $V_n \star P = 0$ if $P = \sum_{m=-k_1}^{k_2} a_m e^{imt}$ ($k_1, k_2 \geq 0$) is a trigonometrical polynomial such that $a_m = 0$ for every integer m with $|m| \leq 2n + 1$.

Finally, we say that two trigonometrical polynomials $\sum a_m e^{imt}$ and $\sum b_m e^{imt}$ have disjoint supports if, for each integer j, either $a_j = 0$ or $b_j = 0$.

LEMMA 3.3. *Let α be a point on the unit circle. Then, for all $1 \leq p < \infty$, there is an infinite-dimensional Banach subspace $B_0 \subset H^p(\mathbb{D})$ such that each $f \in B_0$ is continuous at α and $f(\alpha) = 0$.*

Proof. Without loss of generality we suppose that $\alpha = 1$. We consider, for each $n \geq 1$, the function $(1 - z)^n \in H^p(\mathbb{D})$ satisfying, for every p ($1 \leq p < \infty$),

$$\|(1 - z)^n\|_p = \|(1 - z)\|_{pn}^n \geq \|(1 - z)\|_1^n = (4/\pi)^n > 1.$$

We also observe that if we select any positive number a with $0 < a < \frac{\pi}{3}$, then $|1 - e^{it}| \leq b = b(a) = |1 - e^{ia}| < 1$ for $|t| \leq a$. Therefore, if we define for every $n \geq 1$ the function $f_n(z) = (1 - z)^n / \|(1 - z)^n\|_p$, we find that $|f_n(e^{it})| \leq b^n$ for $|t| \leq a$. Of course, each f_n is a trigonometrical polynomial and $\|f_n\|_p = 1$ for every $n \geq 1$. From these polynomials we may construct by induction a sequence of trigonometrical polynomials $\{P_n\}_{n\geq 1}$ and a subsequence V_{r_n} of the de la Vallée Poussin kernel such that:

(a)   each P_n is in $H^p(\mathbb{D})$ with $\|P_n\|_p = 1$ for every $n \geq 1$;

(b)   the supports of these polynomials are pairwise disjoint; and

(c) $V_{r_n} \star P_j = P_j$ for $j \leq n$ and $V_{r_n} \star R_j = 0$ for $j > n$.

Toward this end, we construct subsequences of positive integers $\{m_n\}_{n\geq 1}$ and $\{r_n\}_{n\geq 1}$ in the following way: $m_1 = 1$ and $r_1 + 1 = 2$. Then, for $n > 1$, we define $m_n = \max(m_{n-1} + n, 2r_{n-1} + 2)$ and $r_n + 1 = m_n + n$. (In fact, it is easy to check by induction that $m_n = 2^{n+1} + 2^{n-1} - 2n - 2$.) By defining $P_n(z) = z^{m_n} f_n(z)$, properties (a)–(c) are easily verified. Properties (b) and (c) imply that, for any positive integers $m < n$ and for any choice of scalars $\{a_n\}_{n\geq 1}$,

$$\left\|\sum_{j=1}^{m} a_j P_j\right\|_p = \left\|V_{r_m} \star \sum_{j=1}^{m} a_j P_j\right\|_p = \left\|V_{r_m} \star \sum_{j=1}^{n} a_j P_j\right\| \leq 3 \left\|\sum_{j=1}^{n} a_j P_j\right\|_p.$$

Therefore, $\{P_n\}_{n\geq 1}$ is a normalized basic sequence in $H^p(\mathbb{D})$ (see [Di, Thm. 1, p. 36]). Precisely, V_{r_n} is the nth projection operator and the basis constant is less than 3. It should be observed that in the Hilbert case ($p = 2$) it would not have been necessary to use the de la Vallée Poussin kernel since in this case $P_n(z) = z^{n(n+1)/2-1} f_n(z)$ form an orthonormal system in $H^2(\mathbb{D})$ and (consequently) a basic sequence.

Let $\mathcal{B}_0$ be the closed linear span generated by $\{P_n\}_{n\geq 1}$. Let us see that each $f \in \mathcal{B}_0$ is continuous on $A = \{e^{it} : |t| < \frac{\pi}{3}\}$. If a is any positive number with $0 < a < \frac{\pi}{3}$, then for $|t| \leq a$ we have $|P_n(e^{it})| = |f_n(e^{it})| < b^n$, where $b = |1 - e^{ia}| < 1$. Thus, if $f = \sum_{m=1}^{\infty} \alpha_m P_m \in \mathcal{B}_0$ and recalling that $\{P_n\}$ is a normalized basis, for every $|t| \leq a$ we have

$$|f(e^{it})| = \left|\sum_{m=1}^{\infty} \alpha_m P_m(e^{it})\right| \leq \sum_{m=1}^{\infty} |\alpha_m| |P_m(e^{it})| < 6\|f\|_p \sum_{m=1}^{\infty} b^n = \frac{6b}{1-b}\|f\|_p.$$

So the convergence is uniform on every compact subset $K \subset A$ and, consequently, f is continuous on A. On the other hand, it is obvious that $f(1) = 0$ for every $f \in \mathcal{B}_0$. Therefore, the proof of the lemma, and of Theorem 3.1, is now complete. $\square$

REMARK 1. We have used the de la Vallée Poussin kernel to give explicitly the space $\mathcal{B}_0$. An alternative proof can be given by using functional analysis. First, observe that in order to prove Lemma 3.3 it suffices to extract from $\{f_n\}_{n\geq 1}$ a basic sequence. For $1 < p < \infty$ we have that the sequence $\{f_n\}_{n\geq 1}$ defined in the proof of Lemma 3.3 is bounded and tends uniformly to zero on compact subsets of the unit disk; hence this sequence tends weakly to zero in $H^p(\mathbb{D})$. Therefore, by means of the Bessaga–Pelczinsky selection principle (see [Di, p. 42]), we may extract a basic sequence from $\{f_n\}_{n\geq 1}$. For the case $p = 1$ there is no subsequence of $\{f_n\}_{n\geq 1}$ converging weakly to zero. Since $H^1(\mathbb{D})$ is a weakly sequentially complete space, we thus have that there is no weakly Cauchy subsequence of $\{f_n\}_{n\geq 1}$. The Rosenthal–Dor l_1 theorem (see [Di, p. 201]) therefore allows us to extract from $\{f_n\}_{n\geq 0}$ a subsequence equivalent to the unit vector basis of l_1. Of course, one can avoid the use of these tools by proving directly that for each p ($1 \leq p < \infty$) a basic sequence can be extracted from $\{f_n\}_{n\geq 1}$ equivalent to the unit vector basis of the sequence space l^p. This may be done by taking out a nearly disjointly supported subsequence from $\{f_n\}_{n\geq 1}$.

REMARK 2. It would also be possible to prove a theorem analogous to Theorem 3.1 for those linear fractional transformations that take the unit disk into itself, have no fixed point in $\mathbb{D}$, and are hyperbolic (see [Sh, pp. 5, 114]).

One might think that, given a hypercyclic operator, there is always a whole infinite-dimensional Banach space that consists (except for zero) entirely of hypercyclic vectors. This is not true, as we will see in the next theorem. We denote by l^p the Banach space of sequences of complex numbers for which the norm

$$\|\{a_n\}\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$$

is finite and with c_0 the Banach space of sequences of complex numbers whose limit is zero, endowed with the norm $\|a_n\|_\infty = \sup_n |a_n|$. We recall that the sequence of unit vectors $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the nth place, is a basis for l^p and c_0. In fact, it is an unconditional basis. The backward shift operator on l^p and c_0 relative to this basis is the bounded linear operator B defined by $B(e_n) = e_{n-1}$ if $n > 1$ and $B(e_1) = 0$. Rolewicz [Ro] proved that if λ is any complex number of modulus greater than 1, then the operator λB is hypercyclic. Gethner and Shapiro [GS] proved that this operator satisfies the hypercyclicity criterion (hypothesis (a) of Theorem 2.2) by taking $X = \text{span}(e_n)$. However, we have the following theorem in which the underlying space is l^p or c_0.

THEOREM 3.4. *Let λ be a complex number of modulus greater than 1; then all Banach spaces of hypercyclic vectors for λB are finite-dimensional.*

Proof. Let us suppose that there is an infinite-dimensional Banach space $\mathcal{B}_1$ of hypercyclic vectors. We will construct a vector x of $\mathcal{B}_1$ such that

$$\lim_{j \to \infty} \|(\lambda B)^j x\| = \infty. \tag{6}$$

Hence x cannot be hypercyclic—a contradiction. By induction we may always choose a strictly increasing sequence of nonnegative integers $\{k_n\}_{n \geq 1}$ and a sequence $\{u_n\}_{n \geq 1} \subset \mathcal{B}_1$ satisfying

$$\|u_n\| = 1, \tag{7}$$

$$u_n = \sum_{i=k_n+1}^{\infty} a_{n,i} e_i, \tag{8}$$

$$\left\| \sum_{i=k_{n+1}+1}^{\infty} a_{m,i} e_i \right\| < \frac{3}{2\pi^2} \frac{1}{(n+1)^2} \quad \text{for } m = 1, \ldots, n, \tag{9}$$

where $k_1 = 0$ and u_1 is any vector of $\mathcal{B}_1$ such that $\|u_1\| = 1$. To obtain (8) we consider the mapping $\Pi_{k_n} : \mathcal{B}_1 \to \mathbb{C}^{k_n}$, which sends each vector to its first k_n coordinates with respect to the basis $\{e_i\}_{i \geq 1}$. Since $\mathcal{B}_1$ is infinite-dimensional, this map cannot be one-to-one. Therefore, there must be vectors of $\mathcal{B}_1$ that are sent to the zero vector, so we have vectors satisfying (8). If we normalize we also get (7). Finally, (9) is obtained by choosing k_{n+1} large enough.

Now, from (7), (8), (9), and the reverse triangle inequality, for every $n \geq 1$ we have

$$\left\| \sum_{i=k_n+1}^{k_{n+1}} a_{n,i} e_i \right\| = \left\| u_n - \sum_{i=k_{n+1}+1}^{\infty} a_{n,i} e_i \right\| > 1 - \frac{3}{2\pi^2 (n+1)^2} > \frac{1}{2}. \tag{10}$$

By applying (8) and (9) again, we have

$$\left\| \sum_{m=1}^{n-1} \frac{1}{m^2} \sum_{i=k_n+1}^{k_{n+1}} a_{m,i} e_i \right\| \leq \sum_{m=1}^{n-1} \frac{1}{m^2} \left\| \sum_{i=k_n+1}^{k_{n+1}} a_{m,i} e_i \right\|$$

$$< \frac{3}{2\pi^2 n^2} \sum_{m=1}^{n-1} \frac{1}{m^2} < \frac{1}{4n^2}. \tag{11}$$

In the second inequality, the fact that $\left\| \sum_{i=k_n+1}^{k_{n+1}} a_{m,i} e_i \right\| \leq \left\| \sum_{i=k_n+1}^{\infty} a_{m,i} e_i \right\|$ has been used. We define $x = \sum_{m=1}^{\infty} (1/m^2) u_m$, which clearly is in B_1. Now, for each positive integer j, $k_{n-1} < j \leq k_n$,

$$\|B^j x\| = \left\| \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{i=j+1}^{\infty} a_{m,i} e_i \right\|$$

$$= \left\| \sum_{m=1}^{\infty} \frac{1}{m^2} \left(\sum_{i=j+1}^{k_n} a_{m,i} e_i + \sum_{i=k_n+1}^{k_{n+1}} a_{m,i} e_i + \sum_{i=k_{n+1}+1}^{\infty} a_{m,i} e_i \right) \right\|$$

$$\geq \left\| \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{k_n+1}^{k_{n+1}} a_{m,i} e_i \right\|$$

$$= \left\| \sum_{m=1}^{n} \frac{1}{m^2} \sum_{k_n+1}^{k_{n+1}} a_{m,i} e_i \right\|$$

$$\geq \left\| \frac{1}{n^2} \sum_{k_n+1}^{k_{n+1}} a_{n,i} e_i \right\| - \left\| \sum_{m=1}^{n-1} \frac{1}{m^2} \sum_{k_n+1}^{k_{n+1}} a_{m,i} e_i \right\|$$

$$> \frac{1}{2n^2} - \frac{1}{4n^2} = \frac{1}{4n^2}.$$

We have applied the fact that $\left\| \sum_{i=j+1}^{\infty} a_i e_{i-j} \right\| = \left\| \sum_{i=j+1}^{\infty} a_i e_i \right\|$, the unconditionality of the basis $\{e_i\}$, the reverse triangle inequality, and the inequalities (10) and (11). Therefore, for every positive integer j, $k_{n-1} < j \leq k_n$, we obtain

$$\|(\lambda B)^j x\| > \frac{|\lambda|^j}{4n^2}.$$

Since $|\lambda| > 1$, we have (6) and so are finished. $\qquad\square$

It should be observed that the same argument of Theorem 3.4 shows that the operator λB does not satisfy condition (b) of Theorem 2.2.

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