On Removable Singularities for the Analytic Zygmund Class

Joan Josep Carmona & Juan Jesús Donaire

1. Introduction and Statement of Results

A complex-valued function f defined on the complex plane $\mathbb{C}$ belongs to the Zygmund class ($f \in \Lambda_*$), or quasismooth class, if it is bounded and there exists a positive constant C such that

$$|f(z+h)+f(z-h)-2f(z)| \leq C|h| \tag{1.1}$$

for all $z, h \in \mathbb{C}$.

The boundedness of f and (1.1) imply the continuity of f. We define the Zygmund norm as $\|f\|_* = \|f\|_\infty + \|f\|_{\Lambda_*}$, where $\|f\|_{\Lambda_*}$ denotes the smallest constant C for which (1.1) holds.

We shall call a compact subset K in $\mathbb{C}$ a removable set for the analytic functions of the Zygmund class (resp. Lipschitz class) provided that every function $f \in \Lambda_*$ (resp. $f \in \text{Lip}_\alpha$) that is analytic on $\mathbb{C} \setminus K$ has an analytic extension to the entire plane.

We recall the definition of Hausdorff measure. A measure function is an increasing continuous function $h(t)$, $t \geq 0$, such that $h(0) = 0$. Let E be a bounded set, and for $0 < \delta \leq \infty$ write

$$\Lambda_h^\delta(E) = \inf\left\{ \sum_{j=1}^{\infty} h(\text{diam}(U_j)) : E \subset \bigcup_{j=1}^{\infty} U_j,\ \text{diam}(U_j) \leq \delta \right\}.$$

Since $\Lambda_h^\delta(E)$ is a decreasing function of δ, the limit

$$\Lambda_h(E) = \lim_{\delta \to 0} \Lambda_h^\delta(E) = \sup_{\delta > 0} \Lambda_h^\delta(E)$$

exists; it is called the Hausdorff measure of E with respect to h. For instance, if $h(t) = t^\alpha$ for some $\alpha > 0$, then we will write Λ_α instead of Λ_h. We will denote by m the planar measure Λ_2. If $\delta = \infty$, $\Lambda_h^\infty = M_h$ is called the Hausdorff content with respect to h. From the definitions it follows that $\Lambda_h(E) = 0$ if and only if $M_h(E) = 0$. See [2] for more information.

Dolzenko [1] proved that K is removable for the analytic functions of Lip_α $(0 < \alpha < 1)$ if and only if $\Lambda_{1+\alpha}(K) = 0$. This result is also true for the extreme case $\alpha = 1$, as was proved by Uy [11]. The limit case $\alpha = 0$ corresponds

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to the BMO class, and Kaufman [4] showed that K is removable for analytic functions in BMO if and only if $\Lambda_1(K)=0$.

In this paper we study the problem of the characterization, in metric or geometric terms, of the removable singularities for the analytic Zygmund class. If a compact set has $m(K)>0$ then it is nonremovable for analytic functions in the Zygmund class. To show this, let ν be the restriction of the Lebesgue measure on K. The Cauchy transform of ν,

$$\hat{\nu}(z)=\int_{\mathbb{C}}\frac{1}{w-z}\,d\nu(w), \tag{1.2}$$

belongs to Λ_* [2, p. 80] but $\bar{\partial}\hat{\nu}=-\pi\nu\neq 0$ in the sense of distributions [2, p. 38]; hence $\hat{\nu}$ is not an entire function. In 1986, Uy [13] found a compact set K with $m(K)=0$ nonremovable for Λ_*. On the other hand, using the fact that the modulus of continuity of Zygmund functions is $O(\delta\log(1/\delta))$ [14, p. 44] and the ideas of Dolzenko [1], one can see that K is removable if $\Lambda_h(K)=0$ where $h(t)=t^2\log(1/t)$. We will show that $\Lambda_h(K)=0$ is not a necessary condition. Lord and O'Farrell [6] showed, making use of an example of Kahane and Uy, that there exists a compact set K nonremovable for Λ_* such that $\Lambda_g(K)=0$ with $g(t)=t^2\sqrt{\log(1/t)}$. Recently, Kaufman [5] showed that a porous set is removable for Λ_*. Following [5], a closed set E is called porous (with parameter $a>0$) if, for each $\delta>0$, there is a covering of E by disjoint open disks $D(z_j,r_j)$ such that $r_j<\delta$ and each disk $D(z_j,r_j)$ contains a disk $D(z_j',ar_j)$ disjoint from E. He also proved that, for any measure function h such that $\lim_{t\to 0^+}h(t)t^{-2}=\infty$, there is a porous set of positive Λ_h measure. This means that $\Lambda_h(K)=0$ is not a necessary condition for Λ_*-removability.

We define the following measure function:

$$\psi(t)=t^2\sqrt{\log(1/t)\log\log\log(1/t)} \quad \text{if } 0<t\le\exp\{-e^e\} \text{ and constant if } t\ge\exp\{-e^e\}. \tag{1.3}$$

The function $\psi(t)/t$, involving an iterated logarithm, plays a fundamental role in Makarov's work on Bloch functions and conformal mappings (see [7; 8; 9]).

Our results are the following.

THEOREM 1. *If $K\subset\mathbb{C}$ is a compact set such that $\Lambda_\psi(K)=0$, then K is Λ_*-removable.*

THEOREM 2. *There exists a compact set K_1 with finite Λ_ψ measure such that K_1 is nonremovable.*

THEOREM 3. *There exists a compact set K_2 Λ_*-removable with $\Lambda_\psi(K_2)=\infty$.*

Let K_1 be the compact set given by Theorem 2. Since K_1 has finite Λ_ψ measure, by the comparison lemma between Hausdorff measures [2, p. 60] one has $\Lambda_\varphi(K_1)=0$ for every measure function φ such that $\lim_{t\to 0}(\varphi(t)/\psi(t))=0$.

This means that the function ψ cannot be replaced by a smaller function in Theorem 1. In other words, Theorem 1 gives in some sense the best sufficient condition for Λ_*-removability. Theorem 3, as has been stated, can be obtained from the aforementioned result of Kaufman. We include it here for two reasons. The first is that the proof of the Λ_*-removability of K_2 is helpful in giving a geometrical proof of the fact that a porous compact set is removable. The other reason is that K_2 is not locally porous at any point. Our results leave open the problem of finding a complete characterization of Λ_*-removable sets, but they tell us that the situation is completely different from the Lip_α and BMO cases.

Throughout this paper we will denote by C certain absolute constants, not necessarily the same in each occurrence.

2. Preliminary Results

In proving Theorem 1, we will use some ideas from Makarov's papers [8; 9]. We will always denote by Q_0 the unit cube in $\mathbb{C}$. For each $n \geq 0$, let $\mathfrak{D}_n$ be the family of dyadic squares

$$\left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right) \times \left[\frac{k_2}{2^n}, \frac{k_2+1}{2^n}\right), \quad 0 \leq k_1, k_2 < 2^n. \tag{2.1}$$

By definition, if $R \in \mathfrak{D}_n$ then there are four disjoint squares of $\mathfrak{D}_{n+1}$ whose union is R. We will need several definitions.

Definition 1. A dyadic martingale (S_n), $n \geq 0$, is a sequence of complex functions defined on Q_0 with the following properties:

(1) the function S_n is constant on every square of $\mathfrak{D}_n$; and
(2) if $R \in \mathfrak{D}_n$ then $\int_R S_n \, dm = \int_R S_{n+1} \, dm$.

Condition (2) implies the following important property of martingales:

$$\int_{Q_0} S_n \, dm = \int_{Q_0} S_0 \, dm, \quad n \geq 0. \tag{2.2}$$

One of the main tools in the theory of martingales is the notion of stopping time. We will say that $\tau: Q_0 \to \mathbb{N} \cup \{\infty\}$ is a stopping time with respect to $(\mathfrak{D}_n)$ if $\{\tau = k\}$ is the union of some squares in $\mathfrak{D}_k$ for every k. A crucial theorem [10, p. 73] in the theory of martingales is the following: If (S_n) is a martingale then the sequence $(S_{\tau \wedge n})$, where $\tau \wedge n = \min\{\tau, n\}$, is also a martingale.

Definition 2. Let f be a Lebesgue integrable complex function defined on Q_0. The conditional expectation $E(f \mid \mathfrak{D}_k)$ is the function defined in each square $Q \in \mathfrak{D}_k$ as

$$E(f \mid \mathfrak{D}_k)(z) = \frac{1}{m(Q)} \int_Q f \, dm \quad \text{if } z \in Q. \tag{2.3}$$

The main ingredient in proving Theorem 1 is the following result.

THEOREM A. *Assume that* (S_n) *is a dyadic real-valued martingale defined on* Q_0 *such that* $|S_n - S_{n-1}| \leq 1$ *for all* n *and* $S_0 = c > 0$. *Then*

$$\Lambda_{\psi}(\{w \in Q_0 : S_n(w) > 0 \text{ for all } n\}) > 0,$$

where ψ *is the function defined in* (1.3).

This theorem is a two-dimensional version of Theorem 3.1 in [9]. The kernel of the proof is the original one of Makarov, but it is necessary to introduce some modifications to adapt it to our situation. We will need several lemmas.

LEMMA 1. *Let* (S_n) *be a dyadic real-valued martingale on* Q_0. *Assume that* $|S_n - S_{n-1}| \leq 1$ *for every* n. *Define* $Z_0 = \exp\{S_0\}$ *and*

$$Z_n = \frac{\exp\{S_n\}}{\prod_{k=1}^{n} E(\exp\{S_k - S_{k-1}\} \mid \mathfrak{D}_{k-1})}, \quad n \geq 1. \tag{2.4}$$

Then (Z_n) *is a dyadic martingale and there exists an absolute constant* $\alpha \geq 1/2$ *such that*

$$Z_n^t \geq \exp\{tS_n - \alpha t^2 n\}, \quad t \geq 0, \; n \geq 0, \tag{2.5}$$

where (Z_n^t) *denotes the martingale* (2.4) *associated to* (tS_n).

Proof. Let $R \in \mathfrak{D}_n$. Observe that each function $E(\exp\{S_k - S_{k-1}\} \mid \mathfrak{D}_{k-1})$ is constant on R for $k \leq n$; therefore, statement (1) of Definition 1 holds. Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$, where $R_j \in \mathfrak{D}_{n+1}$. Denote by $a_j = S_{n+1|R_j}$ and $a = S_{n|R}$. The condition (2) for (S_n) is equivalent to the assertion

$$a = \frac{1}{4} \sum_{j=1}^{4} a_j. \tag{2.6}$$

From this we need only verify that

$$e^a = \frac{1}{4} \sum_{j=1}^{4} \frac{e^{a_j}}{E(\exp\{S_{n+1} - S_n\} \mid \mathfrak{D}_n)\,|_R}.$$

By calculations, the last equality is just (2.3).

To prove (2.5) it is necessary to make an upper estimate on the denominator of (2.4). Then (2.5) will follow from the inequality

$$E(\exp\{t(S_k - S_{k-1})\} \mid \mathfrak{D}_{k-1}) \leq \exp\{\alpha E(t^2 (S_k - S_{k-1})^2 \mid \mathfrak{D}_{k-1})\}, \quad k \geq 1. \tag{2.7}$$

By hypothesis and (2.3), one has that $E(t^2(S_k - S_{k-1})^2 \mid \mathfrak{D}_{k-1}) \leq t^2$; thus (2.7) implies (2.5).

Taking into account the martingale condition (2.6), the inequality (2.7) is equivalent to

$$\sum_{j=1}^{4} e^{x_j} \leq 4 \exp\left\{\frac{\alpha}{4} \sum_{j=1}^{4} x_j^2\right\}, \quad \text{provided } \sum_{j=1}^{4} x_j = 0. \tag{2.8}$$

We check (2.8) in the special case where $x_1 = -x_2 = y$ and $x_3 = -x_4 = z$. In this case, the elementary inequality $\cosh x \le \exp\{x^2/2\}$ gives

$$\cosh y + \cosh z \le 2\exp\{(y^2+z^2)/4\},$$

which is (2.8) with $\alpha = 1/2$.

To show the existence of α for which (2.8) is true, we define the function f on the hyperplane $\Pi = \{(x_1, x_2, x_3, x_4) : \sum_{j=1}^4 x_j = 0\}$ as

$$f(0) = \frac{1}{2}, \qquad f(p) = 4\log\left(\frac{1}{4}\sum_{j=1}^{4} e^{x_j}\right)\|p\|^{-2} \quad \text{if } p \ne 0, \tag{2.9}$$

where $p = (x_1, x_2, x_3, x_4)$ and $\|p\|^2 = \sum_{j=1}^4 x_j^2$. Let us see that $\lim_{p\to\infty} f(p) = 0$ and f is continuous. Jensen's inequality states that $f(p) \ge 0$ if $p \in \Pi$, so $\alpha = \max_{p\in\Pi} f(p)$ is well-defined and (2.8) holds for this constant. By the definition (2.9),

$$0 \le f(p) \le \frac{4\max_{1\le j\le 4}|x_j|}{\|p\|^2} \le \frac{4}{\|p\|},$$

so $\lim_{p\to\infty} f(p) = 0$. The inequality $\log(1+x) \le x$ if $x \ge 0$, the fact that $p \in \Pi$, and Taylor's formula for e^x give that

$$\begin{aligned} f(p) - \frac{1}{2} &\le \frac{4(\frac{1}{4}\sum_{j=1}^4 e^{x_j} - 1) - \frac{1}{2}\|p\|^2}{\|p\|^2} \le \frac{\sum_{j=1}^4 (e^{x_j} - 1 - x_j - \frac{1}{2}x_j^2)}{\|p\|^2} \\ &\le \left(\sum_{j=1}^{4} \frac{e^{\xi_j}-1}{2} x_j^2\right)\|p\|^{-2} \le \max_{1\le j\le 4} |e^{\xi_j} - 1| \to 0 \end{aligned} \tag{2.10}$$

as $p \to 0$ because $|\xi_j| \le |x_j|$.

Fix $0 < \delta < 1$. The inequality $\delta x \le \log(1+x)$, if x is small enough, and an analogous argument to the one made in (2.10) give $\delta - 1 \le f(p) - 1/2$. Letting $\delta \to 1$, we obtain the continuity of f. □

After some computations, one can see that $\frac{1}{2} < \alpha < \frac{2}{3}$. It is interesting to point out that the inequality (2.7) does not hold for general martingales.

LEMMA 2. *Let (S_n) be a dyadic real-valued martingale, let $S_0 = c > 0$, and assume that $|S_n - S_{n-1}| \le 1$ for all $n \ge 1$. Put $S_n^* = \max_{1\le j\le n}|S_j|$. If α is the constant in* (2.8), *then:*

(1) *$m(\{\max(S_1, \dots, S_n) \ge a\}) \le \exp\{-(a-c)^2/4\alpha n\}$ if $n \ge 1$ and $a \ge c$;*
(2) *$m(\{S_n^* \ge a\}) \le 2\exp\{-(a-c)^2/4\alpha n\}$ if $n \ge 1$ and $a \ge c$; and*
(3) *$\int_{Q_0} (S_n^*)^{2p}\, dm \le C^p p!\, n^p$ if $n \ge 1$ and $p \ge 0$, where C only depends on c.*

Proof. Let n be fixed. To prove (1), let us consider the stopping time $\tau = \inf\{k : S_k \ge a\}$ if $\{k : S_k \ge a\}$ is nonempty and $\tau = \infty$ otherwise. Put

$$E = \{z : \max\{S_1(z), \dots, S_n(z)\} \ge a\}.$$

It is clear that z belongs to E if and only if $\tau(z) \le n$.

Fix $t > 0$ and consider (Z_k) the martingale (2.4) associated to the martingale (tS_k). Then, by (2.2) and using Fatou's lemma, we obtain

$$e^{tc} = \lim_{k\to\infty} \int_{Q_0} Z_{k\wedge\tau}\, dm \geq \liminf_{k\to\infty} \int_E Z_{k\wedge\tau}\, dm \geq \int_E Z_\tau\, dm. \tag{2.11}$$

The inequality (2.5) and the definition of τ give

$$Z_\tau \geq \exp\{tS_\tau - \alpha t^2 \tau\} \geq \exp\{ta - \alpha t^2 n\} \text{ on } E.$$

Integrating the previous inequality and using (2.11), it follows that

$$e^{tc} \geq m(E) \exp\{ta - \alpha t^2 n\}$$

for all $t > 0$. The best choice of t is $t = (a-c)/2\alpha n$. With this value, the previous inequality gives (1).

Now, by the definition of S_n^*,

$$\begin{aligned} m(\{S_n^* \geq a\}) &\leq m(E) + m(\{\max\{-S_1, \dots, -S_n\} \geq a\}) \\ &= m(E) + m(\{\max\{2c - S_1, \dots, 2c - S_n\} \geq 2c + a\}). \end{aligned}$$

The martingales (S_n) and $(2c - S_n)$ start at c, so we can apply to them the first part of this lemma. Thus

$$m(\{S_n^* \geq a\}) \leq \exp\left\{-\frac{(a-c)^2}{4\alpha n}\right\} + \exp\left\{-\frac{(a+c)^2}{4\alpha n}\right\} \leq 2\exp\left\{-\frac{(a-c)^2}{4\alpha n}\right\}$$

and we have proved (2).

Let us show (3). Computing the integral by means of the distribution function of S_n^*, we obtain

$$\int_{Q_0} (S_n^*)^{2p}\, dm = 2p \int_0^{2c} t^{2p-1} m(\{S_n^* > t\})\, dt + 2p \int_{2c}^\infty t^{2p-1} m(\{S_n^* > t\})\, dt.$$

We estimate the last integral using (2). Hence the previous sum is bounded by

$$C^p + 4p \int_{2c}^\infty t^{2p-1} \exp\left\{-\frac{(t-c)^2}{4\alpha n}\right\} dt \leq C^p + 2^{2p+1} p \int_0^\infty u^{2p-1} \exp\left\{-\frac{u^2}{4\alpha n}\right\} du.$$

With a change of variable, the previous integral can be calculated in terms of the gamma function. Thus

$$\int_{Q_0} (S_n^*)^{2p}\, dm \leq C^p + (16\alpha)^p n^p p \int_0^\infty x^{p-1} e^{-x}\, dx \leq C^p n^p p!,$$

where we can take $C = 2\max\{4c^2, 16\alpha\}$; this gives (3). □

Proof of Theorem A. We consider the stopping time τ defined by $\tau(\omega) = \inf\{n : S_n(\omega) \leq 0\}$ if the set $\{n : S_n(\omega) \leq 0\}$ is nonempty and $\tau(\omega) = \infty$ otherwise. Define

$$\mu_n = S_{n\wedge\tau}\, dm.$$

Since $(S_{n\wedge\tau})$ is a martingale, (2.2) implies that $\mu_n(Q_0) = c$. The fact that $|S_n - S_{n-1}| \leq 1$ tells us that the function $S_{n\wedge\tau}^-$ takes its values in $[0, 1]$. Hence

$$c \le \|\mu_n^+\| = \mu_n(Q_0) + \mu_n^-(Q_0) \le c+1.$$

Let $\mu_\infty \neq 0$ denote some limit measure (in the weak-star topology) of the sequence (μ_n^+). It is clear that μ_∞ is supported on the set $\{\tau = \infty\} = \{w \mid S_n(w) > 0 \text{ for all } n\}$. Now we can follow Makarov's proof [9, p. 26]. His Lemma 3.4, asserting that

$$\int_{Q_0} (S_n^*)^{2p}\, d\mu_\infty \le C^p p!\, n^p, \tag{2.12}$$

is also true in our case. A proof of (2.12) can be obtained by carefully following Makarov's argument. However, since we are working with dyadic martingales on $\mathbb{C}$, we cannot assert that

$$\int S_{(n-1)\wedge\tau}^{2p-2j}(S_{n\wedge\tau} - S_{(n-1)\wedge\tau})^{2j+1}\, dm = 0 \quad \text{for all } j \le p,$$

as he did. Yet the fact that this integral vanishes for $j = 0$, together with inequality (3) of Lemma 2, becomes crucial for obtaining (2.12). From inequality (2.12) and following the proof of Makarov's Corollary 3.5, we obtain

$$\mu_\infty\{S_k < M\Psi(k) \text{ for all } k \ge 4\} > c/4, \tag{2.13}$$

where $\Psi(k) = \sqrt{k \log\log k}$ and $M > 0$ is a well-chosen number.

Denote by $A = \{\tau = \infty\}$ and $B = \{S_k < M\Psi(k) \text{ for all } k \ge 4\}$. Since μ_∞ is supported on A, we have $\mu_\infty(A \cap B) > c/4$ by (2.13). Let R be a dyadic square of $\mathfrak{D}_n$, $n \ge 4$, intersecting $A \cap B$. Then $\tau(\omega) \ge n$ and $S_n(\omega) \le M\Psi(n)$ for each $\omega \in R$. Therefore

$$\begin{aligned}\mu_\infty(R) &= \lim_{k\to\infty}\left(\int_R S_{k\wedge\tau}\, dm + \int_R S_{k\wedge\tau}^-\, dm\right) \le \lim_{k\to\infty}\int_R S_{k\wedge\tau}\, dm + m(R)\\ &= \int_R S_{n\wedge\tau}\, dm + m(R) = \int_R S_n\, dm + m(R)\\ &\le C\Psi(n) m(R) \le C\psi\left(\frac{1}{2^n}\right).\end{aligned}$$

The last inequality implies that $M_\psi(A \cap B)$ is positive and hence $\Lambda_\psi(A \cap B) > 0$. □

Now we define a martingale associated to a function in the Zygmund class. Let $f \in \Lambda_*(\mathbb{C})$ and $R \in \mathfrak{D}_n$, and fix $z \in R$. We then define

$$M_n(f)(z) = 4^n \int_{\partial R} f(w)\, dw. \tag{2.14}$$

LEMMA 3. *The sequence of functions defined by* (2.14) *is a dyadic martingale. Moreover, there exists an absolute constant C such that*

$$|M_{n+1}(f) - M_n(f)| \le C\|f\|_* \quad \textit{for all } n \ge 0. \tag{2.15}$$

Proof. The first assertion follows from the fact, obtained by cancellation of line integrals, that

$$M_n(f)\,|_R = \frac{1}{4}\sum_{j=1}^{4} M_{n+1}(f)\,|_{R_j},$$

where $R \in \mathfrak{D}_n$ and R_j, $1 \le j \le 4$, are the squares of $\mathfrak{D}_{n+1}$ included in R.

To prove the second assertion, let z be a point in Q_0 and let R and R_1 be the squares of $\mathfrak{D}_n$ and $\mathfrak{D}_{n+1}$ (respectively) that contain z. Denote by R_2, R_3, R_4 the other three squares of $\mathfrak{D}_{n+1}$ included in R. Then

$$|M_{n+1}(f)(z) - M_n(f)(z)| = \left| 4^{n+1}\int_{\partial R_1} f(w)\,dw - 4^n \int_{\partial R} f(w)\,dw \right|,$$

and this is bounded by

$$4^n \sum_{j=2}^{4} \left| \int_{\partial R_1} f(w)\,dw - \int_{\partial R_j} f(w)\,dw \right|. \tag{2.16}$$

We will estimate each of the terms of (2.16) in the same way. Call R_0 one of the cubes R_j, $2 \le j \le 4$. Let us denote by c_0 the geometric center of $R_0 \cup R_1$. Observe that the function $T(w) = 2c_0 - w$ maps ∂R_0 onto ∂R_1. Then, by a change of variables, we have

$$\begin{aligned}\left| \int_{\partial R_1} f(w)\,dw - \int_{\partial R_0} f(w)\,dw \right| &= \left| \int_{\partial R_1} (f(w) + f(2c_0 - w))\,dw \right| \\ &= \left| \int_{\partial R_1} (f(w) + f(2c_0 - w) - 2f(c_0))\,dw \right| \\ &\le \|f\|_* \sup_{w \in \partial R_1} |w - c_0| \Lambda_1(\partial R_1) \le \|f\|_* \frac{C}{4^n}.\end{aligned}$$

Inserting the previous inequality in (2.16), we obtain (2.15). □

We will need a useful characterization of the functions in the Zygmund class. It is probably mentioned somewhere, but we have not found any reference and so will prove it here.

PROPOSITION 1. *Let f be a bounded function in $\mathbb{C}$. Then $f \in \Lambda_*(\mathbb{C})$ if and only if there exists a C such that*

$$|(1-t)f(a) + tf(b) - f((1-t)a + tb)| \le C\varphi(t)|b-a| \tag{2.17}$$

for all $0 \le t \le 1$ and for all $a, b \in \mathbb{C}$, where φ denotes the function $\varphi(0) = \varphi(1) = 0$ and $\varphi(t) = \min\{t, 1-t\}\log(1/\min\{t, 1-t\})$ if $0 < t < 1$.

Note that (2.17) implies the continuity of f and gives us the correct modulus of continuity. Taking $t = 1/2$, (2.17) is exactly (1.1).

Proof. Suppose that $f \in \Lambda_*(\mathbb{C})$, and define for $t \in \mathbb{R}$ the function $g(t) = f(a + t(b-a))$. Since $g \in \Lambda_*(\mathbb{R})$ and $\|g\|_{\Lambda_*} \le \|f\|_{\Lambda_*}|b-a|$, the inequality (2.17) will follow from

$$|(1-t)g(0) + tg(1) - g(t)| \le C\|g\|_* \varphi(t), \quad 0 \le t \le 1. \tag{2.18}$$

By the continuity of the functions g and φ, we need to prove (2.18) only for a dyadic point $t \in [0,1]$. Under these reductions, our goal is to show that

$$\left|\left(1-\frac{k}{2^n}\right)g(0)+\frac{k}{2^n}g(1)-g\left(\frac{k}{2^n}\right)\right| \leq C\|g\|_*\varphi\left(\frac{k}{2^n}\right), \tag{2.19}$$

where k is an odd number $k < 2^n$ and $n > 1$. We must distinguish two cases.

Case 1: $1 \leq k < 2^{n-1}$. Then

$$\begin{aligned}\left|\left(1-\frac{k}{2^n}\right)g(0)+\frac{k}{2^n}g(1)-g\left(\frac{k}{2^n}\right)\right| &\leq \frac{k}{2^n}|g(1)-g(0)|+\left|g(0)-g\left(\frac{k}{2^n}\right)\right| \\ &\leq 2\|g\|_*\frac{k}{2^n}+C\|g\|_*\frac{k}{2^n}\log\frac{2^n}{k} \\ &\leq C\|g\|_*\varphi\left(\frac{k}{2^n}\right).\end{aligned}$$

The last inequality follows because $k/2^n < 1/2$ and $t \leq (1/\log 2)t\log(1/t)$ for all $0 \leq t \leq 1/2$.

Case 2: $2^{n-1} < k < 2^n$. Then

$$\begin{aligned}&\left|\left(1-\frac{k}{2^n}\right)g(0)+\frac{k}{2^n}g(1)-g\left(\frac{k}{2^n}\right)\right| \\ &\quad \leq \left(1-\frac{k}{2^n}\right)|g(1)-g(0)|+\left|g(1)-g\left(\frac{k}{2^n}\right)\right| \\ &\quad \leq 2\|g\|_*\left(1-\frac{k}{2^n}\right)+C\|g\|_*\left(1-\frac{k}{2^n}\right)\log\frac{1}{(1-k/2^n)} \leq C\|g\|_*\varphi\left(1-\frac{k}{2^n}\right),\end{aligned}$$

since $1-k/2^n \leq 1/2$. Therefore, (2.19) has been proved. □

If $f \in \Lambda_*(\mathbb{C})$, a more difficult argument improves (2.17) in the sense that it holds also taking $C = 2\|f\|_{\Lambda_*}/\log 2$.

3. Proofs of the Theorems

Proof of Theorem 1. Let K be a compact set such that $\Lambda_\psi(K) = 0$, and suppose that K is not Λ_*-removable. Then there exists a function $f \in \Lambda_*(\mathbb{C})$ analytic on $\mathbb{C}\backslash K$ such that $\int_{\partial Q} f(z)\,dz \neq 0$ for a certain square Q. By a change of variables, we may assume that Q is the unit cube Q_0. Without loss of generality we suppose that

$$\operatorname{Re}\int_{\partial Q_0} f(z)\,dz = c > 0.$$

Let us consider the complex martingale $(M_n(f))$ defined by (2.14), and put $S_n = \operatorname{Re} M_n(f)$. By (2.15), (S_n) has uniformly bounded increments. Moreover, if $z \in Q_0\backslash K$ then there exists n_0 such that $S_n(z) = 0$ for all $n \geq n_0$. Then the set $\{z: S_n(z) > 0 \text{ for all } n \geq 1\}$ is included in K. Hence we can apply Theorem A to obtain

$$\Lambda_\psi(K \cap Q_0) \geq \Lambda_\psi(\{z: S_n(z) > 0 \text{ for all } n \geq 1\}) > 0.$$

The previous statement contradicts the fact that $\Lambda_\psi(K) = 0$. □

The proof of Theorem 2 requires some preliminary definitions and results.

DEFINITION 3. Let μ be a locally finite positive measure on $\mathbb{C}$. We say that μ is a Zygmund measure if

$$|\mu(Q) - \mu(Q')| \leq Cm(Q) \tag{3.1}$$

for any two adjacent squares Q, Q' of the same size.

The inequality (3.1) implies

$$\mu(Q(h)) \leq Ch^2 \log(1/h), \quad 0 < h < 1/2, \tag{3.2}$$

where $Q(h)$ is any square of side length h. The proof of (3.2) can be obtained with an analogous argument to the study of the modulus of continuity of a Zygmund function [14, p. 44]. Therefore, by (3.2), the Cauchy transform $\hat{\mu}$ of μ, defined by (1.2), is continuous on the whole plane because its modulus of continuity is $O(\delta \log^2(1/\delta))$ [2, p. 76]. We will need also the fact that the Cauchy transform of a Zygmund measure is a function that belongs to $\Lambda_*(\mathbb{C})$ [12].

Now we recall the construction due to Kahane [3] of a compact set, which has become a source of counterexamples. Given $x \in [0,1]$, consider its 4-adic development:

$$x = \sum_{j=1}^{\infty} \frac{x_j}{4^j}, \quad x_j \in \{0, 1, 2, 3\}.$$

We define the independent Bernouillian random variables ϵ_n as follows: If $x_n = 0$ or 3 then $\epsilon_n(x) = -1$; define $\epsilon_n(x) = 1$ otherwise. Let us consider the 4-adic martingale $S_n(x) = 1 + \sum_{j=1}^n \epsilon_j(x)$. The compact set is

$$K = \{x: S_k(x) > 0 \text{ for all } k \geq 1\}.$$

This compact set is usually known by the name of Kahane's compact set. It has the remarkable property, proved by Kahane [3], that K supports a positive singular Zygmund measure μ_K on $\mathbb{R}$. This measure is the limit (in the weak-star topology) of the sequence of probability measures

$$\mu_n = S_{n \wedge \tau}\, dx,$$

where τ is the stopping time $\tau(s) = \inf\{n \geq 1: S_n(s) = 0\}$.

Makarov [7] made a careful study of such compact set and was able to prove that it has finite Λ_ϕ measure, where $\phi(t) = \psi(t)/t$ and ψ is the function defined by (1.3).

Proof of Theorem 2. Let $K_1 = K \times [0,1]$ and write $\nu = \mu_K \times f\,dt$, where dt is the Lebesgue measure on $\mathbb{R}$ and f is a C^1 function with compact support included in $(0,1)$ such that $0 \le f \le 1$. Since $\Lambda_\phi(K) > 0$, we can apply Frostman's theorem [2, p. 62] to find another positive measure $\tilde{\mu} \neq 0$ supported on K such t $\tilde{\mu}(I) \le C\phi(\operatorname{diam} I)$ for all open intervals $I \subset \mathbb{R}$. Let $\tilde{\nu} = \tilde{\mu} \times dt$. If we denote by $Q(z,r)$ the square of center $z = x_1 + ix_2$ and side length $2r$, one has

$$\tilde{\nu}(Q(z,r)) = 2r\tilde{\mu}(x_1 - r, x_1 + r) \le C\psi(2r). \tag{3.3}$$

From (3.3) and using

$$\psi(2r) \le 4\psi(r), \tag{3.4}$$

it follows that $\Lambda_\psi(K_1) > 0$.

To show that K_1 has finite Λ_ψ measure, fix $\epsilon > 0$ and $0 < \delta < 1$. Then there exists a δ-covering (U_j) of K such that

$$\sum_{j=1}^{\infty} \phi(\operatorname{diam} U_j) \le \Lambda_\phi(K) + \epsilon. \tag{3.5}$$

For each j, let us consider the half-open intervals $I_j^k = [k\delta_j, (k+1)\delta_j)$, $0 \le k \le [1/\delta_j]$, where $\delta_j = \operatorname{diam} U_j$. Considering the $\sqrt{2}\delta$-covering $(U_j \times I_j^k)$ and using (3.4) and (3.5), one has

$$\begin{aligned}\Lambda_\psi^{\sqrt{2}\delta}(K_1) &\le \sum_{j,k} \operatorname{diam}(U_j \times I_j^k)\phi(\operatorname{diam}(U_j \times I_j^k)) \\ &\le \sum_j \sqrt{2}\delta_j\left(\left[\frac{1}{\delta_j}\right] + 1\right)\phi(2\delta_j) \le C\sum_j \phi(2\delta_j) \le C\Lambda_\phi(K) + C\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ and $\delta \to 0$ in the last inequality, we obtain $\Lambda_\psi(K_1) \le C\Lambda_\phi(K)$, so $\Lambda_\psi(K_1) < \infty$.

Now we will see that ν is a Zygmund measure. Consider two adjacent squares Q and Q' of side length $l \le 1/2$. First suppose that $Q = I \times J$ and $Q' = I' \times J$. Since μ_K is a Zygmund measure on $\mathbb{R}$, we have

$$|\nu(Q) - \nu(Q')| = |\mu_K(I) - \mu_K(I')| \int_J f\,dt \le Cl^2 = Cm(Q).$$

On the other hand, if $Q = I \times J$ and $Q' = I \times J'$ then

$$|\nu(Q) - \nu(Q')| = \mu_K(I)\left|\int_J f\,dt - \int_{J'} f\,dt\right| = \mu_K(I)\,l\,|f(\zeta) - f(\zeta')|$$

for some points $\zeta \in J$ and $\zeta' \in J'$. Using the estimate $\mu_K(I) = O(l\log(1/l))$ and the fact that f is C^1, we obtain

$$|\nu(Q) - \nu(Q')| \le Cl^3 \log(1/l) \le Cm(Q);$$

therefore ν is a Zygmund measure.

Finally, let h be the Cauchy transform of the measure ν. By Uy's theorem [12], h is a Zygmund function. Also, h is holomorphic on $\mathbb{C}\setminus K_1$, bounded and continuous on $\mathbb{C}$, but it is nonconstant since $h'(\infty) = \lim_{z\to\infty} zh(z) = -\nu(K_1) \neq 0$. □

Proof of Theorem 3. For $0 \le t \le 1$ we define the function φ as

$$\varphi(0) = 0, \qquad \varphi(t) = \frac{t}{\sqrt[4]{\log_2(2/t)}} \quad \text{if } t > 0.$$

The function φ is strictly increasing and strictly convex on $[0, 1]$, so

$$2\varphi(2^{-(n+1)}) < \varphi(2^{-n}), \quad n \ge 0. \tag{3.6}$$

Let $\Psi(s) = (\varphi^{-1}(s))^2$, $0 \le s \le 1$. A straightforward computation shows that

$$\lim_{s\to 0} \frac{\Psi(s)}{\psi(s)} = 0. \tag{3.7}$$

Our compact set will be constructed in a similar way as the usual 1/4 planar Cantor set [2, p. 87]. Let us denote by $Q_1^{(0)} = \bar{Q}_0 = E_0$. By induction, let $E_n = \bigcup_{j=1}^{4^n} Q_j^{(n)}$, where $Q_j^{(n)}$ $(1 \le j \le 4^n)$ are all the closed corner squares of the squares $Q_j^{(n-1)}$, $1 \le j \le 4^{n-1}$, with side length $\varphi(2^{-n})$. Note that this construction is possible in every step by (3.6) and since $\varphi(1) = 1$. Our compact set is $K_2 = \bigcap_{n\ge 1} E_n$. Define the probability measures $\mu_n = (1/m(E_n))\, dm|_{E_n}$. Let μ be the limit of the sequence $\{\mu_n\}$ in the weak-star topology. Then $\operatorname{supp}(\mu) \subset K_2$ and $\mu(Q_j^{(n)}) = 4^{-n}$.

Given a small $r > 0$, let n be such that $\varphi(2^{-n-1}) \le r < \varphi(2^{-n})$. Since each square $Q(z, \varphi(2^{-n}))$ intersects at most eight squares $Q_j^{(n+1)}$, we have

$$\mu(Q(z, r)) \le \mu(Q(z, \varphi(2^{-n})) \le 8\cdot 4^{-n} = 8(2^{-n-1})^2 \le C\Psi(r).$$

The last inequality implies $M_\Psi(K_2) > 0$ and so $\Lambda_\Psi(K_2) > 0$. Using (3.7) and the comparison lemma between Hausdorff measures, we have $\Lambda_\psi(K_2) = \infty$.

Now we will see that K_2 is Λ_*-removable. Let f be a Zygmund function analytic outside K_2. First we will show that

$$\int_{\partial Q_j^{(n)}} f(z)\, dz = 0 \quad \text{for all } n, j. \tag{3.8}$$

Taking into account that each $Q_j^{(n)} \cap K_2$ is geometrically similar to $Q_0 \cap K_2$, it is necessary to prove only that

$$\int_{\partial Q_0} f(z)\, dz = 0.$$

Since f is analytic on $\mathbb{C} \setminus K_2$, the Cauchy integral theorem gives

$$\int_{\partial Q_0} f(z)\, dz = \sum_{j=1}^{4^n} \int_{\partial Q_j^{(n)}} f(z)\, dz \quad \text{for all } n.$$

Fix $n \ge 1$ and $1 \le j \le 4^n$. We shall estimate

$$\int_{\partial Q_j^{(n)}} f(z)\,dz.$$

Write $R = Q_j^{(n)}$. Let c be the middle point of the right side of R. For each $z \in \partial R$, define $z' = T(z)$ as the point of $\mathbb{C}$ such that $(1-t)z + tz' = c$ for some $t > 0$. The number t is chosen in such a way that the interior of the square $T(R)$ is included in $\mathbb{C}\setminus K_2$. A suitable selection of t is made taking

$$t = \frac{\varphi(2^{-n})}{\varphi(2^{-n+1}) - \varphi(2^{-n})}.$$

By (3.7) and by the definition of φ, one has that $1/2 < t < 1$. Since f is holomorphic on the interior of $T(R)$, we conclude that

$$\begin{aligned}
\left|\int_{\partial R} f(z)\,dz\right| &= \left|\int_{\partial R} f(z)\,dz - \int_{T(\partial R)} \frac{t^2}{(1-t)^2} f(z')\,dz'\right| \\
&= \left|\int_{\partial R}\left(f(z) + \frac{t}{1-t} f\left(\frac{c}{t} - \frac{1-t}{t}z\right)\right) dz\right| \\
&= \frac{1}{1-t}\left|\int_{\partial R}\left((1-t)f(z) + tf\left(\frac{c}{t} - \frac{1-t}{t}z\right) - f(c)\right) dz\right|.
\end{aligned}$$

The last term can be estimated using Proposition 1 and the fact that $1/2 < t < 1$. Hence

$$\begin{aligned}
\left|\int_{\partial R} f(z)\,dz\right| &\le C \log\frac{1}{1-t} \int_{\partial R} \frac{1}{t}|c-z|\,|dz| \\
&\le C\varphi(2^{-n})^2 \log\frac{\varphi(2^{-n+1}) - \varphi(2^{-n})}{\varphi(2^{-n+1}) - 2\varphi(2^{-n})}.
\end{aligned}$$

Repeating this argument for each $Q_j^{(n)}$ we obtain

$$\begin{aligned}
\left|\int_{\partial Q} f(z)\,dz\right| &\le C4^n\varphi(2^{-n})^2 \log\frac{\varphi(2^{-n+1}) - \varphi(2^{-n})}{\varphi(2^{-n+1}) - 2\varphi(2^{-n})} \\
&\le C\frac{1}{\sqrt{n+1}} \log\frac{\sqrt[4]{n+1} - \frac{1}{2}\sqrt[4]{n}}{\sqrt[4]{n+1} - \sqrt[4]{n}} = O\left(\frac{\log n}{\sqrt{n}}\right),
\end{aligned}$$

which goes to 0 as n goes to infinity. This gives (3.8).

Since f is bounded, f has a removable singularity at ∞. We must prove that f is constant. Fix $z \notin K_2$ and let Q be a big enough square such that Q contains z and Q_0 in its interior. By Cauchy's theorem,

$$f(\infty) - f(z) = \frac{1}{2\pi i}\int_{\partial Q} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = \sum_{j=1}^{4^n} \frac{1}{2\pi i}\int_{\partial Q_j^{(n)}} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta \tag{3.9}$$

for n big enough. The function $g(\zeta) = (f(\zeta) - f(z))/(\zeta - z)$ belongs to $\Lambda_*(\mathbb{C})$ and is analytic outside K_2. Then (3.8) is true with f replaced by g. Therefore the third term of (3.9) vanishes, so $f(z) = f(\infty)$ for $z \notin K_2$. By continuity, f is constant and the proof is finished. □

Now we sketch the proof of the fact that the compact set K_2 is not locally porous at any point of K_2. Given a square $Q_j^{(n)}$, we label the four subsquares of E_{n+1} included in $Q_j^{(n)}$ with $0, 1, 2, 3$ respectively, starting from the lower left square and continuing counterclockwise. With this notation it is clear that every point in K_2 can be identified with a sequence (ϵ_n), where $\epsilon_n \in \{0, 1, 2, 3\}$. Let us consider the set S of points in K_2 corresponding to the sequences (ϵ_n) for which there exists $n_0(z)$ such that $\epsilon_{2n} = 0$ and $\epsilon_{2n+1} = 2$ if $n \geq n_0(z)$. The set S is dense in K_2. Assume that there exist $z \in S$ and $\delta_0 > 0$ such that $K_2 \cap \bar{D}(z, \delta_0)$ is porous with parameter $a > 0$. Let $\delta < \delta_0$ and $\delta < \varphi(1/2^{n_0(z)})$. For this δ, consider the corresponding covering (D_j) given by the porosity. Choose the disk $D_j(r)$ such that $z \in D_j(r)$ and choose a positive integer n such that $\varphi(1/2^{n+1}) \leq r < \varphi(1/2^n)$. By the choice of z, the biggest open disk included in $D_j(r)$ and disjoint from K_2 has radius not greater than $\varphi(1/2^{n-2}) - 2\varphi(1/2^{n-1})$. So, by the definition of porosity, one has

$$a\varphi\left(\frac{1}{2^{n+1}}\right) \leq ar \leq \varphi\left(\frac{1}{2^{n-2}}\right) - 2\varphi\left(\frac{1}{2^{n-1}}\right).$$

Letting $\delta \to 0$, the previous inequality contradicts the fact that

$$\lim_{x \to 0} \frac{\varphi(2x) - 2\varphi(x)}{\varphi(x/4)} = 0.$$

Now we show how the argument devoted to prove the Λ_*-removability of the compact set K_2 gives us a geometric proof of the fact that a porous set is removable.

Proof. Let K be a porous compact set with parameter $0 < a < 1$, and fix $\epsilon > 0$. Since $m(K) = 0$, there exists a covering $D(z_j, \delta_j)$ of K such that $\sum_{j=1}^{\infty} \delta_j^2 < \epsilon$ and each disk $D(z_j, \delta_j)$ contains a disk $D(z_j', a\delta_j)$ disjoint from K. Fix j and let $S = \partial D(z_j, \delta_j)$. Let $f \in \Lambda_*(\mathbb{C})$ be any function analytic outside K. We show that

$$\left| \int_S f(z)\, dz \right| \leq C\delta_j^2,$$

where C depends on a and $\|f\|_*$.

Set $D = D(z_j, \delta_j)$ and $D_1 = D(z_j', a\delta_j)$. By a geometric argument, there exists a point $p \in \bar{D}_1$ such that the homothety $z' = T(z) = p + a(z - p)$ maps S onto $S_1 = \partial D_1$. As in the previous proof, one has

$$\begin{aligned}\int_S f(z)\, dz &= \int_S f(z)\, dz - \frac{1}{a^2} \int_{S_1} f(z')\, dz' \\ &= \frac{1}{a} \int_S (af(z) + (1-a)f(p) - f((1-a)p + az))\, dz.\end{aligned}$$

By Proposition 1, we have

$$\left| \int_S f(z)\, dz \right| \leq \frac{C}{a} \varphi(z) \int_S |z - p||dz| \leq C\delta_j^2.$$

Proceeding in this way for each $D(z_j, \delta_j)$, we obtain

$$\left| \sum_{j=1}^{\infty} \int_{C(z_j,\delta_j)} f(z)\,dz \right| \leq C\epsilon. \tag{3.10}$$

Now complete the proof as before by considering $g(\zeta)=(f(\zeta)-f(z))/(\zeta-z)$, applying (3.10) instead of (3.8) and letting ϵ approach zero. □

REMARK. It is interesting to point out that the methods of our paper can be used to study the characterization of those compact sets K on the real line with the property that there exists a nonconstant function in the real Zygmund class such that it is constant on each component of the complement of K. Then it is possible to prove, in a similar way, results analogous to ours.

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Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Barcelona
Spain