A Family of Meromorphic Univalent Functions

Evelyn M. Pupplo-Cody & T. J. Suffridge

1. Introduction

In this paper, we consider a family of functions that are meromorphic and univalent in the unit disk $\Delta=\{z\in\mathbf{C}: |z|<1\}$ and that have some rather striking geometric properties. These functions all have the form

$$\mu(z)=\frac{1}{z}+\sum_{k=1}^{n} a_k z^k \quad \text{with } |a_n|=\frac{1}{n}. \tag{1}$$

It is well known that if μ given above is univalent in the disk then

$$\begin{aligned}0 &\neq \mu'(z)\\ &= -\frac{1}{z^2}+\sum_{k=1}^{n} k a_k z^{k-1} = -\frac{1}{z^2}\left(1-\sum_{k=1}^{n} k a_k z^{k+1}\right).\end{aligned}$$

Therefore, $|na_n|\le 1$ and equality is possible only if all zeros of $-z^2\mu'(z)$ lie on the circle $\{|z|=1\}$. In that case, $a_{n-1}=0$ and $(k-1)a_{k-1}=-na_n(n-k)\bar{a}_{n-k}$ [1, p. 166; 2, p. 10]. We will call a function given by (1) a *meromorphic polynomial of degree n.*

Figure 1 gives the image of the disk under mappings by some meromorphic polynomials of this type; that is, all zeros of μ' lie on $\{|z|=1\}$. The image of the disk is, of course, the unbounded component shown. The coefficients $a_1, \dots, a_{n-2}$ are given.

The following theorem is the key to many of the results in this paper.

Theorem 1. *If* $\mu(z)=1/z+\sum_{k=1}^{n-2}[(n-k-1)/n]a_k z^k-(1/n)z^n$ *is univalent in the unit disk* $\{|z|<1\}$, *then* $\operatorname{Re}(z\mu''(z)/\mu'(z)+1)=(n-1)/2$ *for each* z, $|z|=1$, *such that* $\mu'(z)\neq 0$.

Proof. Note that, under the hypotheses, all zeros of μ' lie on the circle $\{|z|=1\}$. As noted earlier, this implies that

$$\bar{a}_k=a_{n-k-1} \quad \text{for } 1\le k\le n-2$$

Received May 12, 1993. Revision received July 20, 1994.

Some of the results in this paper are contained in the first author's dissertation, completed in 1992.

Michigan Math. J. 42 (1995).

and

$$-z^2\mu'(z) = P(z) = 1 - \sum_{k=1}^{n-2} \frac{k(n-k-1)}{n} a_k z^{k+1} + z^{n+1}$$

is a "self-inversive" polynomial. That is,

$$P(z) = \prod_{j=1}^{n+1}(1+ze^{i\alpha_j}) \quad \text{with} \quad \prod_{j=1}^{n+1} e^{i\alpha_j} = 1$$

so that $z^{n+1}\overline{P(1/\bar{z})} = P(z)$. If we set $z = e^{i\theta}$, we see that $P(e^{i\theta}) = e^{i(n+1)\theta}\overline{P(e^{i\theta})}$ so that $e^{-i((n+1)/2)\theta}P(e^{i\theta}) = S(\theta)$ is real. Thus, it follows that $e^{i\theta}\mu'(e^{i\theta}) = e^{i((n-1)/2)\theta}T(\theta)$, where $T(\theta) = -S(\theta)$ is real. Now differentiate with respect to θ and divide by $ie^{i\theta}\mu'(e^{i\theta})$ to get

$$\frac{e^{i\theta}\mu''(e^{i\theta})}{\mu'(e^{i\theta})} + 1 = \frac{n-1}{2} - i\frac{T'(\theta)}{T(\theta)}. \tag{2}$$

This completes the proof. □

Let U_n be the family of univalent meromorphic polynomials (i.e., univalent in the unit disk) with $a_{-1} = 1 = n|a_n|$ and with $a_0 = 0$. Of course, $a_{n-1} = 0$ follows from the fact that all zeros of $z^2\mu'(z)$ lie on $\{|z| = 1\}$. This family was mentioned by Brannan [1; 2]. The real-coefficient case was studied by Schnack [10] and the general case by Mansour [7]. Mansour obtained some necessary geometric conditions for μ to be an extreme point of U_n. He found all extreme points in U_n for $1 \le n \le 5$. We require some results of Mansour that parallel results of Suffridge [12] for polynomials. In some cases these results have proofs that are almost identical to the corresponding results in [12]. Generally, we give the basic idea of the proof in these cases.

We will show that $\bigcup_{n=1}^{\infty} U_n$ is dense in the family U of all functions of the form $\mu(z) = 1/z + \sum_{n=1}^{\infty} a_k z^k$ that are univalent in the disk $\{|z| < 1\}$. It is obvious that U is related to the family Σ of functions that are analytic and univalent in $\{|z| > 1\}$ and of the form $\sigma(z) = z + \sum_{k=1}^{\infty} a_k/z^k$, simply by replacing z with $1/z$.

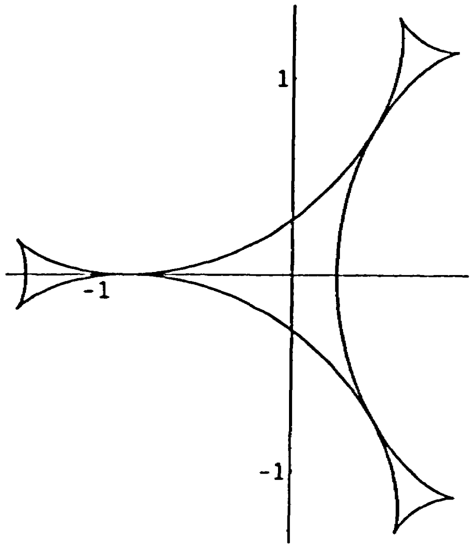
We prove a theorem concerning extreme points in the subclass of U_n consisting of functions that have real coefficients. We also prove Kirwan's conjecture [6, p. 37] that $ka_1 - a_k \le k$ for typically real meromorphic functions with a simple pole of residue 1 at 0 (see also [5] for some results on typically real meromorphic functions). Some of these results are contained in the first author's thesis [8].

2. Previous Results and Other Preliminary Observations

Recall that a function $\mu(z) = 1/z + \sum_{k=1}^{\infty} a_k z^k$ that is analytic in the unit disk is univalent if and only if the equation $[\mu(ze^{i\theta}) - \mu(ze^{-i\theta})]/(2i\sin\theta) = 0$ has

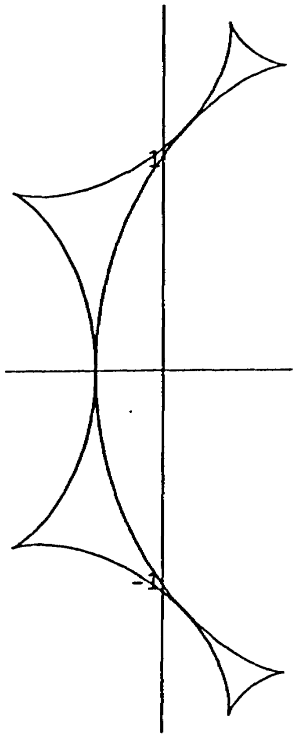
$n = 5$

$$\begin{bmatrix} 0 \\ -0.5656854249 \\ 0 \end{bmatrix}$$



$n = 5$

$$\begin{bmatrix} -0.6 \\ -0.3079201436 \\ -0.2 \end{bmatrix}$$



$n = 7$

$$\begin{bmatrix} 0 \\ 0 \\ 0.4285714286 \\ 0 \\ 0 \end{bmatrix}$$

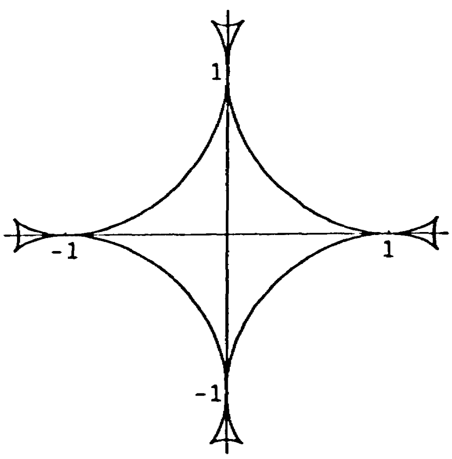
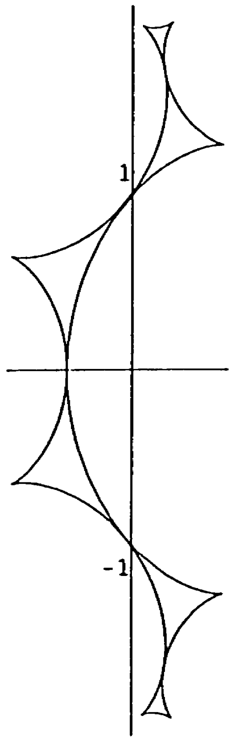


Figure 1a

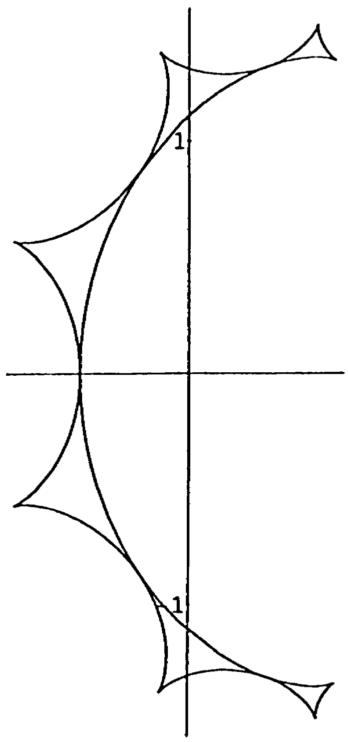
$n = 7$

$$\begin{bmatrix} -0.7186414595 \\ -0.2179341974 \\ 0.0052268942 \\ -0.1089670987 \\ -0.1437282919 \end{bmatrix}$$



$n = 7$

$$\begin{bmatrix} -0.5371104668 \\ -0.3066589278 \\ -0.212610297 \\ -0.1533294639 \\ -0.1074220934 \end{bmatrix}$$



$n = 7$

$$\begin{bmatrix} 0.346629553 \\ 0 \\ -0.4411873935 \\ 0 \\ -0.0693259106 \end{bmatrix}$$

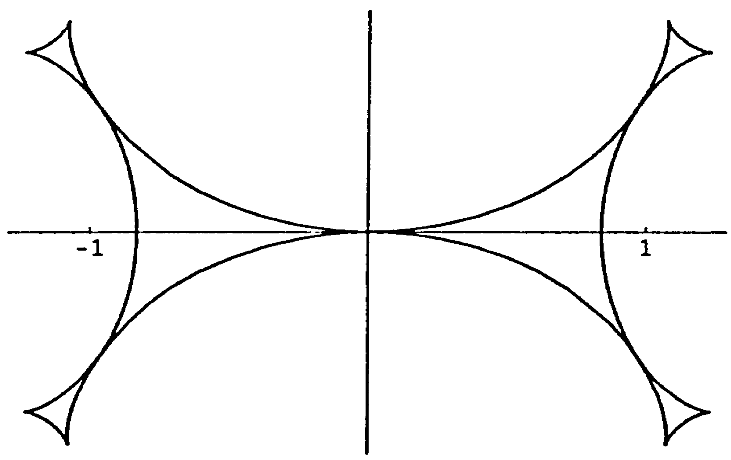


Figure 1b

no solutions for $|z|<1$ and $\theta \in (0, \pi/2)$. We define R_n, $n \geq 3$, to be the family of functions of the form (1) with $a_{n-1}=0$ that satisfy the condition $[\mu(ze^{ik\alpha})-\mu(ze^{-ik\alpha})]/(2i\sin k\alpha) \neq 0$ when $|z|<1$, $\alpha=\pi/(n-1)$, and $k=1,2,\ldots,n-2$. That is, the univalence criterion stated above holds for the discrete set of values of θ, $\pi/(n-1), 2\pi/(n-1), \ldots, (n-2)\pi/(n-1)$ that are uniformly distributed over the interval $[0,\pi]$. Thus, for large n, the functions in the family R_n should be "almost univalent" in some sense. For convenience, we define the operator Δ_k (which actually depends on n as well) on functions of the form (1), setting $\alpha=\pi/(n-1)$ as before, by

$$\Delta_k \mu(z) = \frac{\mu(ze^{ik\alpha})-\mu(ze^{-ik\alpha})}{2i\sin k\alpha} \tag{3}$$

We also define $R_1=\{1/z+az: |a|\leq 1\}$ and $R_2=\{1/z+az^2: |a|\leq 1\}$.

The sense in which the functions in R_n are "almost univalent" for large n is described in the following theorem.

THEOREM 2 [7, Thm. 1.12]. *Suppose* $\{n_k\}_{k=1}^{\infty}$ *is a strictly increasing sequence of positive integers and* $\mu_{n_k} \in R_{n_k}$ *for each* k. *Further, assume* $\lim_{k\to\infty} \mu_{n_k}=\mu$ *uniformly on compact subsets of* $\{|z|<1\}$. *Then* μ *is univalent.*

The proof is essentially identical to that in [12, Thm. 5] for polynomials.

There is a useful one-to-one correspondence between functions in R_n that have $|a_n|=1$ and univalent functions in U_n with $|a_n|=1/n$. That is, results on the family R_n can often be translated into results on the family U_n. Since the family R_n is more tractable, the relationship is fortunate.

We view the family R_n as a subset of the metric space $\mathfrak{M}(\Delta)$ of meromorphic functions on the unit disk using the topology of uniform convergence on compact sets in the spherical metric. Recall that an *extreme point* of a subset A of a vector space over $\mathbb{C}$ or $\mathbb{R}$ is a point $a \in A$ such that if $x, y \in A$ with $x \neq y$ and if $0<t<1$, then $a \neq tx+(1-t)y$. For each n, R_n is a compact subset of $\mathfrak{M}(\Delta)$ and the Krein–Milman theorem [4] applies. That is, R_n is contained in the closed convex hull of its extreme points. Further, if L is a continuous linear functional on R_n then L assumes its maximum modulus and its maximum real part over R_n at an extreme point.

THEOREM 3 [7, Thm. 1.3]. *If* $R(z)=1/z+\sum_{k=1}^{n} a_k z^k$ *is an extreme point of* R_n *such that* $a_n \geq 0$, *then* $a_n=1$ *and* $a_k=\bar{a}_{n-k-1}$ *for* $k=1,2,\ldots,n-2$.

Proof. Suppose $a_n<1$,

$$S(z)=\frac{1}{1+a_n}[\mu(z)+\hat{\mu}(z)], \quad \text{and} \quad T(z)=\frac{1}{1-a_n}[\mu(z)-\hat{\mu}(z)],$$

where

$$\hat{\mu}(z)=z^{n-1}\overline{\mu(1/\bar{z})}. \tag{4}$$

Note that $\Delta_k \hat{\mu}(z)=(-1)^{k-1}(\widehat{\Delta_k \mu})(z)$. Then $|\Delta_k \hat{\mu}(z)/\Delta_k \mu(z)|=1$ on $\{|z|=1\}$ and $|\Delta_k \hat{\mu}(z)/\Delta_k \mu(z)|=a_n$ when $z=0$. It easily follows that S and T are in R_n and

$$\mu = \frac{1+a_n}{2}S + \frac{1-a_n}{2}T.$$

This completes the proof. □

REMARK 1. The restriction $a_n \geq 0$ is not really necessary, since μ is an extreme point if and only if $e^{i\alpha}\mu(ze^{i\alpha})$ is an extreme point. Thus, the conclusion is $|a_n| = 1$ for all extreme points of R_n.

For $\mu \in R_n$ we define

$$\mu^*(z) = \frac{n-1}{n}\mu(z) - \frac{1}{n}z\mu'(z). \tag{5}$$

We will show that the univalent meromorphic polynomials with all zeros on $\{|z| = 1\}$ are precisely the functions μ^* such that $\mu \in R_n$ and $|a_n| = 1$.

LEMMA 1 [7, p. 10]. *If $\mu \in R_n$ then $\mu^* \in R_n$.*

Proof. Observe that $z(\Delta_k\mu)' = \Delta_k(z\mu')$. Therefore,

$$\Delta_k\mu^* = \frac{n-1}{n}\Delta_k\mu - \frac{1}{n}z(\Delta_k\mu)'. \tag{6}$$

Thus, $\Delta_k\mu^*(z) = 0$ if and only if (i) $\Delta_k\mu(z) = 0$ and $(\Delta_k\mu)'(z) = 0$ (i.e., $\Delta_k\mu$ has a zero of multiplicity 2 or more) or (ii) $z(\Delta_k\mu)'(z)/(\Delta_k\mu(z)) = n-1$. However,

$$\Delta_k\mu(z) = -\frac{1}{n}\prod_{j=1}^{n+1}\left(1 - \frac{z}{z_j}\right)$$

where $|z_j| \geq 1$ for each j, so that

$$\operatorname{Re}\frac{z(\Delta_k\mu)'(z)}{\Delta_k\mu(z)} = -1 + \operatorname{Re}\sum_{j=1}^{n+1}\frac{-z/z_j}{1-z/z_j} \leq -1 + \frac{n+1}{2} = \frac{n-1}{2}$$

when $|z| \leq 1$. Thus the theorem is proved for $n > 1$. If $n = 1$ then $\mu^*(z) = i\mu(iz)$ and the lemma follows for this case as well. □

REMARK 2. Note that we have just shown that if $\mu \in R_n$ and z_0 is a zero of $\Delta_k\mu^*(z)$ with $|z_0| = 1$, then z_0 is a zero (of multiplicity at least 2) of $\Delta_k\mu(z)$.

THEOREM 4. *If $\mu(z) = 1/z + \sum_{k=1}^{n-2} a_k z^k + a_n z^n$, $|a_n| = 1$, $a_{n-k-1} = a_n\bar{a}_k$ for $1 \leq k \leq n-2$, and $\mu^* \in R_n$, then $\mu \in R_n$ and μ^* is univalent.*

Proof. Without loss of generality, we may assume $a_n = 1$. By the coefficient relation, $\Delta_k\mu(z)$ is self-inversive and, as in Theorem 1,

$$\operatorname{Re}\left[\frac{z(\Delta_k\mu)'(z)}{\Delta_k\mu(z)}\right] = \frac{n-1}{2}$$

where $|z| = 1$ and $\Delta_k\mu(z) \neq 0$. If $\Delta_k\mu$ has a zero in $\{|z| < 1\}$, it follows that $z(\Delta_k\mu)'(z)/\Delta_k\mu(z)$ assumes every value that is not on the line $\{w: \operatorname{Re}(w) = (n-1)/2\}$. That is, it assumes every value in a neighborhood of infinity, and the image of the circle—namely, the above line—separates the plane into components in which every value is assumed the same number of times. This means that $z(\Delta_k\mu')(z)/\Delta_k\mu(z)=n-1$ for some z_0 with $|z_0|<1$, and hence $\Delta_k\mu^*(z_0)=0$. This contradiction proves that $\mu^*\in R_n$ implies $\mu\in R_n$ when $|a_n|=1$. The exceptional case $n=1$ is easily handled. To prove μ^* is univalent, proceed as follows. By methods used earlier, it is clear that for $0<r<1$,

$$\mu_r(z)=\frac{r}{1+r^{n+1}}[\mu(rz)+z^{n-1}\overline{\mu(r/\bar{z})}]\in R_n.$$

If $r\to 0$ this function becomes $\mu_0(z)=1/z+z^n$, so $\mu_0^*(z)=1/z-(1/n)z^n$ is univalent. As $r\to 1$, $\mu_r(z)$ tends to $\mu(z)$. If $\mu_1^*=\mu^*$ is not univalent, then for some r_0 $(0<r_0<1)$, μ_r^* is univalent when $0\le r\le r_0$ but μ_r^* is not univalent when $r_0<r<r_0+\epsilon$ for some $\epsilon>0$. Either $(\mu_r^*)'(z)=0$ for some $z=z(r)$ with $|z|<1$ when $r_0<r<r_0+\epsilon$, or $\mu_r^*(z_1)=\mu_r^*(z_2)$ with $|z_1|=|z_2|<1$ and $z_1\neq z_2$. Consider the first case. The zeros of the polynomial vary continuously with the coefficients. Because of the coefficient relation, if $(\mu_r^*)'(z)=0$ then $(\mu_r^*)'(1/\bar{z})=0$; that is, the zeros off $|z|=1$ occur in pairs that are inverse points with respect to the unit circle. Thus we conclude $(\mu_{r_0}^*)'$ has a double zero on $|z|=1$. This contradicts the univalence of $\mu_{r_0}^*$.

In the second case, $\mu_{r_0}^*(z_1)=\mu_{r_0}^*(z_2)$ for some z_1 and z_2, with $z_1\neq z_2$ and $|z_1|=|z_2|=1$. By Theorem 1, since the curve $\mu_{r_0}^*(\{|z|=1\})$ has a common tangent line at $\mu_{r_0}^*(z_1)$ and $\mu_{z_0}^*(z_2)$, we must have $z_2=z_1e^{i(2l\pi/(n-1))}$ for some l with $0<l<n-1$. For $r_0<r<r_0+\epsilon$, a contradiction to the fact that $\mu_r^*\in R_n$ is obtained by noting that there are arcs $I_1=\{e^{i\theta}:\theta_1<\theta<\theta_2\}$ and $I_2=\{e^{i\theta}:\theta_3<\theta<\theta_4\}$ with $\mu_r^*(I_j)\subset\mu_r^*(\{|z|<1\})$ for $j=1,2$ and with $\mu_r^*(e^{i\theta_1})=\mu_r^*(e^{i\theta_4})$ and $\mu_r^*(e^{i\theta_2})=\mu_r^*(e^{i\theta_3})$. Therefore the curves $\mu_r^*(I_1)$ and $\mu_r^*(I_2)$ have parallel tangents at appropriate points. This implies that $\mu_r^*(ze^{il\pi/(n-1)})=\mu_r^*(ze^{-il\pi/(n-1)})$ for some z with $|z|<1$ and l with $0<l<n-1$, a contradiction. □

THEOREM 5. *If $\mu(z)=1/z+\sum_{k=1}^{n-2}a_kz^k+a_nz^n$, $|a_n|=1$, and $a_{n-k-1}=a_n\bar{a}_k$ for $1\le k\le n-2$, then $\mu\in R_n$ if and only if $\mu^*\in U_n$. In this case, if $\theta_1<\theta_2<\theta_1+2\pi$ and $\mu^*(e^{i\theta_2})=\mu^*(e^{i\theta_1})$, then $\theta_2-\theta_1=2k\pi/(n-1)$ for some k with $0<k<n-1$; with $\theta=(\theta_1+\theta_2)/2$, $\Delta_k\mu(z)$ has a double zero of multiplicity at least 2 at $e^{i\theta}$.*

Proof. The first part of the theorem follows from Lemma 1 and Theorem 4. The equality $\theta_2-\theta_1=2k\pi/(n-1)$ follows from Theorem 1 and the fact that the curve $\mu^*(\{|z|=1\})$ has a common tangent line at $\mu^*(e^{i\theta_1})$ and $\mu^*(e^{i\theta_2})$. □

LEMMA 2. *Suppose $\{\mu_{n_k}(z)\}$ is a sequence such that n_k is a strictly increasing sequence of positive integers and $\mu_{n_k}\in R_{n_k}$ for each k. Then $\mu_{n_k}\to\mu$ uniformly on compact subsets of $\{|z|<1\}$ if and only if $\mu_{n_k}^*\to\mu$ uniformly on compact subsets of $\{|z|<1\}$.*

Proof. Note that, by assumption, if $\{\mu_{n_k}\}$ converges to μ then $\mu_{n_k}-\mu$ is analytic (not just meromorphic) in the unit disk. The lemma easily follows from the fact that $(1/n_k)|z\mu_{n_k}'(z)+1/z|\to 0$ uniformly on compact subsets of $\{|z|<1\}$. □

THEOREM 6. *The family* $\bigcup_{n=1}^{\infty}\{\mu^*\in U_n : \mu\in R_n$ *and* $a_n=1\}$ *is dense in* U.

Proof. It is sufficient to show that given $f\in U$ there exists $\{\mu_{n_k}\}$ such that $\mu_{n_k}\in R_{n_k}$, $n_k\to\infty$ as $k\to\infty$, and $\mu_{n_k}\to f$ uniformly on compact subsets of $\{|z|<1\}$. Hence $\mu_{n_k}^*\to f$ uniformly on compact subsets of $\{|z|<1\}$. By taking an increasing sequence $\{r_l\}$ with $r_l\to 1$ and forming $r_l f(r_l z)$, we may find a sequence of univalent functions of the form $S_n(z)=1/z+b_1z+\cdots+b_nz^n\in R_n$ that converges to f. Then

$$\mu_{2n+1}(z)=S_n(z)+z^{2n}S_n(1/\bar{z})\in R_{2n+1}$$

is the required sequence. □

Concerning the extreme points of R_n, Mansour proved the following [7, Thm. 1.16].

THEOREM 7. *If* μ *is an extreme point of* R_n *for* $n\geq 3$, *then* $|a_n|=1$ *and the curve* $\{\mu^*(e^{i\theta}) : 0\leq\theta\leq 2\pi\}$ *has* $n-2$ *points of self-tangency.*

COROLLARY. *If* μ^* *is an extreme point of* $\{\mu^*\in U_n$: *all zeros of* $(\mu^*)'$ *lie on* $|z|=1\}$, *then* $\{\mu^*(e^{i\theta}) : 0\leq\theta\leq 2\pi\}$ *has* $n-2$ *points of self-tangency.*

The proof is similar to [12, Thm. 6] for polynomials and to our Theorem 8 for the real-coefficient case.

REMARK 3. In studying the extreme points of R_n or the subset of U_n consisting of μ^* for which all zeros of $(\mu^*)'$ lie on $\{|z|=1\}$, we may assume without loss of generality that $a_n=1$. Thus, for $n=1,2$, the extreme points of R_n are $1/z+z$ and $1/z+z^2$, respectively. The corresponding extreme points in the univalent class are $1/z-z$ and $1/z-\frac{1}{2}z^2$.

APPLICATION OF THEOREM 7. We illustrate the application of Theorem 7 by finding the extreme points of R_4. By the theorem, $\mu^*(e^{i\theta})$ should have two self-tangencies that arise from $\mu^*(e^{i\theta_1})=\mu^*(e^{i\theta_2})$ with $\theta_2-\theta_1=2\pi/3$ or $4\pi/3$. Clearly, we may assume $\theta_2-\theta_1=2\pi/3$. Thus $\Delta_1\mu$ has two double roots on $\{|z|=1\}$. That is,

$$\begin{aligned}-z\Delta_1\mu(z)&=1-a_1z^2-\bar{a}_1z^3+z^5\\&=(1-2te^{i\psi}z+e^{2i\psi}z^2)^2(1+e^{-4i\psi}z).\end{aligned}$$

We conclude that $e^{-4i\psi}-4te^{i\psi}=0$ so that $e^{-5i\psi}=4t$ is real. Since the rotation $e^{\pm 2i\pi/5}\mu(e^{\pm 2i\pi/5}z)$ preserves the property $a_5=1$, it is clear that we may take $\psi=0$ and a_1 real so that $t=\frac{1}{4}$. Thus, $a_1=-\frac{5}{4}$ and the extreme points of R_4 are $\{1/z-\frac{5}{4}e^{i4k\pi/5}z-\frac{5}{4}e^{-i4k\pi/5}z^2+z^4 : k=0,\pm1,\pm2\}$.

By similar arguments, we may show that $\{1/z \pm 2z + z^3\}$ are the extreme points in R_3 and that

$$\left\{\frac{1}{z}+\sqrt{2}z^2+z^5,\ \frac{1}{z}-z+\frac{4\sqrt{3}}{9}z^2-z^3+z^5,\ \frac{1}{z}+i\sqrt{6\sqrt{3}-9}z-i\sqrt{6\sqrt{3}-9}z^3+z^5\right\},$$

together with rotations by $2k\pi/3$ and conjugation of coefficients, are the extreme points in R_5. In the latter case ($n=5$) one shows that the above polynomials are the only possible extreme points. Then, using the fact that for every linear functional F on U_n there must be an extreme point that maximizes $\operatorname{Re} F(\mu)$ over U_n, each one of the above polynomials must be an extreme point. We use $F_1(\mu)=a_2$, $F_2(\mu)=-a_1$, and $F_3(\mu)=-ia_1$ to arrive at that conclusion.

3. Typically Real Meromorphic Functions

A meromorphic function $\mu(z)=1/z+\sum_{k=0}^{\infty} a_k z^k$ is *typically real,* provided μ is analytic in $\{0<|z|<1\}$ and $\mu(z)$ is real, if and only if z is real for all z, $0<|z|<1$. Note that the coefficients $a_0, a_1, \ldots$ must all be real. Also, if $r>0$ is small then $\operatorname{Im}\mu(re^{i\theta})=-(1/r)\sin\theta+O(r)$, so we conclude $(\operatorname{Im}\mu(re^{i\theta}))\cdot\operatorname{Im}(re^{i\theta})\le 0$. Set

$$TR_n^+=\left\{\mu\colon \mu(z)=\frac{1}{z}+\sum_{k=1}^{n-2} a_k z^k+z^n,\ a_{n-k}=a_{k-1},\right.$$
$$\left. 2\le k\le n-1, \text{ and } \mu^* \text{ given by (5) is typically real}\right\}$$

and

$$TR_n^-=\left\{\mu\colon \mu(z)=\frac{1}{z}+\sum_{k=1}^{n-2} a_k z^k-z^n,\ a_{n-k}=-a_{k-1},\right.$$
$$\left. 2\le k\le n-1, \text{ and } \mu^* \text{ given by (5) is typically real}\right\}.$$

Finally, $TR_n=TR_n^+\cup TR_n^-$. Note that TR_n contains $\{\mu\in R_n\colon \mu$ has real coefficients and all zeros of $\mu^{*\prime}$ lie on $\{|z|=1\}\}$. Further, for $n=1$ and 2, TR_n is rather small; that is, $TR_n=\{1/z\pm z^n\}$ when $n=1,2$.

For $n>2$, let V_n^+ be the vector space over the reals spanned by the functions

$$\frac{1}{z}+z^n,\ z+z^{n-2},\ \ldots,\ \begin{cases} z^{n/2-1}+z^{n/2} & \text{if } n \text{ is even,}\\ z^{(n-1)/2} & \text{if } n \text{ is odd,}\end{cases}$$

and let V_n^- be the vector space over the reals spanned by the functions

$$\frac{1}{z}-z^n,\ z-z^{n-2},\ \ldots,\ \begin{cases} z^{n/2-1}-z^{n/2} & \text{if } n \text{ is even,}\\ z^{(n-3)/2}-z^{(n+1)/2} & \text{if } n \text{ is odd.}\end{cases}$$

Then $TR_n^+\subset V_n^+$, $TR_n^-\subset V_n^-$, and the dimensions of V_n^+ and V_n^- are $[(n+1)/2]$ and $[n/2]$ respectively, where $[\cdot]$ is the greatest integer function. We will produce a basis for V^+ and V^- that will allow a quite complete description of TR_n in geometric terms.

Set $\mu_1(z) = (1/z+z)(1+z^{n-1})$ and $\mu_2(z) = (1/z+z)(1-z^{n-1})$. Then

$$\Delta_k \mu_j(z) = -\left(\frac{1-z^2}{z}\right)(1-(-1)^{k+j}z^{n-1})$$

for $j = 1, 2$ and $1 \le k \le n-2$. Clearly, all zeros of $\Delta_k \mu_j(z)$ are on $\{|z| = 1\}$, so μ_j^* is univalent and has all zeros of $(\mu_j^*)'$ on $\{|z| = 1\}$, $j = 1, 2$. Therefore, $\mu_1 \in TR_n^+$ and $\mu_2 \in TR_n^-$. Also, $\Delta_k \mu_j(z)$ has a double zero at $z = 1$ when $k+j$ is even. This means that the curve $\mu_1^*(e^{i\theta})$, $0 \le \theta \le \pi$, is tangent to the real axis when $\theta = k\pi/(n-1)$ and k is odd, and that $\mu_2^*(e^{i\theta})$, $0 \le \theta \le \pi$, is tangent to the real axis when $\theta = k\pi/(n-1)$ and k is even. Figure 2 shows the image of the circle under the mapping $\mu_j^*(z)$ when $n = 5$ and $j = 1, 2$.

Now consider the functions

$$Q_p(z; n) = \frac{z(1-(-1)^p z^{n+1})}{1-2z\cos(p\pi/(n+1))+z^2}, \quad 1 \le p \le n, \tag{7}$$

that were defined in [12, p. 226]. We wish to show $Q_p(z; n-2) \in V_n^+$ when p is odd and $Q_p(z; n-2) \in V_n^-$ when p is even. This follows from the fact that the coefficient of z^j in the expansion of $Q_p(z, n-2)$ is

$$a_j = \frac{\sin(jp\pi/(n-1))}{\sin(p\pi/(n-1))}, \quad 1 \le j \le n-2,$$

so

$$a_{j-1} = (-1)^{p-1} a_{n-j}, \quad 2 \le j \le n-1.$$

$n = 5$

$\frac{1}{z} + \frac{3z}{5} + \frac{z^3}{5} - \frac{z^5}{5}$

$n = 5$

$\frac{1}{z} + \frac{3z}{5} - \frac{z^3}{5} + \frac{z^5}{5}$

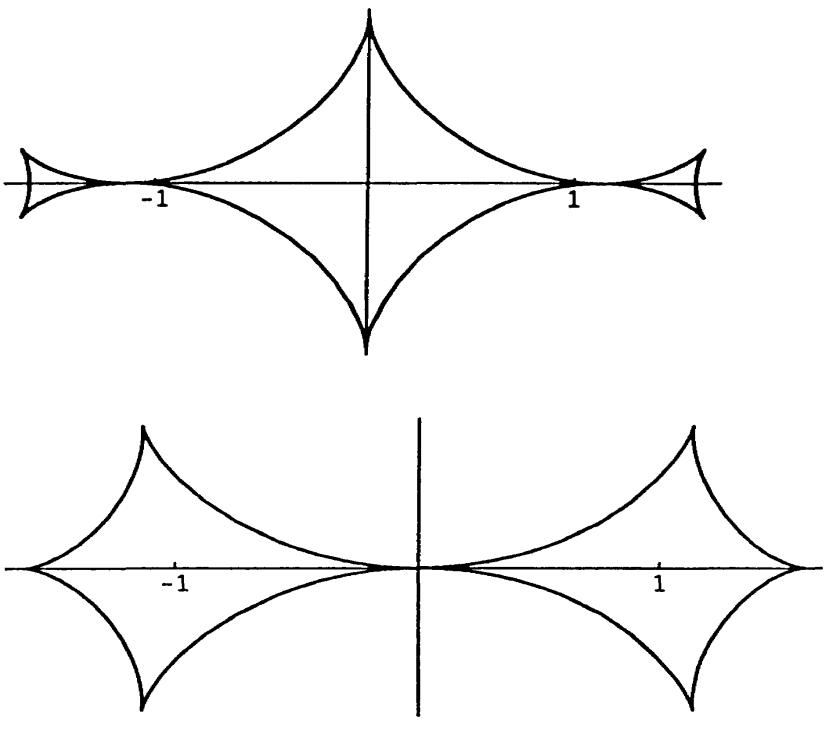


Figure 2

Note that the polynomials

$$P(z;n,j)=\frac{n+1}{n}Q_j(z;n)-\frac{1}{n}zQ_j'(z;n) \tag{8}$$

are the univalent polynomials defined in [11]. It is easy to check that

$$\Delta_k Q_p(1;n-2)=0 \quad \text{if } 1\le k\le n-2,\ k\ne p$$

and

$$\Delta_p Q_p(1;n-2)=\frac{n-1}{2\sin^2 p\pi/(n-1)}.$$

This shows that $\{Q_p(z,n-2): 1\le p\le n-2\}$ is a linearly independent set because $\sum_{p=1}^{n-2} t_p Q_p(z,n-2)=0$ implies $\sum_{p=1}^{n-2} t_p \Delta_k Q_p(1,n-2)=0$; hence

$$t_k\Delta_k Q_k(1,n-2)=0 \quad \text{and} \quad t_k=0$$

when $1\le k\le n-2$. We note also that $\Delta_k Q_p(z;n-2)$ has a double zero at 1 when $k+p$ is even and $k\ne p$.

Now $\{\mu_1(z)\}\cup\{Q_p(z;n-2): p \text{ is odd and } 1\le p\le n-2\}$ is a linearly independent subset of TR_n^+ that has $[(n+1)/2]$ members. It is therefore a basis for TR_n^+. Similarly, $\{\mu_2(z)\}\cup\{Q_p(z;n-2): p \text{ is even and } 2\le p\le n-2\}$ is a basis for TR_n^-. Now assume $\mu\in TR_n^+$. Then

$$\mu=\mu_1-\sum_{\substack{p\,\text{odd}\\ 1\le p\le n-2}} t_p Q_p(z;n-2).$$

We know

$$\mu^*=\mu_1^*-\sum_{\substack{p\,\text{odd}\\ 1\le p\le n-2}} t_p P(z;n-2,p) \quad \text{and} \quad \operatorname{Im}\mu^*(e^{ik\pi/(n-1)})\le 0$$

when k is odd and $1\le k\le n-2$. Since $\Delta_k Q_p(z;n-2)$ has a double zero at $z=1$ when $k+p$ is even and $k\ne p$, it follows that $\operatorname{Im}[P(z;n-2,p)]=0$ when $z=e^{ik\pi/(n-1)}$, k is odd, $k\ne p$, and $1\le k\le n-2$. Also, $\operatorname{Im}\Delta_k\mu_1^*(e^{ik\pi/(n-1)})=0$ when k is odd. Hence $0\ge \operatorname{Im}\mu^*(e^{ik\pi/(n-1)})=-t_k\operatorname{Im}(P(e^{ik\pi/(n-1)};n-2,k)$, so that $t_k\ge 0$. On the other hand,

$$\operatorname{Im}\left[\mu_1^*(re^{i\theta})-\sum_{\substack{p\,\text{odd}\\ 1\le p\le n-2}} t_p P(re^{i\theta};n-2,p)\right]\sin\theta<0$$

for $0<r<1$ and $0<\theta<\pi$ when each $t_p\ge 0$. Thus we have proved the following theorem.

THEOREM 8.

$$TR_n^+=\left\{\mu_1(z)-\sum_{\substack{p\,\text{odd}\\ 1\le p\le n-2}} t_p Q_p(z;n-2): t_p\ge 0 \quad \textit{for each } p\right\}.$$

Similarly,

$$TR_n^-=\left\{\mu_2(z)-\sum_{\substack{p\,\text{even}\\ 2\le p\le n-2}} t_p Q_p(z;n-2): t_p\ge 0 \quad \textit{for each } p\right\}.$$

The representation of $\mu\in TR_n^+$ *given above is unique.*

It is also clear from the preceding arguments that if $\mu \in V_n^+$ and μ has a pole at 0 of residue 1, then $\mu \in TR_n^+$ if and only if $\operatorname{Im} \mu^*(e^{ik\pi/(n-1)}) \leq 0$ when k is odd and $1 \leq k \leq n-2$. The corresponding condition for $\mu \in TR_n^-$ is, of course, that $\operatorname{Im} \mu^*(e^{ik\pi/(n-1)}) \leq 0$ when k is even and $2 \leq k \leq n-2$. Actually, the typically real meromorphic functions that are analytic in $0<|z|<1$ with a simple pole of residue 1 at $z=0$ are the functions that are limits of sequences $\{\mu_{n_k}\}$ such that $n_k \to \infty$ as $k \to \infty$ and $\mu_{n_k} \in TR_{n_k}^+$ (or $TR_{n_k}^-$). We require the following result, which was proved by Goodman [5, Thm. 10] in somewhat more generality. There he considered meromorphic typically real functions that may have more than one pole.

THEOREM 9. *Suppose $\mu(z) = 1/z + \sum_{k=1}^{\infty} a_k z^k$ is analytic in $0<|z|<1$ and typically real. Then $a_1 \leq 1$ with equality if and only if $\mu(z) = 1/z+z$. If $a_1<1$ then there is a typically real analytic function $g(z) = z+b_2z^2+\cdots$ such that $\mu(z) = 1/z+z-(1-a_1)g(z)$. Finally, if $g(z) = z+b_2z^2+\cdots$ is typically real and $c>0$, then $1/z+z-cg(z)$ is typically real.*

Proof. Consider the function $h(z) = 1/z+z-\mu(z)$ where $z=re^{i\theta}$, $0<r<1$, and $0<\theta<\pi$. Suppose, for some r_0 and θ_0, that $0<r_0<1$, $0<\theta_0<\pi$, and $\operatorname{Im} h(z) = -t<0$. Choose r so that $r-1/r > -t/2$ and $1>r>r_0$. Because a harmonic function cannot assume a minimum at an interior point and $\operatorname{Im} h(z) = 0$ when z is real, there is a θ $(0<\theta<\pi)$ such that

$$-t \geq \operatorname{Im} h(re^{i\theta}) = \left(r-\frac{1}{r}\right)\sin\theta - \operatorname{Im}\mu(re^{i\theta});$$

hence

$$\operatorname{Im}\mu(re^{i\theta}) \geq \left(r-\frac{1}{r}\right)\sin\theta + t \geq \frac{t}{2} > 0.$$

This contradicts the fact that μ is typically real. We have proved that the analytic function $1/z+z-\mu(z) = h(z)$ satisfies $\operatorname{Im} h(z) \geq 0$ when $0<r<1$ and $0<\theta<\pi$. Since h is harmonic, either $\operatorname{Im} h(z)>0$ in the upper half-disk or $h(z) \equiv 0$. If $h(z) \equiv 0$ then $\mu(z) = 1/z+z$, so $a_1 = 1$. Otherwise, $h(z) = (1-a_1)z+\cdots$ is typically real. This is not possible unless $1-a_1>0$. The last statement is clear. That is, $\operatorname{Im}(1/z+z-cg(z))\sin\theta \leq 0$ when $0<|z|<1$, so that $\mu(z) = 1/z+z-g(z)$ maps the upper half-disk into the lower half-plane and the lower half-disk into the upper half-plane. Using the minimum or maximum principle for harmonic functions applied to the upper or lower half of the disk yields the conclusion that μ is typically real. Note that this result shows that a_1 does not have a lower bound and that the other coefficients do not have uniform bounds. For each a_1, the other coefficients satisfy the sharp bound $|a_n| \leq n(1-a_1)$. □

THEOREM 10. *Let $\mu(z) = 1/z+\sum_{k=1}^{\infty} a_k z^k$ be analytic in $\{0<|z|<1\}$. Then μ is typically real if and only if there is a sequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} \in TR_{n_k}^+$, $n_k \to \infty$ as $k \to \infty$, and $\mu_{n_k} \to \mu$ uniformly on compact subsets of the disk.*

Proof. Assume that $\mu_{n_k} \in TR^+_{n_k}$ and $\mu_{n_k} \to \mu$ uniformly on compact subsets. Then $\mu^*_{n_k}$ is typically real for each k, and $\mu^*_{n_k} \to \mu$ uniformly on compact subsets; that is, $\mu_{n_k} - \mu^*_{n_k} = (1/n_k)(\mu_{n_k} + z\mu'_{n_k})$ tends to zero uniformly on compact sets under the given assumptions. Therefore μ is typically real.

On the other hand, suppose μ is typically real. Then $\mu(z) = 1/z + z - cg(z)$ for some $c \geq 0$, where g is typically real and $g(z) = z + b_2 z^2 + \cdots$. If $c = 0$, we are done. Otherwise, by a result of Robertson [9],

$$g(z) = \int_0^\pi \frac{z}{1-2z\cos t + z^2}\, dF(t),$$

where F is monotone increasing and $F(\pi) - F(0) = 1$. Then $g(z)$ can be approximated uniformly on compact sets by functions of the form

$$\begin{aligned}&\left[F\left(\frac{\pi}{n-1}\right) - F(0)\right]\frac{z}{1-2z\cos\pi/(n-1)+z^2}\\ &\quad+\left[F\left(\frac{3\pi}{n-1}\right) - F\left(\frac{\pi}{n-1}\right)\right]\frac{z}{1-2z\cos 3\pi/(n-1)+z^2}\\ &\quad+\cdots+\left[F(\pi) - F\left(\frac{(l-2)\pi}{n-1}\right)\right]\frac{z}{1-2z\cos l\pi/(n-1)+z^2},\end{aligned}$$

where l is the largest odd integer $\leq n-2$. Since

$$\left|\frac{z(1+z^{n-1})}{1-2z\cos\theta+z^2} - \frac{z}{1-2z\cos\theta+z^2}\right| = \frac{|z^n|}{|1-2z\cos\theta+z^2} \leq \frac{|z^n|}{(1-|z|)^2}$$

can be made arbitrarily small (by choosing n large) on compact subsets of the disk, μ can be uniformly approximated on compact subsets by functions of the form

$$\left(\frac{1}{z}+z\right)(1+z^{n-1}) - \sum_{\substack{p\text{ odd}\\ 1\leq p\leq n-2}} t_p Q_p(z; n-2) \quad \text{where } t_p \geq 0$$

for each p. These are functions of TR^+_n. The proof is now complete. □

Notice that there are meromorphic typically real functions that assume every value in the extended plane. For example, let

$$f(z) = \frac{1}{z} + z - c\left[\frac{z}{(1-z)^2} + \frac{z}{(1+z)^2}\right], \quad c > 0.$$

Then f is rational of degree 6. Also, $f(1/z) = f(z)$. Since f assumes every value six times (counting multiplicities) and the image of the interior is the same as the image of the exterior (including the point at ∞), we need only examine the boundary values. It is easy to check that f is real on $\{|z| = 1\}$ and that it assumes every real value exactly twice (and the value ∞ twice with multiplicity 2 each time). Therefore f assumes every nonreal complex value exactly three times, every real value twice, and the value ∞ one time in the open disk.

It was conjectured by Kirwan and Schober [6, p. 37] that for functions in the family Σ (and therefore in U) that $\operatorname{Re}(ka_1 - a_k) \le k$ where $\mu(z) = 1/z + \sum_{l=1}^{\infty} a_l z^l \in U$. We prove that the inequality holds for typically real meromorphic functions, which proves the result for U and Σ in the restricted class of functions with real coefficients.

THEOREM 11. *If* $\mu(z) = 1/z + \sum_{l=1}^{\infty} a_l z^l$ *is analytic in* $\{0 < |z| < 1\}$ *and typically real, then* $ka_1 - a_k \le k$.

Proof. By Theorem 10, it is sufficient to prove the inequality for functions in TR_n^+. If $\mu \in TR_n^+$ then

$$\mu(z) = \mu_1(z) - \sum_{\substack{p\,\text{odd} \\ 1 \le p \le n-2}} t_p Q_p(z; n),$$

where each $t_p \ge 0$. Then

$$a_1 = 1 - \sum t_p \quad \text{and} \quad a_k = \sum t_p \frac{\sin(kp\pi/(n-1))}{\sin(p\pi/(n-1))}.$$

Therefore

$$ka_1 - a_k = k - \sum t_p \left(k + \frac{\sin(kp\pi/(n-1))}{\sin(p\pi/(n-1))} \right) \le k,$$

because $|\sin(kp\pi/(n-1))/\sin(p\pi/(n+1))| \le k$. The last inequality follows by induction using the fact that

$$\frac{\sin k\theta}{\sin\theta} = \cos(k-1)\theta + \cos\theta \, \frac{\sin(k-1)\theta}{\sin\theta}. \qquad \square$$

4. Extreme Points in σ_n

Let σ_n denote the subclass of the family R_n consisting of those members of R_n that have real coefficients, and let $a_n = \pm 1$. Let σ_n^+ be the family of $\mu \in \sigma_n$ such that $a_n = 1$ and σ_n^- the family of $\mu \in \sigma_n$ such that $a_n = -1$. If $\mu \in \sigma_n^+$ then

$$\mu^*(z) = \frac{1}{z} + \sum_{k=1}^{n-2} \frac{n-k-1}{n} a_k z^k - \frac{1}{n} z^n,$$

and if $\mu \in \sigma_n^-$ then

$$\mu^*(z) = \frac{1}{z} + \sum_{k=1}^{n-2} \frac{n-k-1}{n} a_k z^k + \frac{1}{n} z^n.$$

That is, the last coefficient of μ^* has the opposite sign to that of μ. We now prove some lemmas that lead to a necessary condition for a function to be an extreme point in σ_n^+ or σ_n^-.

LEMMA 3. *The functions*

$$\mu_1(z) = (1/z + z)(1 + z^{n-1}) \quad \textit{and} \quad \mu_2(z) = (1/z + z)(1 - z^{n-1})$$

are extreme points in σ_n^+ *and* σ_n^-, *respectively.*

Proof. Since $\sigma_n^{\pm} \subset TR_n^{\pm}$, Theorem 8 is applicable. Since the numbers t_p given in Theorem 8 are nonnegative, the lemma clearly follows. □

LEMMA 4. *Suppose* $\mu = \mu_j - tQ_k(z; n-2)$ *where* $j = 1$ *or* 2 *and* $k - j$ *is even,* $1 \le k \le n-2$. *Then*

(a) $\mu \in \sigma_n$ *if and only if*

$$0 \le t \le \left(2 \sin \frac{k\pi}{n-1} \sin \frac{\pi}{n-1}\right)^2;$$

(b) μ *is an extreme point of* σ_n *if and only if equality holds on either side in* (a); *and*

(c) *in case equality holds on the right in* (a), $\Delta_1 \mu$ *has a double zero for some nonreal* z, $|z| = 1$, *and hence the curve* $\mu^*(e^{i\theta})$ *has a self-tangency in both the upper and lower half-planes.*

Proof. By elementary algebra,

$$-z\Delta_l \mu(z) = [1 + (-1)^{j-1+l} z^{n-1}][1 - z^2](1 + tP(z))$$

where

$$P(z) = z^2 \Big/ \left[\left(1 - 2z \cos \frac{(k+1)\pi}{n-1} + z^2\right)\left(1 - 2z \cos \frac{(k-l)\pi}{n-1} + z^2\right)\right],$$

so

$$-z\Delta_l \mu(z) = P(z)[1 + (-1)^{j-1+l} z^{n-1}][1 - z^2]\left(\frac{1}{P(z)} + t\right). \tag{9}$$

It is sufficient to show that for $1 \le l \le n-2$, the function given by (9) has $n+1$ zeros on $\{|z| = 1\}$ if and only if t satisfies the inequality in (a). Clearly, $P(z)(1 + (-1)^{j-1+l} z^{n-1})(1 - z^2)$ has $n-3$ zeros on the circle. It remains to show that $1/P(z) + t$ has four zeros on the circle if and only if $0 \le t \le 4(\sin k\pi/(n-1) \sin \pi/(n-1))^2$. Let $z = e^{i\theta}$. The expression $1/P(z) + t$ becomes

$$\left[4\left(\cos\theta - \cos \frac{(k+l)\pi}{n-1}\right)\left(\cos\theta - \cos \frac{(k-l)\pi}{n-1}\right) + t\right].$$

For positive t, this expression is positive when $\theta = 0$ or $\theta = \pi$. Set $\cos\theta = \cos k\pi/(n-1) \cos l\pi/(n-1)$. The expression $1/P(z) + t$ is now

$$-4\left(\sin \frac{k\pi}{n-1} \sin \frac{l\pi}{n-1}\right)^2 + t \le -4\left(\sin \frac{k\pi}{n-1} \sin \frac{\pi}{n-1}\right)^2 + t.$$

Thus, $1/P(z) + t$ has two zeros on the upper semicircle and two on the lower semicircle when $t \le 4(\sin k\pi/(n-1) \sin \pi/(n-1))^2$ (counting multiplicities), with a double zero at $\cos\theta = \cos k\pi/(n-1) \cos \pi/(n-1)$ when equality holds and $l = 1$. This proves (c). On the other hand, if

$$t > 4\left(\sin \frac{k\pi}{n-1} \sin \frac{\pi}{n-1}\right)^2,$$

the expression becomes

$$4\left(\cos\theta - \cos\frac{k\pi}{n-1}\cos\frac{\pi}{n-1}\right)^2 + t - 4\left(\sin\frac{k\pi}{n-1}\sin\frac{\pi}{n-1}\right)^2 > 0$$

for all θ. Therefore $\mu \notin \sigma_n$. This proves (a).

To prove (b), using Theorem 8 we choose t as large as possible to obtain equality on the right in (a). Then $0 < s < 1$ and $\mu = s\psi_1 + (1-s)\psi_2$ for ψ_1 and ψ_2 in σ_n imply $\psi_1 = \psi_2 = \mu$, and we are done. □

REMARK 4. The functions μ_1^*, μ_2^*, and μ^*, where μ is given in Lemma 4 with equality on the right in (a), were introduced by Schnack [10].

THEOREM 12. *Let $\mu = \mu_j - \sum t_k Q_{n-2}(z,k)$, where $j = 1$ or 2 and the sum is taken over all k ($1 \le k \le n-2$) for which $k-j$ is even (such k are said to be* allowable). *If μ is an extreme point of σ_n, then the following hold.*

(a) *If $t_k > 0$ for each allowable k, then among the zeros of $-z\Delta_l\mu(z)$ for $1 \le l \le (n-1)/2$ there are $[(n-1)/2]$ double zeros, $z = e^{i\theta}$ with $0 < \theta < \pi$, and an equal number with $0 > \theta > -\pi$. This yields $n-2$ self-tangencies on the curve $\mu^*(e^{i\theta})$, $0 \le \theta \le 2\pi$. If n is odd, then $-z\Delta_{(n-1)/2}\mu(z)$ is an even function and the double zeros occur in pairs that account for only one self-tangency of the curve $\mu^*(e^{i\theta})$.*

(b) *For each allowable k such that $t_k = 0$, the curve $\mu^*(e^{i\theta})$, $0 < \theta < \pi$, is tangent to the real axis at $\theta = k\pi/(n-1)$. If at least one t_k is zero for an allowable k, then for each nonzero t_k there is an l, $1 \le l < (n-1)/2$, such that $-z\Delta_l\mu(z)$ has a double zero at $z = e^{i\theta}$ for some θ, $0 < \theta < \pi$. Thus, if $t_k = 0$ for p allowable values of k and $[(n-j)/2] - p$ of the t_k are nonzero, then the curve $\mu^*(e^{i\theta})$, $0 \le \theta \le 2\pi$, has $2\,[(n-j)/2] - p$ self-tangencies unless $j = 1$ and $p = 0$.*

Note that the exceptional case, $j = 1$ and $p = 0$, was described in (a).

Proof. It is sufficient to show that for an extreme point in σ_n given by $\mu = \mu_j - \sum t_k Q_{n-2}(z,k)$, if there are p allowable t_k that are not zero, among zeros of the polynomials $-z\Delta_l\mu(z)$, $1 \le l \le (n-1)/2$, then there are p double zeros $z = e^{i\theta}$ with $0 < \theta < \pi$. The basic idea is as follows. The zeros of $-z\Delta_l\mu(z)$ vary continuously with the numbers t_k. If each t_k is nonnegative then μ^* maps the upper half-disk into the lower half-plane and the lower half-disk into the upper half-plane. Starting from a $\mu \in \sigma_n$ (e.g. with all $t_k = 0$), if we vary the t_k then μ will remain in σ_n unless two zeros of some $-z\Delta_l\mu(z)$ coalesce on $\{|z| = 1\}$; then one goes outside the disk and one inside as inverse points relative to the unit circle (i.e., if the relevant zeros are z_1 and z_2 then $z_1\bar{z}_2 = 1$). This holds because of the coefficient relation. Now suppose p of the allowable t_k are positive and the remaining t_k are zero. Assume $\mu \in \sigma_n$ and assume there are fewer than p double roots among the roots of $-z\Delta_l\mu(z) = 0$, $1 \le l \le (n-1)/2$, $z = e^{i\theta}$, $0 < \theta < \pi$. We view the nonzero t_k as variables, and we wish to construct $S(z) = \sum_{s_k \ne 0} s_k Q_{n-2}(z,k) \not\equiv 0$ so that $\mu \pm \epsilon S \in \sigma_n$ for some $\epsilon > 0$. Then $\mu = \frac{1}{2}(\mu + \epsilon S) + \frac{1}{2}(\mu - \epsilon S)$, so μ is not an extreme point. We determine the p real numbers s_k with $t_k \ne 0$ as follows. If $-z\Delta_l\mu(z)$ has a double zero at $z = e^{i\theta}$, we require that $\Delta_l S(z)$ have a zero at $z = e^{i\theta}$. Since $\Delta_l S(z) = e^{i((n-1)/2)\theta} R(\theta)$ where R is real, this yields a real linear equation in the p variables, s_k for each double zero of some $-z\Delta_l \mu(z)$. To obtain p equations, require that $\Delta_l S(1) = K$ ($l = 1, 2, \ldots$) for sufficiently many values of l, so that the total number of linear equations in the p values of s_k is p. Here $K = 0$ if the determinant of coefficients of the s_k is 0; $K = 1$ otherwise. In any case, there is a nontrivial solution for the s_k. Now, for sufficiently small $\epsilon > 0$, $\mu \pm \epsilon S$ is typically real. Also, for $1 \le l \le (n-1)/2$, for each double zero of $-z\Delta_l \mu(z)$ on $\{|z| = 1\}$ there is a simple zero of $-z\Delta_l(\mu \pm \epsilon S)(z)$. The double zero cannot move off $\{|z| = 1\}$ because it would become a pair of zeros that are inverse points with respect to the unit circle. However, S was constructed so that one of these zeros will remain on the circle. This completes the proof. □

From Lemmas 3 and 4, the extreme points of σ_n^+ $(1 \le n \le 4)$ and σ_n^- $(1 \le n \le 5)$ are determined. Denoting the extreme points of σ_n^+ and σ_n^- by $E\sigma_n^+$ and $E\sigma_n^-$, respectively, we have

$E\sigma_n^+ = \{\mu_1(z)\}, \quad n = 1, 2;$

$E\sigma_n^+ = \{\mu_1(z), \mu_1(z) - 4(\sin \pi/(n-1))^4 Q_{n-2}(z, 1)\}, \quad n = 3, 4;$

$E\sigma_n^- = \{\mu_2(z)\}, \quad n = 1, 2, 3;$ and

$E\sigma_n^- = \{\mu_2(z), \mu_2(z) - 4(\sin(2\pi/(n-1)) \sin(\pi/(n-1))^2 Q_{n-2}(z, 2)\}, \quad n = 4, 5.$

For $n = 5$, $\mu \in \sigma_5^+$ is given by

$$\begin{aligned} \mu(z) &= \mu_1(z) - t_1 Q_3(z, 1) - t_3 Q_3(z, 3) \\ &= 1/z + (1 - t_1 - t_3)z + (t_3 - t_1)\sqrt{2} z^2 + (1 - t_1 - t_3)z^3 + z^5. \end{aligned}$$

For extreme points, either $\mu = \mu_1$ or μ is given by Lemma 4 (in this case $-z\Delta\mu(z)$ has three double roots and it follows that $t_1 = 1$ and $t_3 = 0$ or that $t_1 = 0$ and $t_3 = 1$), or $-z\Delta_1 \mu$ has two double roots and $-z\Delta_2 \mu$ has two double roots. In the latter case,

$$1 - a_1 z^2 - a_2 \sqrt{2} z^3 - a_1 z^4 + z^6$$

and

$$1 - a_1 z^2 + a_1 z^4 - z^6 = (1 - z^2)(1 + (1 - a_1) z^2 + z^4)$$

both have double roots: $a_1 = 1 - t_1 - t_3$ and $a_2 = (t_3 - t_1)\sqrt{2}$. We conclude that $a_1 = -1$ because $(1 + (1 - a_1) z^2 + z^4)$ must be a square and

$$\cos 3\theta + \cos \theta - \frac{a_2 \sqrt{2}}{2} = 0$$

has a double root, $0 < \theta < \pi$ (writing the first condition in trigonometric form using $a_1 = -1$). This means $3 \sin 3\theta + \sin \theta = 0$, so $12 \cos^2 \theta - 2 = 0$, $\cos^2 \theta = 1/6$, and $a_2 = \pm 4\sqrt{3}/9$. Thus, we have the following theorem.

THEOREM 13. *For $n=3,4,5$, the extreme points of σ_n are as follows.*

(a) *If $n=3$, then $\mu(z)=1/z+a_1z+z^3$ for $a_1=\pm 2$ and $\mu(z)=1/z-z^3$ are extreme points in σ_3^+ and σ_3^-, respectively.*

(b) *If $n=4$, then $\mu(z)=1/z+a_1z+a_1z^2+z^4$ for $a_1=1,-5/4$ are the extreme points in σ_4^+. In σ_4^-, the extreme points are $-\mu(-z)$, where μ is an extreme point of σ_4^+.*

(c) *If $n=5$, then $\mu(z)=1/z+a_1z+a_2z^2+a_1z^3+z^5$ for $(a_1,a_2)=(1,0)$ or $(0,\pm\sqrt{2})$ or $(-1,\pm 4\sqrt{3}/9)$ when μ is an extreme point of σ_5^+. The extreme points of σ_5^- are $\mu(z)=1/z+a_1z-a_1z^3-z^5$ where $a_1=\pm 1$.*

Of course, these results yield bounds on the coefficients of univalent meromorphic polynomials μ, with real coefficients and all zeros of μ' on $\{|z|=1\}$ for $3\le n\le 5$. The sharp bounds are

$$-2/3\le a_1\le 2/3 \quad \text{for } n=3,$$

$$-5/8\le a_1\le 1/2 \quad \text{for } n=4,$$

and

$$-3/5\le a_1\le 3/5 \quad \text{and} \quad |a_2|\le 2\sqrt{2}/5 \quad \text{for } n=5$$

(see [10]), with $\mu(z)=1/z+a_1z+a_2z^2+\frac{1}{3}a_1z^3-\frac{1}{5}z^5$.

Now consider $n=6$. Since $\mu\in\sigma_6^+$ if and only if $-\mu(-z)\in\sigma_6^-$, we need only consider σ_6^+. Write $\mu(z)=1/z+a_1z+a_2z^2+a_2z^3+a_1z^4+z^6$. We know that μ is an extreme point when $(a_1,a_2)=(1,0)$,

$$(a_1,a_2)=\left(\frac{5\sqrt{5}-7}{8},-\frac{5(\sqrt{5}-1)}{8}\right), \quad \text{and} \quad (a_1,a_2)=\left(-\frac{1}{4},\frac{5(\sqrt{5}-1)}{8}\right),$$

by Lemmas 3 and 4. The remaining extreme points in σ_6^+ occur when both $-z\Delta_1\mu(z)$ and $-z\Delta_2\mu(z)$ have two double roots. That is,

$$1-a_1z^2-\frac{\sqrt{5}+1}{2}a_2z^3-\frac{\sqrt{5}+1}{2}a_2z^4-a_1z^5+z^7$$
$$=(1-2tz+z^2)^2(1+4tz+4tz^2+z^3)$$

and

$$1-a_1z^2-\frac{\sqrt{5}-1}{2}a_2z^3+\frac{\sqrt{5}-1}{2}a_2z^4+a_1z^5-z^7$$
$$=(1-2sz+z^2)^2(1+4sz-4sz^2-z^3)$$

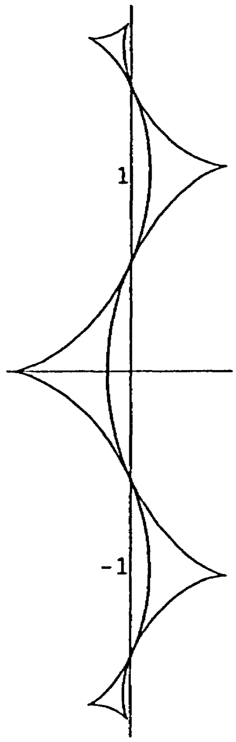
where $-1<t<1$ and $-1<s<1$. We have $a_1=12t^2-4t-2=12s^2+4s-2$ and

$$a_2=-\frac{\sqrt{5}-1}{2}(16t^3-16t^2+4t+1)=-\frac{\sqrt{5}+1}{2}(16s^3+16s^2+4s+1).$$

From a_1 we get $12(t^2-s^2)-4(t+s)=0$ and $4(t+s)(3(t-s)-1)=0$. Then, $t=-s$ or $t=s+\frac{1}{3}$. If $t=-s$, we observe that $a_2=0$ so $16t^3-16t^2+4t+1=$

$n = 6$

$$\begin{bmatrix} -0.7605206751 \\ 0 \\ 0 \\ -0.1901301688 \end{bmatrix}$$



$n = 6$

$$\begin{bmatrix} -0.5690892159 \\ -0.3098299566 \\ -0.2065533044 \\ -0.142272304 \end{bmatrix}$$

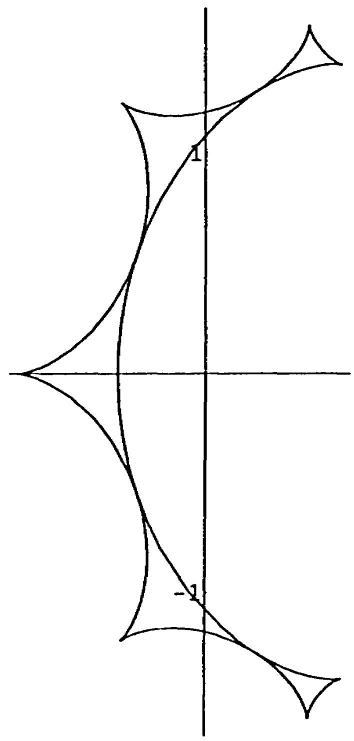
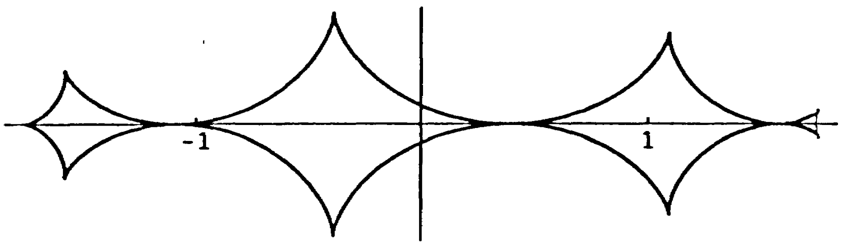


Figure 3

0. In case $t = s + \frac{1}{3}$, we also obtain a cubic equation in t. In each case, we obtain exactly one value of t for which the resulting m is in the family σ_6. The values are approximately, $(a_1, a_2) = (-1.1408, 0)$ and $(a_1, a_2) = (-.8536, -.6197)$. Figure 3 shows roughly the image of the disk under the mapping μ^* in each case.

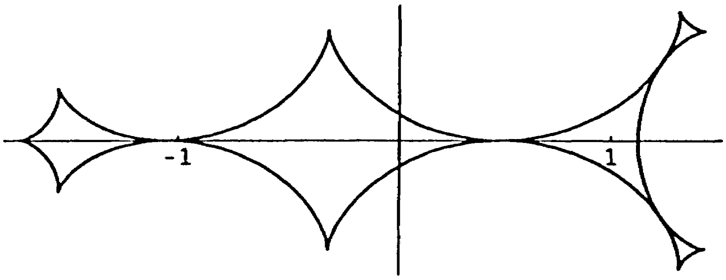
$n = 8$

$$\begin{pmatrix} 0.75 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.125 \end{pmatrix}$$



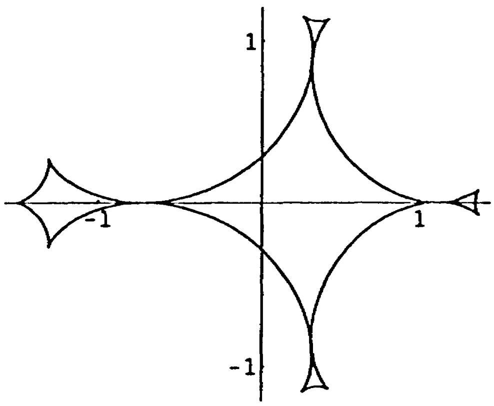
$n = 8$

$$\begin{pmatrix} 0.643680053 \\ -0.1596516038 \\ -0.1592658349 \\ -0.1194493762 \\ -0.0638606415 \\ 0.1072800088 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} 0.2131993762 \\ -0.1990822936 \\ 0.2869871179 \\ 0.2152403384 \\ -0.0796329174 \\ 0.0355332294 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} 0.4047813007 \\ 0.3587338974 \\ -0.127721283 \\ -0.0957909623 \\ 0.143493559 \\ 0.0674735501 \end{pmatrix}$$

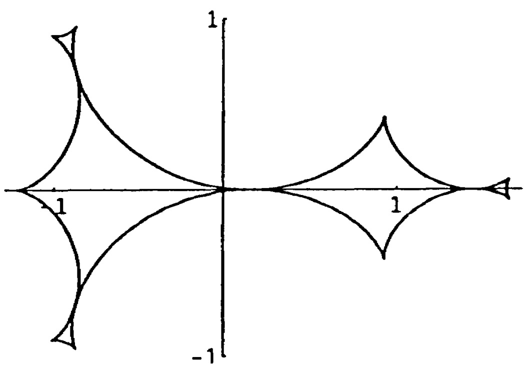


Figure 4a

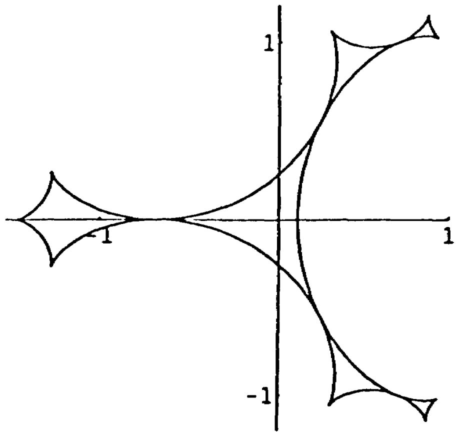
We summarize the results for $n = 6$ in the following theorem.

THEOREM 14. *The extreme points in σ_6^+ are the functions*

$$\mu(z) = 1/z + a_1 z + a_2 z^2 + a_2 z^3 + a_1 z^4 + z^6$$

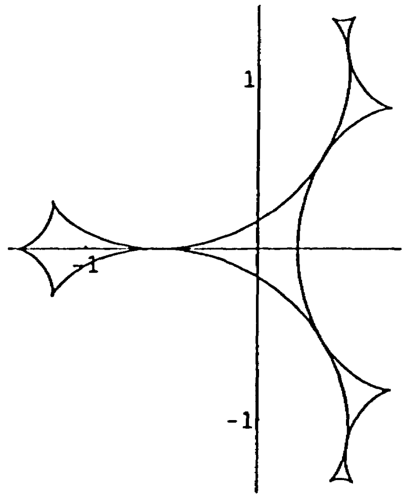
$n = 8$

$$\begin{pmatrix} 0.0668081443 \\ -0.5017028103 \\ -0.0811399951 \\ -0.0608549963 \\ -0.2006811241 \\ 0.0111346907 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} -0.066759068 \\ -0.5017131476 \\ 0.0792946228 \\ 0.0594709671 \\ -0.200685259 \\ -0.0111265113 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} -0.475739221 \\ 0.2099619006 \\ 0.2289690847 \\ 0.1717268135 \\ 0.0839847602 \\ -0.0792898702 \end{pmatrix}$$

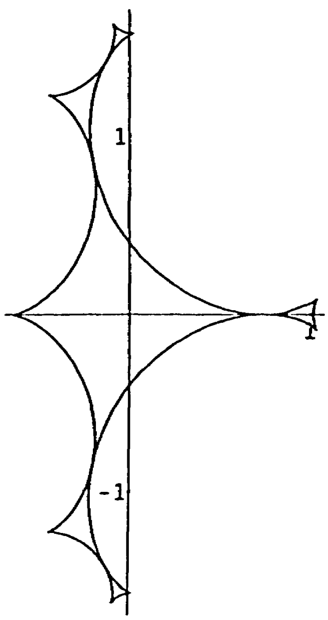


Figure 4b

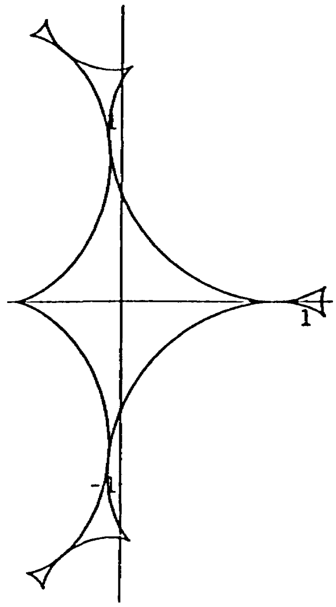
where

$$(a_1, a_2) \in \left\{(1,0), \left(\frac{\sqrt{5}-7}{8}, -\frac{5(\sqrt{5}-1)}{8}\right), \left(-1/4, \frac{5(\sqrt{5}-1)}{8}\right), (\alpha, 0), (\beta, r)\right\},$$

where $\alpha \approx -1.1408$, $\beta \approx -.8536$, *and* $\gamma \approx -.6197$.

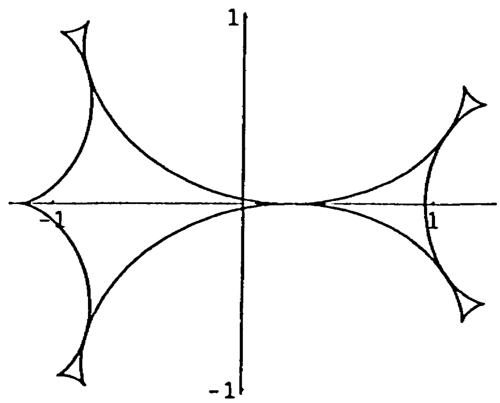
$n = 8$

$$\begin{pmatrix} -0.4331521273 \\ 0.3625734659 \\ 0.1184257464 \\ 0.0888193098 \\ 0.1450293864 \\ -0.0721920212 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} 0.2843468157 \\ 0.1958185529 \\ -0.3001698035 \\ -0.2251273526 \\ 0.0783274212 \\ 0.0473911359 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} -0.506627872 \\ -0.2997798241 \\ -0.2146979248 \\ -0.1610234436 \\ -0.1199119297 \\ -0.0844379787 \end{pmatrix}$$

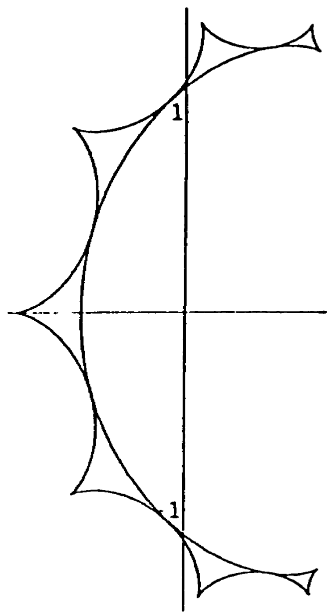
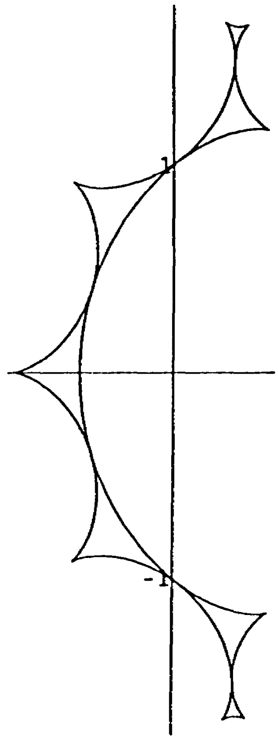


Figure 4c

$n = 8$

$$\begin{pmatrix} -0.6423170874 \\ -0.3175894794 \\ -0.07303584 \\ -0.05477688 \\ -0.1270357918 \\ -0.1070528479 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} -0.8236494327 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.1372749054 \end{pmatrix}$$

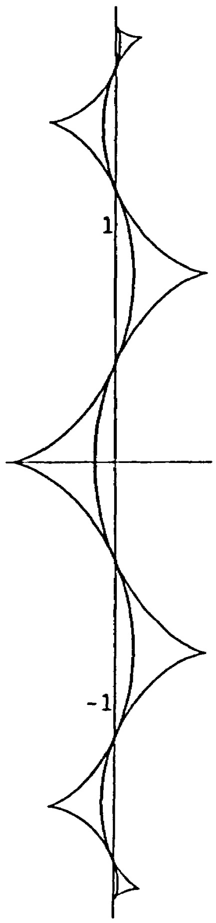
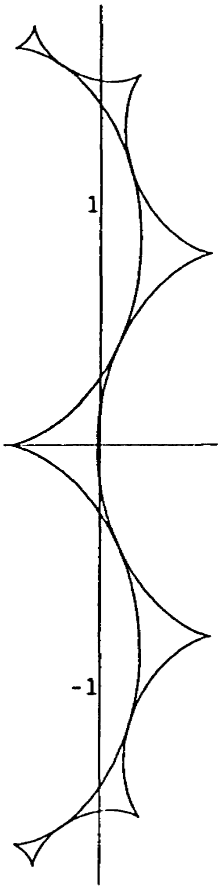


Figure 4d

$n = 8$

$$\begin{pmatrix} -0.7674121895 \\ 0.1338254826 \\ -0.1015690693 \\ -0.076176802 \\ 0.053530193 \\ -0.1279020316 \end{pmatrix}$$



$n = 8$

$$\begin{pmatrix} -0.5305095812 \\ 0.0073972263 \\ -0.289787091 \\ -0.2173403182 \\ 0.0029588905 \\ -0.0884182635 \end{pmatrix}$$

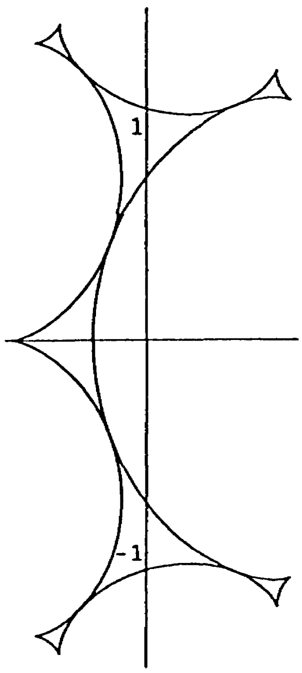


Figure 4e

Again, using the connection between univalent meromorphic polynomials and σ_6^*, we have the following theorem.

THEOREM 15. *If* $\mu(z) = 1/z + a_1 z + a_2 z^2 + \frac{2}{3} a_2 z^3 + \frac{1}{4} a_1 z^4 - \frac{1}{6} z^6$ *is univalent in* $0 < |z| < 1$ *and all zeros of* μ' *lie on* $\{|z| = 1\}$, *then* a_1 *and* a_2 *satisfy the sharp inequalities*

$$\frac{2}{3}c \le a_1 \le \frac{2}{3} \quad \textit{and} \quad -\frac{5(\sqrt{5}-1)}{16} \le a_2 \le \frac{5(\sqrt{5}-1)}{16},$$

where $-1.1408 \approx c = 12t^2 - 4t - 2$ *and* t *is the real root of the equation* $16t^3 - 16t^2 + 4t + 1 = 0$.

Figure 4 shows the variety of possible image domains for μ^* when μ is an extreme point in σ_8^+.

References

[1] D. A. Brannan, *Coefficient regions for univalent polynomials of small degree,* Mathematika 14 (1967), 165–169.

[2] ———, *Results, old and new, on univalent polynomials,* unpublished manuscript.

[3] V. F. Cowling and W. C. Royster, *Domains of variability for univalent polynomials,* Proc. Amer. Math. Soc. 19 (1968), 767–772.

[4] N. Dunford and J. T. Schwartz, *Linear operators I: General theory,* Pure Appl. Math., 7, Interscience, New York, 1958.

[5] A. W. Goodman, *Functions typically real and meromorphic in the unit circle,* Trans. Amer. Math. Soc. 81 (1956), 92–105.

[6] W. E. Kirwan and G. Schober, *New inequalities from old ones,* Math. Z. 180 (1982), 19–40.

[7] S. Mansour, *On extreme points in two families of functions with univalent sequential limits,* Dissertation, University of Kentucky, 1972.

[8] E. M. Pupplo-Cody, *A structural formula for a class of typically real functions and some consequences,* Dissertation, University of Kentucky, 1991.

[9] M. S. Robertson, *On the coefficients of a typically real function,* Bull. Amer. Math. Soc. 41 (1935), 565–572.

[10] D. H. Schnack, *On the coefficient bounds for a class of meromorphic univalent function,* Dissertation, University of Kentucky, 1969.

[11] T. J. Suffridge, *On univalent polynomials,* J. London Math. Soc. 44 (1969), 496–504.

[12] ———, *Extreme points in a class of polynomials having univalent sequential limits,* Trans. Amer. Math. Soc. 163 (1972), 225–237.

E. M. Pupplo-Cody
Mathematics Department
Marshall University
Huntington, WV 25755

T. J. Suffridge
Mathematics Department
University of Kentucky
Lexington, KY 40506