Self-Similar Solutions of the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation

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1. Introduction

In this paper we consider self-similar solutions to the nonlinear Schrödinger equation

$$iu_t+\Delta u+\epsilon|u|^\alpha u=0, \tag{1.1}$$

where $u=u(t,x)$ is a complex-valued function of $t\in\mathbb{R}$ (or a subset of $\mathbb{R}$) and $x\in\mathbb{R}^n$, α is a positive real number, and $\epsilon:=\pm1$. It is known that if $\epsilon=-1$ and $\alpha<4/(n-2)$, or if $\epsilon=+1$ and $\alpha<4/n$, then all solutions (in $H^1(\mathbb{R}^n)$ for example) are global. If $\epsilon=+1$ and $4/n\le\alpha<4/(n-2)$, then there exist nonglobal solutions. From this point of view, the value $\alpha=4/n$ in (1.1) is called the *critical power.*

If $u(t,x)$ is a solution of (1.1), then for all $\lambda>0$, the rescaled function

$$u_\lambda(t,x):=\lambda^{-i\omega+2/\alpha}u(\lambda^2t,\lambda x), \tag{1.2}$$

where $\omega\in\mathbb{R}$ is fixed, is also a solution of (1.1). A solution u of (1.1) is *self-similar* with respect to rescaling if $u\equiv u_\lambda$ for all $\lambda>0$ (and some fixed choice of $\omega\in\mathbb{R}$). It is straightforward to check that u, defined for $t>0$, is a self-similar solution (with respect to rescaling) of (1.1) if and only if u is of the form

$$u(t,x)=t^{i\omega/2-1/\alpha}v(x/\sqrt{t}), \tag{1.3}$$

where $v:\mathbb{R}^n\to\mathbb{C}$ satisfies

$$\Delta v-\frac{i}{2}\left(y\cdot\nabla v+\frac{2}{\alpha}v\right)-\frac{\omega}{2}v+\epsilon|v|^\alpha v=0. \tag{1.4}$$

The function v is called the *profile* of the self-similar solution u. Self-similar solutions of (1.1) have played an important role in the study of the blow-up behavior of nonglobal solutions to (1.1) with $\epsilon=1$ (see [11; 12; 13; 14; 16]). Just recently, Johnson and Pan [9] and Kopell and Landman [10] have rigorously analyzed the asymptotic behavior for large $|y|$ of the profiles of such self-similar solutions. In particular, Kopell and Landman [10] treat the

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supercritical powers, while Johnson and Pan [9] treat both the critical and supercritical powers.

In the first part of this paper, we present an analysis of the asymptotic behavior of (radially symmetric) solutions to (1.4) for the critical power $\alpha = 4/n$ and $\epsilon = 1$, that is,

$$\Delta v - \frac{i}{2}\left(y\cdot\nabla v + \frac{n}{2}v\right) - \frac{\omega}{2}v + |v|^{4/n}v = 0, \tag{1.5}$$

where $v = v(y) := v(r)$ for $y \in \mathbb{R}^n$, $r := |y|$, and $\omega \in \mathbb{R}$ is a parameter. Our principal result concerning (1.5) is the following.

1.1. THEOREM. *Let $v\colon \mathbb{R}^n \to \mathbb{C}$ $(n \geq 1)$ be a nontrivial radially symmetric regular solution of equation* (1.5). *Then there exists a nonzero complex number c and a real number θ such that*

$$v(r) = c\exp\left(\frac{ir^2}{8}\right)r^{-n/2}\sin\left(\frac{r^2}{8} - \omega\log\frac{r}{2\sqrt{2}} + \theta\right) + O\left(\frac{1}{r^{(n/2)+2}}\right)$$

as $r \to \infty$. Furthermore, the error estimate $O(1/r^{(n/2)+2})$ cannot in general be improved.

This result is close to Theorem 1 in Johnson and Pan [9]. The essential differences are that the methods here apply to all dimensions, including $n = 1$, and that a slightly better error estimate is obtained. (Theorem 1 of [9] gives $O(1/r^{(n/2)+2-(2/n)})$ as the error term.) Also, though this is probably not essential, we consider all real values of the parameter ω, not just positive values. Finally, the treatment here is perhaps less technical than in [9], and the asymptotic form of a solution appears naturally in an integral formulation of a second-order ordinary differential equation; see (3.9) and (3.10) below. The most technical part of our proof of Theorem 1.1 is Section 3b, where we show, by explicitly calculating the leading term in the asymptotic expansion of the integal in (3.9), that the error estimate $O(1/t)$ in (3.10) cannot in general be improved. Here, we were inspired by work of Li [15] and Gui, Ni, and Wang [7], who analyzed the asymptotic behavior of positive solutions to certain elliptic equations in $\mathbb{R}^n$.

It is well known (cf. Ginibre and Velo [6], Weinstein [20]) that the nonlinear Schrödinger equation (1.1) with the critical power $\alpha = 4/n$, that is,

$$iu_t + \Delta u + \epsilon|u|^{4/n}u = 0, \tag{1.6}$$

admits a special conservation law, called the *pseudo-conformal* conservation law, and is invariant under a corresponding transformation. More precisely, and we use here the formulation in Section 3 of Weinstein [20], if $u = u(t,x)$ is a solution of (1.6), defined for t in some subset of the real numbers and all $x \in \mathbb{R}^n$, and if $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{R})$, the set of real matrices with determinant 1, then $(Au)(t,x)$ is likewise a solution, where

$$(Au)(t,x) := (a+bt)^{-n/2}u\left(\frac{c+dt}{a+bt}, \frac{x}{a+bt}\right)\exp\left(\frac{ib|x|^2}{4(a+bt)}\right). \tag{1.7}$$

In other words, formula (1.7) defines a group action of $\mathbf{SL}_2(\mathbb{R})$ on complex-valued functions $u(t,x)$, leaving invariant the set of solutions to (1.6). By abuse of notation, we denote this action simply by Au, where $A \in \mathbf{SL}_2(\mathbb{R})$. In addition, multiplying u by a complex number of modulus one also leaves invariant the set of solutions to (1.6); and so we obtain a group action of $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{S}^1$ on the set of solutions to (1.6). This action includes the transformation (1.2).

The existence of a larger class of transformations leaving the set of solutions invariant suggests a broader definition of self-similar solutions. Roughly speaking, by a self-similar solution to (1.6) we will mean a solution u which is fixed by a 1-parameter subgroup of $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{S}^1$. Recall that the Lie algebra of $\mathbf{SL}_2(\mathbb{R})$ is $\mathfrak{sl}_2(\mathbb{R})$, the set of traceless 2×2 real matrices.

1.2. Definition. Let $B \in \mathfrak{sl}_2(\mathbb{R})$, $B \neq 0$. A solution u of (1.6) is called *self-similar with respect to* B if there exists $\omega \in \mathbb{R}$ such that $e^{sB}u \equiv e^{is\omega}u$ for all $s \in \mathbb{R}$.

It turns out, as we shall prove in Section 5, that such self-similar solutions of (1.6) fall into three classes (up to transformation by $A \in \mathbf{SL}_2(\mathbb{R})$): those given by (1.3); standing wave solutions (i.e. solutions such as $u(t,x) := e^{i\omega t}\varphi(x)$, where $\Delta\varphi - \omega\varphi + \epsilon|\varphi|^\alpha\varphi = 0$); and solutions of the form

$$u(t,x) := (1+t^2)^{-n/4} \exp\left(\frac{it|x|^2}{4(1+t^2)}\right) e^{i\omega \arctan t}\varphi\left(\frac{x}{(1+t^2)^{1/2}}\right), \tag{1.8}$$

where $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ satisfies the elliptic equation

$$-\Delta\varphi + \tfrac{1}{4}|y|^2\varphi + \omega\varphi = \epsilon|\varphi|^{4/n}\varphi. \tag{1.9}$$

Solutions to (1.6) of the form (1.9) are analogous to the solutions of the heat equation given in Olver [17, ex. 3.17, p. 212].

Unlike (1.5), it is not true that all solutions of (1.9) exhibit essentially the same asymptotic behavior; and so we look for solutions which are well-behaved as $|y| \to \infty$. One also has the choice of seeking radially symmetric solutions by classical "shooting" type arguments for ODEs, or of variational methods; it is this latter approach that we take here. This has the advantage of giving solutions to (1.9) with both signs $\epsilon = \pm 1$. Also, one obtains rather easily a sharp condition for the existence of a positive solution, as well as the existence of an infinite family of solutions. In addition, as we shall see, one obtains solutions to (1.9) when $\mathbb{R}^n$ is replaced with an open cone Ω and with φ satisfying a Dirichlet or Neumann boundary condition on $\partial\Omega$.

1.3. Theorem. *If* $\epsilon = 1$, *then for all fixed* $\omega \in \mathbb{R}$, *equation* (1.9) *has an infinite sequence of solutions in* $C^2(\mathbb{R}^n)$ *and having exponential decay as* $|y| \to \infty$. *Moreover, there exists a positive solution if and only if* $\omega > -n/2$ (*in which case this solution is in* C^∞). *Furthermore, when* $n=1$ *or* $n=2$, *all such solutions are in* $C^\infty(\mathbb{R}^n)$.

1.4. THEOREM. *If $\epsilon=-1$, then for all fixed $\omega\in\mathbb{R}$ such that $\omega<-n/2$, equation* (1.9) *has a unique positive solution in $H^1(\mathbb{R}^n)$. Moreover, this solution is in C^∞ and has exponential decay as $|y|\to\infty$. Equation* (1.9) *with $\epsilon=-1$ does not have any nontrivial solution in $H^1(\mathbb{R}^n)$ when $\omega\geq -n/2$.*

If $\omega<-n/2-k+1$ for some integer $k\geq 1$, then (1.9) *with $\epsilon=-1$ has at least m_k pairs of solutions in $H^1(\mathbb{R}^n)$, where $m_k:=\sum_{j=1}^{k}\binom{n+j-2}{n-1}$. All these solutions are in $C^2(\mathbb{R}^n)$ and have an exponential decay as $|y|\to\infty$.* (*When $n=1$ or $n=2$ all these solutions are in $C^\infty(\mathbb{R}^n)$.*)

The rest of the paper is organized as follows. Theorem 1.1 is proved in Sections 2, 3, and 4. In Section 2 we make various transformations of equation (1.5) to arrive at an equation which lends itself more easily to analysis, which we carry out in Section 3. In Section 3a, we show essentially that the constant c in Theorem 1.1 is not equal to zero if the solution v is nontrivial. (We show the equivalent statement for the "reduced" equation studied in Section 3.) In Section 3b, again in terms of this other equation, we calculate explicitly the second term in the asymptotic expansion, showing in particular that the decay rate of the error term is in general sharp. As indicated in Section 4, this information translates into a higher-order expansion for both $v(r)$ and $v'(r)$, which we do not make explicit. We remark that it is only at the very end of Section 4 that regularity of v at $r=0$ is used; all the results in Sections 2, 3, and 4 up through formula (4.2) are correct for radially symmetric solutions of (1.5) defined for $r>0$, or (with trivial modification) for r sufficiently large. If $n=1$, then (4.2) gives the asymptotic behavior of solutions of (1.5) for either $y>0$ or $y<0$ with $r=|y|$ sufficiently large.

In Section 5 we study the broader definition of self-similar solutions given above, and in particular we show that all such solutions belong to one of the three classes mentioned (see Proposition 5.2). In Section 6 we describe the variational formulation of (1.9) and give the proofs of Theorems 1.3 and 1.4, as well as some additional information. The variational arguments we use are completely standard. The only novelty lies, perhaps, in that we apply them to equation (1.9). As a result, at times we omit some of the details of the proofs.

Finally, in Section 7 we make some remarks on the Schrödinger evolution equation associated with (1.9), that is,

$$iv_s+\Delta v-\tfrac{1}{4}|y|^2v+\epsilon|v|^{4/n}v=0 \tag{1.10}$$

for $s\in\mathbb{R}$ (or an interval of $\mathbb{R}$) and $y\in\mathbb{R}^n$. We exhibit some interesting features of solutions to (1.10) and their interpretation in terms of solutions to (1.6).

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2. Preliminary Reductions and Estimates for Equation (1.5)

Setting

$$v(y)=\exp(i|y|^2/8)\varphi(y), \tag{2.1}$$

we see that equation (1.5) for v is equivalent to the following equation for φ:

$$\Delta\varphi+\frac{|y|^2}{16}\varphi-\frac{\omega}{2}\varphi+|\varphi|^{4/n}\varphi=0. \tag{2.2}$$

We consider radial solutions of (2.2) and so we set, by abuse of notation, $\varphi(r)=\varphi(y)$, where $r=|y|$. Equation (2.2) then becomes

$$\varphi''(r)+\frac{n-1}{r}\varphi'(r)+\frac{r^2}{16}\varphi(r)-\frac{\omega}{2}\varphi(r)+|\varphi(r)|^{4/n}\varphi(r)=0. \tag{2.3}$$

Finally, we set

$$\varphi(r)=r^{-\alpha}z(r^2/8), \tag{2.4}$$

where $\alpha\geq 0$ is to be determined. A tedious but straightforward calculation shows that equation (2.3) for φ is equivalent to

$$z''(t)+\frac{(n-2\alpha)}{2t}z'(t)+z(t)-\frac{\omega}{t}z(t)$$
$$-\frac{\alpha(n-\alpha-2)}{4t^2}z(t)+\frac{2}{8^{2\alpha/n}}\cdot\frac{|z(t)|^{4/n}z(t)}{t^{1+(2\alpha/n)}}=0. \tag{2.5}$$

If we set

$$E_\alpha(t,z,z'):=\frac{1}{2}|z'|^2+\frac{1}{2}|z|^2-\frac{\omega}{2t}|z|^2$$
$$-\frac{\alpha(n-\alpha-2)}{8t^2}|z|^2+\frac{2}{(2+4/n)8^{2\alpha/n}}\cdot\frac{|z|^{2+(4/n)}}{t^{1+(2\alpha/n)}}, \tag{2.6}$$

then a solution $z(t)$ of equation (2.5) verifies the relation

$$\frac{d}{dt}E_\alpha(t,z(t),z'(t))=-\frac{(n-2\alpha)}{2t}|z'(t)|^2+\frac{\omega}{2t^2}|z(t)|^2$$
$$+\frac{\alpha(n-\alpha-2)}{4t^3}|z(t)|^2-\frac{2(1+2\alpha/n)}{(2+4/n)8^{2\alpha/n}}\cdot\frac{|z(t)|^{2+(4/n)}}{t^{2+(2\alpha/n)}}. \tag{2.7}$$

Note that $E_\alpha(t,z,z')$ is bounded below (for $t\geq t_0>0$); thus it follows from (2.6) and (2.7) that if $\alpha\leq n/2$ then a solution of (2.5) can not blow up in finite time. Indeed, if either $|z(t)|$ or $|z'(t)|$ goes to infinity at a finite time then so does $E_\alpha(t,z(t),z'(t))$. On the other hand, if either $|z(t)|$ or $|z'(t)|$ goes to infinity at a finite time then $\frac{d}{dt}E_\alpha(t,z(t),z'(t))$ stays bounded above near the blow-up time, which makes it impossible for $E_\alpha(t,z(t),z'(t))$ to go to infinity. Thus, any local solution of (2.5), say on $(0,t_0)$, can be continued for all $t>0$; so any local solution of (2.3) on $(0,r_0)$ can be continued for all $r>0$.

Now let φ be a solution of equation (2.3) on $(0,\infty)$, and, for each $\alpha\geq 0$, let $z:=z_\alpha$ be given by the relation (2.4). Thus, $z_\alpha(t)$ satisfies equations (2.5) and (2.7), with of course the same value of α. First consider $\alpha=0$. It is clear from equation (2.7) that there exists $A>0$ such that if $|z_0(t)|\geq A$, then $\frac{d}{dt}E_0(t,z_0(t),z_0'(t))\le 0$. In other words, $\frac{d}{dt}E_0(t,z_0(t),z_0'(t))\ge 0$ only where $|z_0(t)|\le A$. It follows immediately from (2.7) that the positive part of $\frac{d}{dt}E_0(t,z_0(t),z_0'(t))$ is integrable on $(1,\infty)$. Since E_0 is bounded below, it follows also that the negative part of $\frac{d}{dt}E_0(t,z_0(t),z_0'(t)))$ is integrable on $(1,\infty)$. Since $\frac{d}{dt}E_0(t,z_0(t),z_0'(t))$ is therefore integrable on $(1,\infty)$, it follows that $E_0(t,z_0(t),z_0'(t))$ has a limit as $t\to\infty$, and in particular that $z_0(t)$ is bounded on $(1,\infty)$. From (2.4), with $\alpha=0$, we see that φ is bounded on $(1,\infty)$ and therefore that $|z_\alpha(t)|\le Ct^{\alpha/2}$ on $(1,\infty)$ for all $\alpha\ge 0$. Thus, if $0\le \alpha<1$ then the three last terms in (2.7), with $z=z_\alpha$, are all integrable on $(1,\infty)$. As long as $\alpha\le n/2$, it follows as above that $E_\alpha(t,z_\alpha(t),z_\alpha'(t))$ has a limit as $t\to\infty$, and in particular that $|z_\alpha(t)|$ is bounded on $(1,\infty)$. From (2.4) we see, as long as $0\le\alpha<1$ and $\alpha\le n/2$, that $r^\alpha|\varphi(r)|$ is bounded on $(1,\infty)$. Again from (2.4), this gives a better estimate on each of the z_αs, which can be used again in (2.7) to show that z_α is bounded for even larger values of α. This argument can be iterated until we get to $\alpha=n/2$, allowing us to conclude that $E_{n/2}(t,z_{n/2}(t),z_{n/2}'(t))$ has a limit as $t\to\infty$. This implies in particular the following crucial result.

2.1. PROPOSITION. *If $z_{n/2}$ is a solution of* (2.5) *with $\alpha=n/2$, then $z_{n/2}(t)$ and $z_{n/2}'(t)$ are bounded on* $(1,\infty)$.

3. Asymptotic Analysis of z in the Case $\alpha=n/2$

In what follows, we drop the subscript α, and consider only $\alpha=n/2$; so z is as in (2.4) with $\alpha=n/2$:

$$\varphi(r)=r^{-n/2}z(r^2/8),\quad z(t):=z_{n/2}(t). \tag{3.1}$$

If φ is a solution of equation (2.3), then

$$z''(t)+z(t)-\frac{\omega}{t}z(t)-\frac{n(n-4)}{16t^2}z(t)+\frac{|z(t)|^{4/n}z(t)}{4t^2}=0. \tag{3.2}$$

We wish to write equation (3.2) as a system of two first order ODEs, but instead of taking z and z' as the unknown functions, we consider the system in z and y where

$$z'(t)=\left(1-\frac{\omega}{2t}\right)y(t). \tag{3.3}$$

We thus obtain the following system of ODEs, equivalent to equation (3.2):

$$\frac{d}{dt}\begin{pmatrix} z\\ y\end{pmatrix}=\begin{pmatrix} 0 & 1-\omega/2t\\ -1+\omega/2t & 0\end{pmatrix}\begin{pmatrix} z\\ y\end{pmatrix}+\left(1-\frac{\omega}{2t}\right)^{-1}\cdot\frac{1}{t^2}\begin{pmatrix} 0\\ F(z,y)\end{pmatrix}, \tag{3.4}$$

where

$$F(z,y)=\left(\frac{n(n-4)}{16}+\frac{\omega^2}{4}\right)z-\frac{\omega}{2}y-\frac{1}{4}|z|^{4/n}z, \tag{3.5}$$

and where t is sufficiently large so that $1-\omega/2t\neq 0$.

Next we let $U(t)$ denote the unitary group of matrices

$$U(t)=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{3.6}$$

and we set

$$T(t,r)=U\left(t-r-\frac{\omega}{2}\log\frac{t}{r}\right)=U\left(t-\frac{\omega}{2}\log t\right)U\left(-r+\frac{\omega}{2}\log r\right). \tag{3.7}$$

Indeed, $T(\cdot,\cdot)$ is the propagator associated with the "unperturbed" part of system (3.4). It is straightforward to check, using (3.4), (3.6), and (3.7), that

$$\frac{d}{dr}T(s,r)\begin{pmatrix} z(r) \\ y(r) \end{pmatrix}=T(s,r)\left(1-\frac{\omega}{2r}\right)^{-1}\frac{1}{r^2}\begin{pmatrix} 0 \\ F(z(r),y(r)) \end{pmatrix}.$$

and therefore

$$T(s,t)\begin{pmatrix} z(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} z(s) \\ y(s) \end{pmatrix}+\int_s^t T(s,r)\left(1-\frac{\omega}{2r}\right)^{-1}\frac{1}{r^2}\begin{pmatrix} 0 \\ F(z(r),y(r)) \end{pmatrix}dr. \tag{3.8}$$

Since the $T(s,r)$ are all unitary matrices and $F(z(r),y(r))$ is bounded as $r\to\infty$, it follows that

$$\lim_{t\to\infty} T(s,t)\begin{pmatrix} z(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}$$

exists (in $\mathbb{C}^2$). In fact, we have the formula

$$\begin{pmatrix} z(t) \\ y(t) \end{pmatrix}=T(t,s)\begin{pmatrix} a \\ b \end{pmatrix}-\int_t^\infty T(t,r)\left(1-\frac{\omega}{2r}\right)^{-1}\frac{1}{r^2}\begin{pmatrix} 0 \\ F(z(r),y(r)) \end{pmatrix}dr. \tag{3.9}$$

In particular,

$$\begin{pmatrix} z(t) \\ y(t) \end{pmatrix}-T(t,s)\begin{pmatrix} a \\ b \end{pmatrix}=O\left(\frac{1}{t}\right) \quad \text{as } t\to\infty. \tag{3.10}$$

Of course the complex numbers a and b depend on the particular choice of s, and we are free to choose any convenient sufficiently large value for s. For simplicity, we require that $s-(\omega/2)\log s$ be an integer multiple of 2π; it follows that $T(t,s)=U(t-(\omega/2)\log t)$. Formula (3.10), along with (3.3), then translates as

$$z(t)=a\cos\left(t-\frac{\omega}{2}\log t\right)+b\sin\left(t-\frac{\omega}{2}\log t\right)+O\left(\frac{1}{t}\right); \tag{3.11}$$

$$y(t)=-a\sin\left(t-\frac{\omega}{2}\log t\right)+b\cos\left(t-\frac{\omega}{2}\log t\right)+O\left(\frac{1}{t}\right); \tag{3.12}$$

$$z'(t)=\left(1-\frac{\omega}{2t}\right)y(t)=-a\sin\left(t-\frac{\omega}{2}\log t\right)+b\cos\left(t-\frac{\omega}{2}\log t\right)+O\left(\frac{1}{t}\right). \tag{3.13}$$

To show that formula (3.10) gives the correct asymptotic behavior of z, we need to show two things: that the constants a and b are not both zero (unless $z\equiv 0$); and that the estimate $O(1/t)$ can not be improved.

3a. The Constants a and b

To show that a and b cannot both be equal to zero, we make yet another transformation, setting

$$\varphi(r)=r^{-(n-1)/2}x(r); \tag{3.14}$$

hence

$$x(r)=r^{-1/2}z\left(\frac{r^2}{8}\right), \tag{3.15}$$

and

$$x'(r)=\frac{-1}{2}r^{-3/2}z\left(\frac{r^2}{8}\right)+\frac{r^{1/2}}{4}\left(1-\frac{4\omega}{r^2}\right)y\left(\frac{r^2}{8}\right). \tag{3.16}$$

Equation (2.3) for φ is then equivalent to

$$x''(r)+\left[\frac{r^2}{16}-\frac{\omega}{2}-\frac{(n-1)(n-3)}{4r^2}\right]x(r)+\frac{1}{r^{2(n-1)/n}}|x(r)|^{4/n}x(r)=0. \tag{3.17}$$

If we let

$$\begin{aligned}H(r,x,x'):=&\frac{1}{2}|x'|^2+\frac{r^2}{32}|x|^2-\frac{\omega}{4}|x|^2\\&-\frac{(n-1)(n-3)}{8r^2}|x|^2+\frac{1}{2+4/n}\cdot\frac{|x|^{2+(4/n)}}{r^{2(n-1)/n}},\end{aligned} \tag{3.18}$$

then a solution $x(t)$ of (3.17) satisfies

$$\frac{d}{dr}H(r,x(r),x'(r))=|x(r)|^2\left[\frac{r}{16}+\frac{(n-1)(n-3)}{4r^3}-\frac{(n-1)}{(n+2)}\cdot\frac{|x(r)|^{4/n}}{r^{(3n-2)/n}}\right]. \tag{3.19}$$

Suppose that $z(t)$ is a nontrivial solution of (3.2), and so $x(r)$ given by (3.14) or (3.15) is a nontrivial solution of (3.17). Since (by Proposition 2.1) $z(t)$ is bounded as $t\to\infty$, it follows from (3.15) that $x(r)$ decays to zero as $r\to\infty$. It then follows from (3.19) that $\frac{d}{dr}H(r,x(r),x'(r))>0$ for r sufficiently large. Moreover, it is clear that for r sufficiently large, $H(r,x(r),x'(r))>0$. Thus $H(r,x(r),x'(r))$ converges to a strictly positive (finite or infinite) limit as $r\to\infty$. On the other hand, if the constants a and b in (3.10) are both zero, we see from (3.15) and (3.16) that $H(r,x(r),x'(r))$ decays to zero as $r\to\infty$. This contradiction shows that a and b cannot both be equal to zero. □

3b. A More Precise Asymptotic Estimate

In this section we wish to analyze the integral term in (3.9). Since by formula (3.7)

$$T(t,r)=U\left(t-\frac{\omega}{2}\log t\right)U\left(-r+\frac{\omega}{2}\log r\right),$$

the matrix in t can be removed from the integral. Thus, it suffices to determine the asymptotic behavior of

$$\int_t^\infty U\left(-r+\frac{\omega}{2}\log r\right)\left(1-\frac{\omega}{2r}\right)^{-1}\frac{1}{r^2}\begin{pmatrix}0\\ F(z(r),y(r))\end{pmatrix}dr. \tag{3.20}$$

Using (3.5), (3.11), and (3.12), and the fact that z is bounded as $r\to\infty$, we rewrite $F(z(r),y(r))$ as

$$F(z(r),y(r))=F\left(Z\left(r-\frac{\omega}{2}\log r\right),Y\left(r-\frac{\omega}{2}\log r\right)\right)+O\left(\frac{1}{r}\right), \tag{3.21}$$

where

$$Z(s):=a\cos s+b\sin s, \qquad Y(s):=-a\sin s+b\cos s.$$

The term $O(1/r)$ in (3.21), when substituted into the integral (3.20), gives a total contribution to the integral of the order $O(1/r^2)$. The remaining part of the integral is precisely

$$\int_t^\infty U\left(-r+\frac{\omega}{2}\log r\right)\left(1-\frac{\omega}{2}\right)^{-1}\frac{1}{r^2}\begin{pmatrix}0\\ F(Z(r-(\omega/2)\log r),Y(r-(\omega/2)\log r))\end{pmatrix}dr.$$

We immediately make the change of variables $s:=r-(\omega/2)\log r$, which yields

$$\int_{t-(\omega/2)\log t}^\infty U(-s)\frac{1}{s^2}\begin{pmatrix}0\\ F(Z(s),Y(s))\end{pmatrix}ds+O\left(\frac{\log t}{t^2}\right). \tag{3.22}$$

To determine the asymptotic behavior of the integral in (3.22), we make use of the following elementary result.

3.1. LEMMA. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous periodic function whose mean value over a single period is equal to m. It follows that, as $\tau\to\infty$,*

$$\int_\tau^\infty h(s)\,\frac{ds}{s^2}=\frac{m}{\tau}+O\left(\frac{1}{\tau^2}\right).$$

Proof. Let $h(s)=m+g(s)$, so that g has mean value zero over a single period. If G is a primitive of g, then integration by parts yields

$$\int_\tau^\infty h(s)\,\frac{ds}{s^2}=m\int_\tau^\infty\frac{ds}{s^2}+\int_\tau^\infty g(s)\,\frac{ds}{s^2}=\frac{m}{\tau}+\left[\frac{G(s)}{s^2}\right]_\tau^\infty+2\int_\tau^\infty G(s)\,\frac{ds}{s^3}.$$

The result follows, since the assumptions imply that G is bounded. □

It follows from this lemma, and from the definitions (3.5) and (3.6) of F and $U(s)$ respectively, that

$$\int_\tau^\infty U(-s)\frac{1}{s^2}\begin{pmatrix}0\\ F(Z(s),Y(s))\end{pmatrix}ds$$
$$=\frac{1}{\tau}\left\{\frac{n(n-4)}{32}-\frac{\omega^2}{8}\begin{pmatrix}-b\\ a\end{pmatrix}-\frac{\omega}{4}\begin{pmatrix}a\\ b\end{pmatrix}+\begin{pmatrix}-bA_n(a,b)\\ aB_n(a,b)\end{pmatrix}\right\}+O\left(\frac{1}{\tau^2}\right). \tag{3.23}$$

where

$$\begin{pmatrix} A_n(a,b) \\ B_n(a,b) \end{pmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} \sin^2 s \\ \cos^2 s \end{pmatrix} |a\cos s + b\sin s|^{4/n}\, ds. \tag{3.24}$$

Note that the constants $A_n(a,b)$ and $B_n(a,b)$ are positive, as long as a and b are not both zero.

Using (3.23) to estimate (3.22), and keeping in mind that

$$\frac{1}{t} - \frac{1}{t-(\omega/2)\log t} = O\left(\frac{\log t}{t^2}\right),$$

we obtain the following more refined consequence of formula (3.9):

$$\begin{pmatrix} z(t) \\ y(t) \end{pmatrix} - T(t,s)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{t} U\left(t - \frac{\omega}{2}\log t\right)\left\{\left(\frac{n(n-4)}{32} - \frac{\omega^2}{8}\right)\begin{pmatrix} b \\ -a \end{pmatrix} + \frac{\omega}{4}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} bA_n(a,b) \\ -aB_n(a,b) \end{pmatrix}\right\} + O\left(\frac{\log t}{t^2}\right) \tag{3.25}$$

as $t \to \infty$. The point of all this is to show that the $O(1/t)$ term in (3.10) cannot in general be improved. While it is possible to imagine just the right combination of parameters so that all the terms on the right side of (3.25) before the $O((\log t)/t^2)$ in fact cancel each other, this will not happen all the time. Indeed, if $n \geq 4$ and $\omega = 0$, and if the solution is nontrivial so that $a^2 + b^2 \neq 0$, then these terms do not all cancel. Though we do not offer a proof, it seems quite likely that the cancellation of all these terms is the exception rather than the rule.

Writing (3.25) out in its two components, and grouping the various constants together, we obtain

$$z(t) = a\cos\left(t - \frac{\omega}{2}\log t\right) + b\sin\left(t - \frac{\omega}{2}\log t\right) + \frac{c}{t}\cos\left(t - \frac{\omega}{2}\log t\right) + \frac{d}{t}\sin\left(t - \frac{\omega}{2}\log t\right) + O\left(\frac{\log t}{t^2}\right); \tag{3.26}$$

$$y(t) = -a\sin\left(t - \frac{\omega}{2}\log t\right) + b\cos\left(t - \frac{\omega}{2}\log t\right) - \frac{c}{t}\sin\left(t - \frac{\omega}{2}\log t\right) + \frac{d}{t}\cos\left(t - \frac{\omega}{2}\log t\right) + O\left(\frac{\log t}{t^2}\right); \tag{3.27}$$

$$\begin{aligned} z'(t) &= \left(1 - \frac{\omega}{2t}\right) y(t) \\ &= -a\left(1 - \frac{\omega}{2t}\right)\sin\left(t - \frac{\omega}{2}\log t\right) + b\left(1 - \frac{\omega}{2t}\right)\cos\left(t - \frac{\omega}{2}\log t\right) \\ &\quad - \frac{c}{t}\sin\left(t - \frac{\omega}{2}\log t\right) + \frac{d}{t}\cos\left(t - \frac{\omega}{2}\log t\right) + O\left(\frac{\log t}{t^2}\right). \end{aligned} \tag{3.28}$$

4. Conclusion of the Proof of Theorem 1.1

It is a simple matter to translate the results of the previous section to give the asymptotic behavior for φ and for v. Taking into account only (3.11), we obtain

$$\varphi(r)=r^{-n/2}\left\{a\cos\left(\frac{r^2}{8}-\omega\log\frac{r}{2\sqrt{2}}\right)+b\sin\left(\frac{r^2}{8}-\omega\log\frac{r}{2\sqrt{2}}\right)\right\} + O\left(\frac{1}{r^{2+(n/2)}}\right); \tag{4.1}$$

$$v(r)=r^{-n/2}\exp\left(\frac{ir^2}{8}\right)\left\{a\cos\left(\frac{r^2}{8}-\omega\log\frac{r}{2\sqrt{2}}\right)+b\sin\left(\frac{r^2}{8}-\omega\log\frac{r}{2\sqrt{2}}\right)\right\} + O\left(\frac{1}{r^{2+(n/2)}}\right). \tag{4.2}$$

(Recall that a and b are complex numbers.) These estimates can be carried out to the next order by using (3.26) instead of (3.11). Also, asymptotic estimates for the derivatives $\varphi'(r)$ and $v'(r)$ can be obtained from (3.1), (2.1), and either (3.13) or (3.28). We leave it to the reader to make these formulas explicit.

One might be tempted to express the cosine and sine using the exponential function, thereby giving

$$v(r)=a'r^{-n/2+i\omega}+b'r^{-n/2-i\omega}\exp\left(\frac{ir^2}{4}\right)+O\left(\frac{1}{r^{2+(n/2)}}\right). \tag{4.3}$$

This seems to imply that $v(r)$ could exhibit two strikingly different types of behavior. For regular (radially symmetric) solutions of (1.5) in $\mathbb{R}^n$, $n\geq 2$, this conclusion is not correct. Indeed, if v is a regular radially symmetric solution of (1.5), then φ, given by (2.1), satisfies the initial value problem associated to (2.3) with data $\varphi(0)=\varphi_0\in\mathbb{C}$ and $\varphi'(0)=0$. Since equation (2.3) is invariant under multiplication by a complex number of modulus one, and since $\varphi_0\in\mathbb{R}$ implies that $\varphi(r)\in\mathbb{R}$ for all $r>0$, it follows that φ is always a fixed complex multiple of a real-valued function. Thus, the constants a and b in (4.1) must be the same complex multiple of two real numbers. This implies that neither a' nor b' in (4.3) can equal zero (for nontrivial solutions). Furthermore, factoring out the common complex divisor of a and b and using trigonometric identities, one can rewrite (4.1) as

$$\varphi(r)=cr^{-n/2}\sin\left(\frac{r^2}{8}-\omega\log\frac{r}{2\sqrt{2}}+\theta\right)+O\left(\frac{1}{r^{2+(n/2)}}\right), \tag{4.4}$$

where c is a nonzero complex number (since a and b are not both zero if the solution is nontrival) and θ is a real number. If $n=1$ then this reasoning is correct only if we require that $\varphi'(0)=0$ (which is the case for a symmetric solution), or more generally that $\varphi(0)$ and $\varphi'(0)$ be the same complex multiple of two real numbers.

5. Pseudo-Conformal Self-Similar Solutions

In this section we study the notion of self-similar solution as given by Definition 1.2. One immediately verifies the following proposition.

5.1. PROPOSITION. *The solution u of* (1.6) *is self-similar with respect to $B \in \mathfrak{sl}_2(\mathbb{R})$ if and only if $P^{-1}u$ is a self-similar solution with respect to $P^{-1}BP \in \mathfrak{sl}_2(\mathbb{R})$ for all $P \in \mathbf{SL}_2(\mathbb{R})$. Furthermore, if u is a self-similar solution of* (1.6) *with respect to $B \in \mathfrak{sl}_2(\mathbb{R})$, then u is a self-similar solution with respect to kB for all nonzero $k \in \mathbb{R}$.*

Two self-similar solutions u and v such that $u = Pv$ for some $P \in \mathbf{SL}_2(\mathbb{R})$ are called *pseudo-conformally equivalent.* To identify all self-similar solutions of (1.6), it suffices to identify all equivalence classes of pseudo-conformally equivalent self-similar solutions. By the previous proposition, this amounts to identifying the equivalence classes in $\mathfrak{sl}_2(\mathbb{R})$ under the relation

$$A \sim B \Leftrightarrow \exists P \in \mathbf{SL}_2(\mathbb{R}) \text{ and } k \in \mathbb{R}\setminus\{0\} \text{ such that } A = kP^{-1}BP. \tag{5.1}$$

It is easy to see that the self-similar solutions considered earlier in this paper - that is, such that $u = u_\lambda$ where u_λ is given by (1.2) - are in fact self-similar with respect to $B_1 := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, self-similar solutions with respect to $B_2 := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ are precisely standing wave solutions of (1.6) - that is, solutions of the form $e^{i\omega t}\varphi(x)$, where $\Delta\varphi - \omega\varphi + \epsilon|\varphi|^{4/n}\varphi = 0$. (This, by the way, shows that the blowing up solution of (1.6) obtained from a standing wave by a pseudo-conformal transformation is indeed a self-similar solution.) It turns out that all self-similar solutions are pseudo-conformally equivalent either to one of these two examples or else to a third example which we discuss below.

5.2. PROPOSITION. *Every $B \in \mathfrak{sl}_2(\mathbb{R})$, $B \neq 0$, is related under $\sim$ to one of the following three matrices:*

$$B_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_3 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. In order to determine the equivalence classes of matrices in $\mathfrak{sl}_2(\mathbb{R})$ under the relation $A \sim B$ defined in (5.1), we use the "real version" of the Jordan canonical form of a matrix; we refer the reader to Chapter 6 of Hirsch and Smale [8] for an excellent discussion of this canonical form (in particular Theorem 2 of Section 4). One remark before we begin. When one considers putting a matrix B in canonical form, one means finding an invertible matrix P such that $P^{-1}BP$ has the desired form. Multiplying P by a suitable scalar allows us to assume that $\det P = \pm 1$. Since the relation $\sim$ is less restrictive than similitude of matrices, we will see below that further conjugation by the matrix $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ will enable us to choose $P \in \mathbf{SL}_2(\mathbb{R})$.

Let $B \in \mathfrak{sl}_2(\mathbb{R})$, $B \neq 0$. Since $\operatorname{tr} B = 0$, it follows that the eigenvalues of B are $\pm\lambda$. If $\det B < 0$, then λ is a nonzero real number and the Jordan form of B is $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. Since this matrix is unchanged by conjugation with the matrix J, we may assume that the matrix P used to put B in its canonical form is of determinant 1. Thus $B \sim B_1$.

Suppose next that $\det B = 0$, and so $\lambda = 0$. The Jordan canonical form of B is then precisely B_2. Since $J^{-1} B_2 J = -B_2$, it follows that $B \sim B_2$.

Finally, if $\det B > 0$ then $\lambda = i\gamma$, where γ is a nonzero real number. The real canonical form of B is then γB_3. Again, since $J^{-1} B_3 J = -B_3$, it follows that $B \sim B_3$. This completes the proof. □

Thus, to identify all the self-similar solutions of (1.6), it remains only to study self-similar solutions with respect to B_3. The 1-parameter subgroup of $\mathbf{SL}_2(\mathbb{R})$ generated by B_3 is precisely $\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$. Thus, $u(t,x)$ is self-similar with respect to B_3 if there exists $\omega \in \mathbb{R}$ such that

$$(\cos s - t \sin s)^{-n/2} u\left(\frac{\sin s + t \cos s}{\cos s - t \sin s}, \frac{x}{\cos s - t \sin s}\right) \exp\left(\frac{-i|x|^2 \sin s}{4(\cos s - t \sin s)}\right) \equiv e^{is\omega} u(t,x) \quad (5.2)$$

for all $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. In particular, if we set $t := -\tan s$, where $-\pi/2 < s < \pi/2$, then (5.2) reduces to

$$u(t,x) = (1+t^2)^{-n/4} \exp\left(\frac{it|x|^2}{4(1+t^2)}\right) e^{i\omega \arctan t} u\left(0, \frac{x}{(1+t^2)^{1/2}}\right).$$

On the other hand, one verifies that a function of the form

$$u(t,x) := (1+t^2)^{-n/4} \exp\left(\frac{it|x|^2}{4(1+t^2)}\right) e^{i\omega \arctan t} \varphi\left(\frac{x}{(1+t^2)^{1/2}}\right), \quad (5.3)$$

where $\varphi : \mathbb{R}^n \to \mathbb{C}$, does indeed satisfy the invariance condition (5.2). Finally, a mildly unpleasant calculation shows that $u(t,x)$ of the form (5.3) is a solution of (1.6) if and only if φ is a solution of (1.9).

6. The Variational Problem

We denote by X the Hilbert space of real-valued functions

$$X := \{u;\ u \in H^1(\mathbb{R}^n),\ |\cdot| u \in L^2(\mathbb{R}^n)\}, \quad (6.1)$$

endowed with the norm $\|u\|_*^2 := \|\nabla u\|^2 + \|(1+|\cdot|^2)^{1/2} u\|^2$ (here $|\cdot|$ denotes the function $x \mapsto |x|$ and $\|\cdot\|$ is the classical norm in $L^2(\mathbb{R}^n)$). By abuse of notation, we will write xu for the function $x \mapsto xu(x)$. Throughout this section $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(\mathbb{R}^n)$. The norm in $L^p(\mathbb{R}^n)$ will be designated by $\|\cdot\|_p$. We begin by collecting the properties of X which we shall need.

6.1. PROPOSITION. *Let X be defined by* (6.1).

(i) *The embedding of X into $L^p(\mathbb{R}^n)$ is compact for all $p \geq 2$ such that $(n-2)p < 2n$.*

(ii) *The constant $c_* := \inf\{\|\nabla u\|^2 + \|xu\|^2;\ u \in X,\ \|u\|^2 = 1\}$ is achieved and $c_* = n$.*

(iii) *The norm defined by*

$$\|u\|_X^2 = \|\nabla u\|^2 + \tfrac{1}{4}\|xu\|^2 \tag{6.2}$$

is an equivalent norm on X.

(iv) *Let $p > 2$ if $n = 1, 2$ or $p \in (2, 2n/(n-2)]$ if $n \geq 3$. For all $\epsilon > 0$ there exists C_ϵ such that for all $u \in X$,*

$$\|u\|^2 \leq C_\epsilon \|u\|_p^2 + \epsilon \|u\|_X^2. \tag{6.3}$$

Proof. (i) For $p = 2$ and any $R > 0$, one has $\int_{|x| \geq R} |u(x)|^2\, dx \leq R^{-2}\|xu\|^2$. Since for a fixed $R > 0$, $H^1([|x| < R])$ is compactly embedded in $L^2([|x| < R])$, one can easily conclude that the embedding $X \subset L^2(\mathbb{R}^n)$ is compact. On the other hand, as $X \subset H^1(\mathbb{R}^n)$, the Sobolev embedding theorem implies $X \subset L^q(\mathbb{R}^n)$ for $q > 2$ and $(n-2)q < 2n$. Choosing $2 < p < q$ and interpolating between $L^2(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, we conclude that the embedding $X \subset L^p(\mathbb{R}^n)$ is compact.

(ii) If $(u_j)_j$ is a sequence in X such that $\|u_j\| = 1$ and $\|\nabla u_j\|^2 + \|xu_j\|^2 \downarrow c_*$, by the compactness of the embedding $X \subset L^2(\mathbb{R}^n)$ we may assume that $u_j \rightharpoonup \psi_1$ in X-weak and $u_j \to \psi_1$ in $L^2(\mathbb{R}^n)$. As $\|\nabla\psi_1\|^2 + \|x\psi_1\|^2 \leq c_*$ and $\|\psi_1\| = 1$, we conclude that c_* is achieved. (In fact, one can easily check that $-\Delta\psi_1 + |x|^2\psi_1 = c_*\psi_1$ and conclude that $c_* = n$ and $\psi_1(x) = c_0 \exp(-|x|^2/2)$, as we shall see below for an analogous equation.)

(iii) This is an immediate consequence of (ii).

(iv) If $u \in X$ then by the Sobolev embedding theorem one has $u \in L^p(\mathbb{R}^n)$; next we choose $R > 0$ so that $R^{-2} = \epsilon$, and write:

$$\begin{aligned}\int_{\mathbb{R}^n} u^2(x)\, dx &= \int_{|x| \geq R} u^2(x)\, dx + \int_{|x| < R} u^2(x)\, dx \\ &\leq \epsilon \int_{|x| \geq R} |x|^2 u^2(x)\, dx + \operatorname{meas}([|x| < R])^{1-2/p} \|u\|_p^2,\end{aligned}$$

from which one gets (6.3) with $C_\epsilon := C(n) R^{n(1-2/p)}$. □

Next, on $L^2(\mathbb{R}^n)$ we define an unbounded self-adjoint operator L by setting

$$D(L) = \{u \in X;\ Lu := -\Delta u + \tfrac{1}{4}|x|^2 u \in L^2(\mathbb{R}^n)\}.$$

In other words, for $u \in D(L)$, Lu is the unique element of $L^2(\mathbb{R}^n)$ such that $\langle \nabla u, \nabla v\rangle + \frac{1}{4}\langle xu, xv\rangle = \langle Lu, v\rangle$ for all $v \in X$. Since X endowed with the inner product $\langle \nabla u, \nabla v\rangle + \frac{1}{4}\langle xu, xv\rangle$ is a Hilbert space, dense in $L^2(\mathbb{R}^n)$, it follows that L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ with $D(L)$ dense in both $L^2(\mathbb{R}^n)$ and X. This operator L is in fact the classical *harmonic oscillator* operator. (The reader can consult Chapter 2 of Faris [5] as a reference for self-adjoint operators defined via sesquilinear forms. In particular, note that $D(L^{1/2}) = X$.) Furthermore, since $D(L)$ with its graph norm is continuously embedded in the Hilbert space X, which in turn is compactly embedded in $L^2(\mathbb{R}^n)$, it follows that the resolvent operator of L is a compact operator on $L^2(\mathbb{R}^n)$. Thus, the eigenvalues of L form an increasing sequence of non-negative real numbers, and $L^2(\mathbb{R}^n)$ is the direct sum of the corresponding eigenvalues. Note that $\langle Lu, u\rangle = \|u\|_X^2$ for all $u \in D(L)$.

6.2. Remark. If, instead of the whole space $\mathbb{R}^n$, one considers a domain Ω which is a cone with vertex at the origin, it can be easily checked that the analysis of Section 5 is still valid. In particular one may seek self-similar solutions to the equation (1.6) which are expressed in the form (5.3), with φ satisfying equation (1.9) on Ω. If one defines $X(\Omega)$ (resp. $X_0(\Omega)$) as the space of functions $u \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$) such that $|\cdot|u \in L^2(\Omega)$, it is straightforward to see that analogous results as in the Proposition 6.1 hold for these spaces. Therefore, using the same arguments as below, one can prove that equation (1.9) on Ω, with $\epsilon = 1$ and homogenous Dirichlet boundary conditions, possesses infinitely many solutions in $X_0(\Omega)$ (or in $X(\Omega)$, with Neumann boundary conditions $\partial\varphi/\partial\mathbf{n} = 0$ on $\partial\Omega$). Similarly, equation (1.9) on Ω, with $\epsilon = -1$ and homogenous Dirichlet boundary conditions, possesses one positive solution for $\omega < -\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the smallest eigenvalue of L on $X_0(\Omega)$ (or on $X(\Omega)$, with Neumann boundary conditions $\partial\varphi/\partial\mathbf{n} = 0$ on $\partial\Omega$). These are easy adaptations of the arguments we will develop below, and so we do not give the details here. Nevertheless, it is interesting to note that the analysis given in Section 7 can also be applied in this case.

6.3. Proposition. *The lowest eigenvalue of L is $n/2$, and the corresponding eigenspace is the span of $\varphi_1(x) := \exp(-|x|^2/4)$. In particular, L is a strictly positive operator, and for all $u \in X$ one has*

$$(n/2)\|u\|^2 \le \|u\|_X^2.$$

The eigenvalues of L are the numbers $\lambda_k = n/2 + k - 1$ for $k \ge 1$, having multiplicity $\binom{n+k-2}{n-1}$.

Proof. Let $\lambda_1 \ge 0$ be the lowest eigenvalue of L. It follows that $\lambda_1\|u\|^2 \le \|u\|_X^2$, with equality holding if u is a corresponding eigenfunction. Replacing u by $|u|$, one sees that the infimum is realized for a nonnegative function. Since the eigenspaces corresponding to other eigenvalues are all orthogonal to the λ_1 eigenspace, it follows that every nonnegative eigenfunction must have eigenvalue λ_1. One checks easily that $\exp(-|x|^2/4)$ is an eigenfunction of L with eigenvalue $n/2 = \lambda_1$.

As a matter of fact, setting $w(y) := \exp(|y|^2/2)$ and denoting by $L^2(w) := L^2(w\,dy)$ the weighted Lebesgue space on $\mathbb{R}^n$ with respect to the measure $w(y)\,dy$, one sees that $v \mapsto \varphi := w^{-1/2}v$ is a unitary isometry between $L^2(\mathbb{R}^n)$

and $L^2(w)$. Also, one can easily check that if we consider the operator $L_0\varphi = -\operatorname{div}(w\nabla\varphi)$ acting on $L^2(w)$ and having the domain $D(L_0) := \{\varphi \in L^2(w); L_0\varphi \in L^2(w)\}$, then L_0 is a self-adjoint operator on $L^2(w)$; also, the following equivalence holds for any $(u, v, \lambda) \in L^2(\mathbb{R}^n) \times L^2(w) \times \mathbb{R}$:

$$v = w^{-1/2}u \text{ and } Lu = \lambda u \Leftrightarrow u = w^{1/2}v \text{ and } L_0 v = (n/2 + \lambda)v.$$

The spectral analysis of L_0, carried out for example in Section 2 of Escobedo and Kavian [4] for an analogous operator, yields the corresponding analysis for the operator L. The eigenvalues of L_0 are $\tilde{\lambda}_k := n + k - 1$, with eigenfunctions $\tilde{\varphi}_{k,\beta}(x) := \partial^\beta \exp(-|x|^2/2)$ where the multi-index $\beta \in \mathbb{N}^n$ has length $|\beta| = k - 1$, each eigenvalue $\tilde{\lambda}_k$ having multiplicity $\binom{n+k-2}{n-1}$. Thus one can see that all the eigenvalues of L are given by the statement of Proposition 6.3 (and the corresponding eigenfunctions are the Hermite functions $\varphi_{k,\beta}(x) := \exp(|x|^2/4)\partial^\beta \exp(-|x|^2/2)$ with $|\beta| = k - 1$). □

We define, for $p > 2$ such that $(n-2)p < 2n$ and $\omega \in \mathbb{R}$, the following two functionals on X:

$$J(u) := \|u\|_X^2 + \omega\|u\|^2, \qquad F(u) := \|u\|_p^p.$$

One easily sees that J and F are of class $C^2(X, \mathbb{R})$.

6.4. PROPOSITION. *J satisfies the Palais–Smale condition on the manifold*

$$S := \{u \in X; F(u) = 1\}.$$

In particular, J possesses an infinite sequence of critical values $(c_k)_{k\geq 1}$ defined as

$$c_k := \inf_{A \in \mathfrak{B}_k} \max_{v \in A} J(v), \tag{6.4}$$

with $\mathfrak{B}_k := \{h(S^{k-1}); h: S^{k-1} \to S$ continuous and odd$\}$. Moreover, $c_k \uparrow +\infty$ as $k \uparrow \infty$.

Proof. We prove first that J satisfies the Palais–Smale condition on S. If $(u_j, \mu_j)_j$ is a sequence in $S \times \mathbb{R}$ such that $J(u_j) \to c \in \mathbb{R}$ and

$$\eta_j := -\Delta u_j + \tfrac{1}{4}|x|^2 u_j + \omega u_j - \mu_j |u_j|^{p-2}u_j \to 0 \text{ in } X',$$

then we must prove that $(u_j, \mu_j)_j$ contains a convergent subsequence. Calculating $\langle \eta_j, u_j\rangle_{X',X}$ one sees that $\langle \eta_j, u_j\rangle_{X',X} = J(u_j) - \mu_j$. We see immediately that $|\mu_j| \leq C(1 + \|u_j\|_X)$ for some constant depending on ω. Next, as $\|u_j\|_X^2 = J(u_j) - \omega\|u_j\|^2$, using (6.3), with $\epsilon > 0$ small enough so that $\epsilon|\omega| < 1/2$, we conclude

$$\|u_j\|_X^2 \leq C(1 + \|u_j\|_X),$$

and finally that $(u_j)_j$ is bounded, as well as $(\mu_j)_j$. We may assume therefore (after extracting a subsequence) that $\mu_j \to \mu$ in $\mathbb{R}$, $u_j \rightharpoonup u$ in X-weak, and (by Proposition 6.1(i)) that $u_j \to u$ strongly in $L^q(\mathbb{R}^n)$ for all $q \geq 2$ such that

$(n-2)q<2n$. In particular, noting that $Lu_j=\eta_j-\omega u_j+\mu_j|u_j|^{p-2}u_j$ and using the fact that $L^{p/(p-1)}(\mathbb{R}^n)$ is continuously embedded in X', one sees that $Lu_j\to-\omega u+\mu|u|^{p-2}u$ strongly in X'. As L induces an isomorphism between X and X', we conclude that $u_j\to L^{-1}(-\omega u+\mu|u|^{p-2}u)$ strongly in X. This proves that J satisfies the Palais-Smale condition on S.

It is now classical that J possesses an infinite sequence of critical values on S, and that in particular the c_ks defined by (6.4) are critical values of J on S and that $c_k\to+\infty$ as $k\to\infty$ (see e.g. Rabinowitz [19] and Palais [18]). □

We prove now the following corollary, which, upon setting $p=2+4/n$, establishes in particular Theorem 1.3, except for the regularity and decay properties.

6.5. COROLLARY. *For all $\omega\in\mathbb{R}$ and all $p>2$ such that $(n-2)p<2n$, the equation*

$$-\Delta\varphi+\tfrac{1}{4}|x|^2\varphi+\omega\varphi=|\varphi|^{p-2}\varphi. \tag{6.5}$$

possesses an infinite sequence of solutions $(v_k)_{k\geq k_0}$ in X such that $\|v_k\|_X\to\infty$. There exists a positive solution of (6.5) *if and only if $\omega>-\lambda_1=-n/2$; in this case $c_1>0$.*

Proof. Let u_k be a critical point of J on S such that $J(u_k)=c_k>0$; hence for some $\mu_k\in\mathbb{R}$,

$$Lu_k+\omega u_k=\mu_k|u_k|^{p-2}u_k.$$

Taking the duality product with u_k, we get $c_k=J(u_k)=\mu_k>0$; thus, setting $v_k:=\mu_k^{1/(p-2)}u_k$, one verifies that

$$-\Delta v_k+\tfrac{1}{4}|x|^2v_k+\omega v_k=|v_k|^{p-2}v_k.$$

As $c_k\leq c_{k+1}$, the c_ks being unbounded, there exists $k_0\geq1$ such that $c_{k_0}>0$, so there are infinitely many solutions to (6.5).

If (6.5) has a positive solution, then taking the duality product with $\varphi_1(x)=e^{-|x|^2/4}$ yields

$$(\lambda_1+\omega)\langle\varphi,\varphi_1\rangle=\langle L\varphi+\omega\varphi,\varphi_1\rangle_{X',X}=\langle\varphi^{p-1},\varphi_1\rangle>0,$$

and so $\lambda_1+\omega>0$. As $J(v)\geq(\lambda_1+\omega)\|v\|^2$, if $\omega>-\lambda_1$ then $c_1=\min_{v\in S}J(v)>0$. On the other hand, for all $v\in S$, one has $|v|\in S$ and $J(|v|)\leq J(v)$; thus the minimum c_1 is achieved at a function $v\geq0$, satisfying $Lv+\omega v=c_1v^{p-1}$. Then, using the strong maximum principle (for the operator $L+\omega$), one concludes that $v>0$. This yields a positive solution to (6.5) when $\omega>-\lambda_1$. □

Incidentally, we have proved that the equation

$$-\Delta\varphi+\tfrac{1}{4}|x|^2\varphi+\omega\varphi+|\varphi|^{p-2}\varphi=0, \tag{6.6}$$

has a positive solution in X if and only if $\omega<-\lambda_1=-n/2$. To see that (6.6) has no nontrivial solution in X when $\omega\geq-n/2$, it suffices to take the duality product of (6.6) with φ, the alleged solution, and use Proposition 6.3. This already establishes much of the Theorem 1.4.

Furthermore, denoting by $d_j := \binom{n+j-2}{n-1}$ the dimension of the λ_j eigenspace and setting $m_k := \sum_{j=1}^{k} d_j$, one can check easily that for $\omega < -\lambda_k$ one has $c_i < 0$ for all $i \le m_k$ (c_i is defined in (6.4)). This implies in turn that (6.6) has at least m_k pairs of solutions. Indeed, whenever u_i is a critical point of J on S such that $J(u_i) = c_i < 0$, by the same argument as above we get that $Lu_i + \omega u_i = c_i |u_i|^{p-2} u_i$. Hence, if $v_i := \pm |c_i|^{1/(p-2)} u_i$ then v_i satisfies equation (6.6). We have therefore proved the following result.

6.6. COROLLARY. *For $p > 2$ such that $(n-2)p < 2n$, and for $\omega < -n/2 - k + 1 = -\lambda_k$, equation (6.6) has at least m_k pairs of solutions in X, where $m_k := \sum_{j=1}^{k} \binom{n+j-2}{n-1}$.*

6.7. REMARK. In fact, using for example the same techniques as Clark [3] (see also Rabinowitz [19, Thm. 9.1] or Escobedo and Kavian [4]), one can prove directly that the functional

$$G(u) := \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + \frac{1}{4} |x|^2 u^2(x) + \omega u^2(x) \right) dx + \frac{1}{p} \int_{\mathbb{R}^n} |u(x)|^p \, dx$$

possesses (at least) m_k pairs of critical points if $\omega < -\lambda_k$ and p is as in Corollary 6.6. Furthermore, one can prove that for all $p > 2$, equation (6.6) has a unique positive solution if and only if $\omega < -n/2$ (to get a positive solution in this case, it suffices to minimize G on $X \cap L^p(\mathbb{R}^n)$; to see that this positive solution is unique, see e.g. [4] for an analogous equation).

6.8. REMARK. If $v \in X$ satisfies (6.5) then one can prove—by multiplying (6.5) by $\zeta_n(x) x \cdot \nabla u$, where ζ_n is a cut-off function, and integrating by parts —that u satisfies

$$\int_{\mathbb{R}^n} \left[\frac{n-2}{2} |\nabla u|^2 + \frac{n+2}{8} |x|^2 u^2 + \frac{n\omega}{2} u^2 \right] dx = \frac{n}{p} \int_{\mathbb{R}^n} |u|^p \, dx;$$

$$\int_{\mathbb{R}^n} \left[|\nabla u|^2 + \frac{1}{4} |x|^2 u^2 + \omega u^2 \right] dx = \int_{\mathbb{R}^n} |u|^p \, dx.$$

When $n \ge 3$ and $p = 2n/(n-2) =: 2^*$, the Sobolev critical exponent, one easily deduces that $\int_{\mathbb{R}^n} (2|x|^2 + \omega) u^2 \, dx = 0$. Therefore (6.5) does not have any nontrivial solution for $\omega \ge 0$. Nevertheless, using the method introduced by Brezis and Nirenberg (cf. [4, §5]), one can establish the existence of λ_* with $0 < \lambda_* < \lambda_1 = n/2$ such that for $\omega \in (-\lambda_1, -\lambda_*)$, equation (6.5) has a positive solution in X. As this equation is not of interest regarding the pseudo-conformally invariant Schrödinger equation, we do not give the proof of this assertion.

We now turn our attention to regularity and the decay rate of the solution $v \in X$ to

$$Lv + \omega v = \epsilon |v|^{p-2} v. \tag{6.7}$$

Standard local regularity results, together with the Sobolev embeddings and a bootstrap argument, imply that $v \in C^2(\mathbb{R}^n)$. In fact, when $p-2$ is an even integer so that $s \mapsto |s|^{p-2}s$ is C^∞, or when the solution v is positive, the same arguments show that v is C^∞. As far as solutions to equation (1.9) are considered, we see that positive solutions are C^∞ for any $n \geq 1$, and that all solutions are C^∞ when $n=1$ or $n=2$.

6.9. PROPOSITION. *Let $(n-2)p \leq 2n$, $\omega \in \mathbb{R}$, and $\epsilon = \pm 1$. If $v \in X$ is a solution to* (6.7), *then $v \in C_0^2(\mathbb{R}^n)$ and*

$$|v(x)|+|Dv(x)|+|D^2v(x)| \leq C(\delta)\exp(-(1-\delta)|x|^2/4)$$

for all $x \in \mathbb{R}^n$ and all $\delta \in (0,1)$. Moreover, when $\epsilon = +1$ and $\omega > -n/2 = -\lambda_1$, or when $\epsilon = -1$ and $\omega \geq -n/2$, one can take $\delta = 0$.

(Here we denote by $C_0^2(\mathbb{R}^n)$ the set of functions $\varphi \in C^2(\mathbb{R}^n)$ such that $\partial^\beta \varphi(x)$ goes to zero as $|x| \to \infty$, for all multi-indices β with $|\beta| \leq 2$.)

Proof. First we prove that if $v \in X$ is a solution to (6.7) then $v(x) \to 0$ as $|x| \to \infty$. (This is obvious when $n=1$, knowing that $X \subset H^1(\mathbb{R}) \subset C_0(\mathbb{R})$, the set of continuous functions on $\mathbb{R}$ going to zero at infinity.)

Let $\zeta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function so that $0 \leq \zeta \leq 1$ and $\zeta(x)=1$ for $|x| \leq 1$. We begin by noting that - multiplying (6.7) by $|v|^{m-1}v\zeta(x/j)$, integrating, and letting $j \to \infty$ - one has $v \in L^q(\mathbb{R}^n)$ for all $q \in [2,\infty)$. In particular, if $h \in H^1(\mathbb{R}^n)$ is defined by

$$-\Delta h + h = |v|^{p-1} + (1+|\omega|)|v|,$$

then since $|v|^{p-1}+(1+|\omega|)|v| \in L^q(\mathbb{R}^n)$ for $2 \leq q < \infty$ it follows that $h \in W^{2,q}(\mathbb{R}^n)$ for all such qs. By the Sobolev embedding and Morrey inequalities, choosing q large enough, we conclude that $h \in C_0(\mathbb{R}^n)$.

Using Kato's inequality (i.e., that for $v \in H^1(\mathbb{R}^n)$ the inequality $\Delta|v| \geq \operatorname{sign}(v)\Delta v$ holds as distributions on $\mathbb{R}^n$) one obtains, upon setting $z := |v|$, the inequality $-\Delta z + z \leq -\Delta h + h$. Therefore by the maximum principle (for the operator $-\Delta + 1$) we have $z \leq h$, and so $z(x)$ goes to zero as $|x| \to \infty$.

Next we observe that if $c(x) := \frac{1}{4}|x|^2 + \omega - \epsilon|v|^{p-2}$, by Kato's inequality and (6.7) we have

$$-\Delta z + c(x)z \leq 0 \text{ in } \mathfrak{D}'(\mathbb{R}^n).$$

On the other hand, if we set $\psi_*(x) := \exp(-(1-\delta)|x|^2/4)$ for $0 \leq \delta < 1$, then a straightforward calculation shows that

$$-\Delta\psi_* + c(x)\psi_* = \left(\frac{\delta(2-\delta)}{4}|x|^2 + \frac{n(1-\delta)}{2} + \omega - \epsilon|v|^{p-2}\right)\psi_*.$$

Now assume that $\delta > 0$, or $\delta = 0$ but $\omega > -n/2$ when $\epsilon = 1$ and $\omega \geq -n/2$ when $\epsilon = -1$. We may choose $R > 0$ large enough so that for $|x| \geq R$ we have

$$c(x) > 0 \quad \text{and} \quad \frac{\delta(2-\delta)}{4}|x|^2 + \frac{n(1-\delta)}{2} + \omega - \epsilon|v|^{p-2} \geq 0.$$

Then choosing $C(\delta) := \max_{|x| \le R} z(x)/\psi_*(x)$ and letting $\psi(x) := C(\delta)\psi_*(x)$, we have

$$-\Delta z + c(x) z \le 0 \le -\Delta\psi + c(x)\psi \ \text{ in } \mathfrak{D}'([|x| > R]);$$

applying the maximum principle, we conclude that $z(x) \le \psi(x)$ for $|x| \ge R$. (Here we would like to thank the referee for having slightly simplified our original proof.)

The proof of the proposition is completed by applying the same arguments to the derivatives Dv and $D^2 v$ (which satisfy analogous elliptic equations obtained by differentiating (6.7)). □

6.10. REMARK. We conclude this section with the following observation. It is known that equation (1.6) with $\epsilon = 1$ admits positively global solutions having initial data of arbitrarily large L^2 norm (see Proposition 2.3 in [2]). It is not known, however, if equation (1.6) with $\epsilon = 1$ admits solutions which are both positively and negatively (in time) global, and which have initial data of arbitrarily large L^2 norm. On the other hand, for each $\omega \in \mathbb{R}$, a solution of (1.9) with $\epsilon = 1$ is an initial value for the positively and negatively global solution (5.3) of (1.6). It would be very interesting to find solutions of (1.9) with $\epsilon = 1$ having arbitrarily large L^2 norm. As a partial result in this direction, Th. Cazenave (private communication) has shown that for every $r > 2$, if φ_ω is solution of (1.9) with $\epsilon = 1$ then one has $\lim_{\omega \to \infty} \|\varphi_\omega\|_r = \infty$.

7. Continuation of Global Solutions beyond Infinity

In this section we continue with the notation established in the previous sections. If $u(t,x)$ is a solution to (1.6), then one can check (via the same mildly cumbersome calculation as the one at the end of Section 5) that $v(s,y)$, defined by

$$u(t,x) =: (1+t^2)^{-n/4} \exp\left[\frac{it|x|^2}{4(1+t^2)}\right] v\left(\arctan t, \frac{x}{(1+t^2)^{1/2}}\right), \tag{7.1}$$

satisfies $v(0,y) = u(0,x)$ and

$$iv_s + \Delta v - \tfrac{1}{4}|y|^2 v + \epsilon |v|^{4/n} v = 0 \tag{7.2}$$

(where $t = \tan s$ and $x = (1+t^2)^{1/2} y$). An existence and uniqueness theorem for solutions in X can be proved for (7.2), provided $v(0,\cdot) \in X$. In particular, one can prove existence of global solutions to (7.2) for initial data $v(0)$ having sufficiently small norm in $L^2(\mathbb{R}^n)$.

Interestingly, while u and v are defined respectively for $t \in \mathbb{R}$ and $s \in \mathbb{R}$, $u(t,x)$ depends on $v(s,y)$ for s defined in an interval of length π; for example, $-\pi/2 < s < \pi/2$. In other words, a global solution of u forms only a small part of what might be a global solution for v. For example, the global solutions of (1.6) given by (5.3) correspond to global solutions of (7.2) given by $v(s,y) := e^{i\omega s}\varphi(y)$, that is, standing waves for (7.2). Furthermore, for such a standing wave solution of (7.2), the resulting solution $u(t,x)$ of (1.6) does not reflect the values of $v(s,y)$ for all $s \in \mathbb{R}$, just those of the form $s = \arctan t$. For different branches of the arctan function, different solutions of (1.6) are obtained from the same solution v of (7.2). More generally, if the $L^2(\mathbb{R}^n)$ norms of the initial values for (1.6) and (7.2) are sufficiently small, then the resulting solutions u and v are global. Thus again, one solution v of (7.2) gives rise to an infinity of solutions u of (1.6), one for each branch of the arctan function. Since the global solution v of (7.2) is determined by any *one* of these infinitely many solutions of (1.6), it follows that any one of these solutions u of (1.6) can be thought of as a continuation "beyond infinity" of the others. Note further that while every solution of the linear version of (7.2), i.e. with $\epsilon = 0$, is periodic with period 4π, or 2π when the dimension n is even (as can be seen by its eigenfunction expansion in X), the standing wave solutions of (7.2) with $\epsilon = \pm 1$ have period $2\pi/|\omega|$.

The fact that transformation (7.1) translates between solutions of (1.6) and (7.2) has already been observed in the linear case (i.e. $\epsilon = 0$) for $n = 1$. Indeed, this transformation is a special case of formula (1.3) in Theorem 1 of [21], taking into account that the operator L (and the semigroup generated by L) appear in [21] in a unitarily equivalent form on L^2 with respect to a Gauss measure on $\mathbb{R}$.

If one adapts to equation (7.2) the standard calculation used to prove the existence of nonglobal solutions to (1.6), one obtains (see Remark 9.2.9 in Cazenave [1, p. 206])

$$F''(s) + F(s) = 16E(s),$$

where $F(s) := \|yv(s,\cdot)\|^2$ and E is the energy of the solution v; that is,

$$E(s) := \frac{1}{2}\|\nabla v(s)\|^2 + \frac{1}{8}\|yv(s,\cdot)\|^2 - \frac{\epsilon}{\alpha+2}\|v(s)\|_{\alpha+2}^{\alpha+2},$$

which is a constant in time s, $E(s) = E(0)$. It follows that

$$F(s) = 4E(0) + (F(0) - 4E(0))\cos 2s + \tfrac{1}{2}F'(0)\sin 2s.$$

The solution of (7.2) cannot be global if $F(s) \leq 0$ for some value of s; in particular this is the case if $E(0) \leq 0$. Unfortunately this does not give anything new for solutions to (1.6), which is not surprising. Indeed, if u and v are related by (7.1) then $\|xu(t)\|^2 = (1+t^2)\|yv(\arctan t)\|^2$, and so an explicit formula for $\|xu(t)\|^2$ gives the same information as an explicit formula for $\|yv(s)\|^2$.

Finally, we should point out that the transformation (7.1), and the fact that classical global solutions of (7.2) are bounded in $C([-\pi/2, \pi/2], X)$, imply that the related global solutions of (1.6) satisfy

$$\|u(t)\|_p = (1+t^2)^{n/2p - n/4}\|v(\arctan t)\|_p \leq C(1+t^2)^{n/2p-n/4}.$$

Indeed, further information about global solutions to (1.6) can be culled from the transformation (7.1); but it seems that such information has already been obtained in Cazenave and Weissler [2]. The only novelty is perhaps the fact that this analysis can be carried out for some other domains than the whole space $\mathbb{R}^n$, namely a domain Ω which is a cone with vertex at the origin.

References

[1] Th. Cazenave, *An introduction to nonlinear Schrödinger equations,* Textos de Métodos Matematicos, 22, Instituto de Matemática, Universidade Federal do Rio de Janeiro (Brasil), 1990.

[2] Th. Cazenave and F. B. Weissler, *The structure of solutions to the pseudo-conformally invariant nonlinear Schrödinger equation,* Proc. Royal Soc. Edinburgh Sect. A 117 (1991), 251–273.

[3] D. C. Clark, *A variant of the Lusternik-Schnirelman theory,* Indiana Univ. Math. J. 22 (1972), 65–74.

[4] M. Escobedo and O. Kavian, *Variational problems related to self similar solutions of the heat equation,* Nonlinear Anal. 11 (1987), 1103–1133.

[5] W. G. Faris, *Self-adjoint operators,* Lecture Notes in Math., 433, Springer, Berlin, 1975.

[6] J. Ginibre and G. Velo, *Sur une équation de Schrödinger non linéaire avec interaction non locale,* Nonlinear partial differential equations and their applications (Collège de France Seminar), vol. II (H. Brezis, J. L. Lions, eds.), pp. 155–199, Pitman, Boston, 1982.

[7] C. Gui, W.-M. Ni, and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in* $\mathbb{R}^n$, preprint.

[8] M. W. Hirsch and S. Smale, *Differential equations, dynamical systems and linear algebra,* Academic Press, New York, 1974.

[9] R. Johnson and X. Pan, *On an elliptic equation related to the blow-up phenomenon in the non-linear Schrödinger equation,* Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 763–782.

[10] N. Kopell and M. J. Landman (to appear).

[11] M. J. Landman, G. C. Papanicolaou, C. Sulem, and P. L. Sulem, *Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension,* Phys. Rev. A (3) 38 (1988), 3837–3843.

[12] B. J. LeMesurier, G. Papanicolaou, C. Sulem, and P. L. Sulem, *The focusing singularity of the nonlinear Schrödinger equation,* Directions in Partial Differential Equations (M. G. Crandall, P. H. Rabinowitz, R. E. Turner, eds.), pp. 159–201, Academic Press, New York, 1987.

[13] ———, *Local structure of the self-focusing singularity of the nonlinear Schrödinger equation,* Phys. D 32 (1988), 210–226.

[14] ———, *Focusing and multi-focusing solutions of the nonlinear Schrödinger equation,* Phys. D 31 (1988), 78–102.

[15] Y. Li, *Asymptotic behavior of positive solutions of equation* $\Delta u + K(x)u^p = 0$ *in* $\mathbb{R}^n$, J. Differential Equations 95 (1992), 304–330.

[16] D. W. McLaughlin, G. C. Papanicolaou, C. Sulem, and P. L. Sulem, *Focusing singularity of the cubic Schrödinger equation,* Phys. Rev. A (3) 34 (1986), 1200–1210.

[17] P. J. Olver, *Applications of Lie groups to differential equations,* Graduate Texts in Math., 107, Springer, New York, 1986.

[18] R. S. Palais, *Critical point theory and the minimax principle,* Proc. Sympos. Pure Math. (Global Analysis), 15, pp. 185–212, Amer. Math. Soc., Providence, RI, 1970.
[19] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conf. Ser. Math. 65, Amer. Math. Soc., Providence, RI, 1986.
[20] M. I. Weinstein, *On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations,* Comm. Partial Differential Equations 11 (1986), 545–565.
[21] F. B. Weissler, *Two-point inequalities, the Hermite semigroup, and the Gauss-Weierstrass semigroup,* J. Funct. Anal. 32 (1979), 102–121.

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