Removable Singularities for L^p CR Functions

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1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$ with C^2-smooth boundary $b\Omega$. A closed subset X of $b\Omega$ is said to be *removable for continuous CR functions* if, for each function f continuous on $b\Omega\setminus X$ satisfying the tangential Cauchy–Riemann equations in the weak sense on $b\Omega\setminus X$, there exists a function F holomorphic in Ω continuously assuming the boundary values f on $b\Omega\setminus X$. There have been many interesting results proved recently relating removability to convexity with respect to various function spaces. For example, define the $O(\bar{\Omega})$-hull of $X\subset b\Omega$ to be the set $\hat{X}_\Omega$ of all points $p\in\bar{\Omega}$ such that $|\phi(p)|\leq \max\{|\phi(z)|: z\in X\}$ for all functions ϕ holomorphic in a neighborhood of $\bar{\Omega}$. The following result of Stout [10] will be important for us: Let Ω be a strictly pseudoconvex domain in $\mathbf{C}^n$, X a compact subset of $b\Omega$. If f is a continuous CR function on $b\Omega\setminus X$, then there exists a function holomorphic in $\Omega\setminus\hat{X}_\Omega$, continuous on $\bar{\Omega}\setminus\hat{X}_\Omega$, with $F=f$ on $b\Omega\setminus X$. In particular, Stout's theorem implies that if $X=\hat{X}_\Omega$ (we say X is *$O(\bar{\Omega})$-convex*) then X is removable. In $\mathbf{C}^2$, the converse is also true: If X is contained in the boundary of a strictly pseudoconvex domain and X is removable for continuous CR functions, then X is $O(\bar{\Omega})$-convex. Stout's paper [11] gives an excellent survey of results on removable singularities for CR functions.

We wish to study removable singularities for other classes of CR functions. Fix p, $1\leq p\leq\infty$, and let σ be the induced $(2n-1)$-dimensional Euclidean measure on $b\Omega$. Let us say that $X\subset b\Omega$ is *removable for L^p CR functions* if, for each $f\in L^p(b\Omega, d\sigma)$ satisfying the tangential Cauchy–Riemann equations on $b\Omega\setminus X$, there exists F in the Hardy space $H^p(\Omega)$ with boundary values f σ-almost everywhere on $b\Omega\setminus X$. In view of Stout's theorem above, it is reasonable to direct our attention first to $O(\bar{\Omega})$-convex subsets of $b\Omega$. Even in the simplest case, where $\Omega=B$ is the unit ball in $\mathbf{C}^n$ and $O(\bar{\Omega})$-convexity is equivalent to polynomial convexity, such sets can be quite large —there exist polynomially convex subsets of bB with positive σ-measure (see [11]). We shall restrict our attention to sets of $(2n-1)$-dimensional measure zero. On the other hand, if the Hausdorff dimension of X is sufficiently small, then the arguments of [11] for the case of L^∞ functions can be adapted

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to show that X is removable for L^1 CR functions. One notable result on relatively "large" sets is that of Kytmanov [7], who proved that peak sets of holomorphic functions satisfying a Lipschitz estimate on $\bar{\Omega}$ are removable for L^1 CR functions. If $\Omega = B$, these sets are polynomially convex and have σ measure zero, but can have Hausdorff dimension arbitrarily close to $2n-1$. Another class of such sets is given in Theorem 1 below: polynomially convex subsets of a totally real submanifold of $b\Omega$ are removable for L^1 CR functions. Both Kytmanov's result and Theorem 1 are established by showing that each L^1 CR function on $b\Omega \backslash X$ is actually a CR function on $b\Omega$. Our main purpose in this paper is to use a different technique. We consider polynomially convex sets X contained in the intersection of $b\Omega$ with a family of analytic varieties. Suppose $f \in L^p(b\Omega)$ and f is a CR function on $b\Omega \backslash X$. With the aid of Stout's theorem we obtain an extension F of f to Ω. This extension is "locally" H^p in a sense made precise below. We then estimate the growth of F near the singularity set by estimating F on each variety, using classical facts about H^p functions in one variable. If X is contained in a sufficiently small set of these analytic varieties then we can conclude that $F \in H^p$. In this way we can show that certain relatively large subsets of $b\Omega$ are removable for L^1 CR functions. The precise results are given in Theorem 2 and Corollary 1 of Section 3. In Section 2 we collect some basic facts about H^p functions and establish the estimates we need on integration over families of analytic varieties.

2. Preliminaries

Ω will denote a bounded domain in $\mathbf{C}^n$ with C^2 boundary $b\Omega$. That is, there exists a real-valued defining function ρ of class C^2 in a neighborhood of $\bar{\Omega}$ such that $\Omega = \{z : \rho(z) < 0\}$ and $d\rho \neq 0$ on $b\Omega = \{z : \rho(z) = 0\}$. Let W be a relatively open subset of $b\Omega$. A function $f \in L^1_{\text{loc}}(W)$ is a *CR function on W* if $\int_W f \bar{\partial} \phi = 0$ for every smooth $(n, n-2)$ form ϕ with compact support in W. The class of CR functions on W will be denoted $\text{CR}(W)$. For $\epsilon > 0$, let $\Omega_\epsilon = \{z : \rho(z) < -\epsilon\}$. The induced Euclidean measure on $b\Omega$ (resp. $b\Omega_\epsilon$) is denoted by $d\sigma$ ($d\sigma_\epsilon$). We recall some facts (see [9]) about the Hardy spaces on Ω: $H^p(\Omega)$ $(1 \le p < \infty)$ is the set of functions F holomorphic in Ω such that

$$\sup_{\epsilon > 0} \int_{b\Omega_\epsilon} |F(z)|^p \, d\sigma_\epsilon(z) < \infty.$$

The resulting space is independent of the choice of defining function ρ. If $F \in H^p(\Omega)$, then $\lim_{\epsilon \to 0} F(z - \epsilon \nu_z)$ exists for σ-almost all $z \in b\Omega$, where ν_z is the outward-pointing normal to $b\Omega$ at z; furthermore,

$$\lim_{\epsilon \to 0} \int_{b\Omega} |F(z - \epsilon \nu_z) - f(z)|^p \, d\sigma(z) = 0. \tag{1}$$

Conversely, if $b\Omega$ is strictly pseudoconvex and $f \in L^p(b\Omega) \cap \text{CR}(b\Omega)$, then there exists $F \in H^p(\Omega)$ satisfying (1). This follows from the following corresponding local extension property, which seems to be well known (for a sketch of the proof in the case of the ball, see [1]): Let q be a point of strict pseudoconvexity of $b\Omega$, V a neighborhood of q in $b\Omega$. Then there exists a neighborhood W of q in $\mathbf{C}^n$ such that, for each $f \in L^p(V) \cap \mathrm{CR}(V)$, there exists a function F holomorphic in $W \cap \Omega$ with

$$\sup_{\epsilon>0} \int_{b\Omega_\epsilon \cap W} |F(z)|^p \, d\sigma_\epsilon(z) < \infty \tag{2}$$

and

$$\lim_{\epsilon\to 0} \int_{b\Omega \cap W} |F(z-\epsilon\nu_z) - f(z)|^p \, d\sigma(z) = 0. \tag{3}$$

Let Ω_0 be an open subset of Ω, U a relatively open subset of $b\Omega$, and $f \in L^p(\Omega)$. We say that $F \in H(\Omega_0)$ is a *U-local H^p extension of f to Ω_0* if for each $q \in U$ there exists a neighborhood W of q in $\mathbf{C}^n$, with $W \cap \Omega \subset \Omega_0$ and with $W \cap b\Omega \subset U$, such that (2) and (3) hold.

Now suppose $b\Omega$ is strictly pseudoconvex, and let X be a closed subset of $b\Omega$. If $f \in L^p(b\Omega) \cap \mathrm{CR}(b\Omega \backslash X)$, then by the remarks above we can construct a local extension $\tilde{F}$ of f to an open subset of Ω whose closure contains $b\Omega \backslash X$. By pulling in the boundary of Ω off X, leaving X fixed, we can construct a new strictly pseudoconvex domain $\tilde{\Omega} \subset \Omega$, with $X \subset b\tilde{\Omega}$ and $b\tilde{\Omega} \backslash X \subset \Omega$, such that $\tilde{F}$ is a continuous CR function on $b\tilde{\Omega} \backslash X$. By Stout's theorem, $\tilde{F}$ extends to $\tilde{\Omega} \backslash \hat{X}_{\tilde{\Omega}}$. Since $\hat{X}_{\tilde{\Omega}} \subset \hat{X}_\Omega$, we obtain an extension to $\Omega \backslash \hat{X}_\Omega$. We summarize this remark (which appears in [11] in the context of L^∞ functions) as follows.

LEMMA 1. *Let Ω be a bounded domain in $\mathbf{C}^n$ with strictly pseudoconvex boundary $b\Omega$ of class C^2, and let X be a closed subset of $b\Omega$. If $f \in L^p(b\Omega) \cap \mathrm{CR}(b\Omega \backslash X)$, then there exists a $(b\Omega \backslash X)$-local H^p extension of f to $\Omega \backslash \hat{X}_\Omega$.*

Next we derive an a priori estimate for integrals of holomorphic functions over families of analytic disks in Ω. Let V be an open set in $\mathbf{C}^{n-1}$. Δ will denote the unit disk $\{\zeta : |\zeta| < 1\}$ in the plane, $b\Delta$ its boundary. We define a *smooth family of analytic disks in Ω with parameter space V* to be the image of a map Ψ such that

(i) Ψ is a diffeomorphism of a neighborhood of $\bar{V} \times \bar{\Delta}$ with an open subset of $\mathbf{C}^n$,

(ii) for each $c \in V$, $\Psi(c, \zeta)$ is a holomorphic function of ζ for $\zeta \in \Delta$, and

(iii) $\Psi(V, \Delta) \subset \Omega$, $\Psi(V, b\Delta) \subset b\Omega$, and each disk meets $b\Omega$ transversally in the sense that $d(\rho \circ \Psi) \neq 0$ on $\bar{V} \times b\Delta$,

where for $A \subset \bar{V}$ and $B \subset \bar{\Delta}$, $\Psi(A, B) = \{z \in \mathbf{C}^n : z = \Psi(c, \zeta)$ for some $c \in A$, $\zeta \in B\}$. In case $B = \bar{\Delta}$, we simply write Ψ_A for $\Psi(A, \bar{\Delta})$. By abuse of language, we also refer to Ψ itself as a smooth family of analytic disks.

LEMMA 2. *Let Ω be a bounded domain with strictly pseudoconvex boundary $b\Omega$ of class C^2, and let Ψ be a smooth family of analytic disks in Ω with* *parameter space V. Then for fixed p, $1 \le p < \infty$, there exist positive constants C and ϵ_0 depending on p, Ω, and Ψ such that for all subsets K of V, all $0 < \epsilon < \epsilon_0$, and all g holomorphic in a neighborhood of Ψ_K,*

$$\int_{\Psi_K \cap b\Omega_\epsilon} |g(z)|^p \, d\sigma_\epsilon(z) \le C \int_{\Psi_K \cap b\Omega} |g(z)|^p \, d\sigma(z). \tag{4}$$

Proof. Fix a defining function ρ for Ω. For each $c \in V$ and $\epsilon \ge 0$, set

$$\gamma_{\epsilon, c} = \{\zeta \in \bar{\Delta} : \rho \circ \Psi(c, \zeta) = -\epsilon\}.$$

By assumption (iii) on Ψ, $b\Delta = \gamma_{0,c}$, and the gradient field of $\rho \circ \Psi$ is nonzero on $b\Delta$ and orthogonal to $b\Delta$. The integral curves of this field can be used to construct a diffeomorphism of $b\Delta$ with $\gamma_{\epsilon,c}$ for all sufficiently small ϵ and all $c \in V$. Let $z(\epsilon, c, \theta)$ be the unique point in $\gamma_{\epsilon,c}$ corresponding to $e^{i\theta}$ under this diffeomorphism. Let S_θ denote the interior of the triangular region with vertices $e^{i\theta}$, $e^{i(\theta - \pi/4)}/\sqrt{2}$, $e^{i(\theta + \pi/4)}/\sqrt{2}$. We may choose ϵ_0 small enough so that $z(\epsilon, c, \theta) \in S_\theta$ for all $\epsilon < \epsilon_0$, $c \in V$, and $\theta \in [0, 2\pi)$. For g defined on Δ, set $Mg(\theta) = \sup\{|g(z)| : z \in S_\theta\}$. Then for $1 \le p < \infty$ (see [6, p. 246]) there exists a constant k depending on p such that for all $g \in H^p(\Delta)$,

$$\int_0^{2\pi} (Mg(\theta))^p \, d\theta \le k \int_0^{2\pi} |g(\theta)|^p \, d\theta.$$

Given g holomorphic in a neighborhood of Ψ_K, $g \circ \Psi(c, \zeta)$ is holomorphic in Δ and smooth in $\bar{\Delta}$. Thus for each p, $1 \le p < \infty$, and for fixed $(c, \epsilon) \in K \times [0, \epsilon_0]$,

$$\begin{aligned}\int_0^{2\pi} |g \circ \Psi(c, z(\epsilon, c, \theta))|^p \, d\theta &\le \int_0^{2\pi} |M(g \circ \Psi)(c, e^{i\theta})|^p \, d\theta \\ &\le C \int_0^{2\pi} |g \circ \Psi(c, e^{i\theta})|^p \, d\theta, \end{aligned} \tag{5}$$

where C depends only on p. Now, for any $K \subset V$ and g holomorphic in a neighborhood of Ψ_K, we have

$$\int_{\Psi_K \cap b\Omega_\epsilon} |g(z)|^p \, d\sigma_\epsilon(z) \le \int_K \int_0^{2\pi} |g \circ \Psi(c, z(c, \epsilon, \theta))|^p H(\epsilon, c, \theta) \, d\theta \, dc,$$

where the pull-back of the form σ_ϵ to $\{(c, z(\epsilon, c, \theta)) : c \in K, \theta \in [0, 2\pi]\}$ is $H(\epsilon, c, \theta) \, d\theta \, dc$, and $dc = (1/2\pi i)^{n-1} d\bar{c}_1 dc_1 \cdots d\bar{c}_{n-1} dc_{n-1}$ is Lebesgue measure on $\mathbf{C}^{n-1}$. By assumption (iii) again, each disk meets $b\Omega_\epsilon$ transversally for all sufficiently small ϵ, and so we can assume there exist positive constants k_1 and k_2 such that $k_2 \le H(\epsilon, c, \theta) \le k_1$ for all $(\epsilon, c, \theta) \in [0, \epsilon_0] \times K \times [0, 2\pi]$. By (5), the latter integral is less than

$$\begin{aligned} &Ck_1 \int_K \int_0^{2\pi} |g \circ \Psi(c, e^{i\theta})|^p \, d\theta \, dc \\ &\quad \le \frac{Ck_1}{k_2} \int_K \int_0^{2\pi} |g \circ \Psi(c, z(c, e^{i\theta})|^p H(0, c, \theta) \, d\theta \, dc \cdot \end{aligned}$$

$$= \frac{Ck_1}{k_2} \int_{\Psi_K \cap b\Omega} |g(z)|^p \, d\sigma(z).$$

The proof is complete. □

3. Main Results

THEOREM 1. *Let Ω be a bounded domain in $\mathbf{C}^n$ with boundary $b\Omega$ of class C^2, M a totally real imbedded submanifold of $b\Omega$, and let X be a polynomially convex subset of M. Then X is removable for L^1 CR functions.*

The following proof was suggested to one of us by an anonymous reviewer of a grant proposal.

Proof. By the remarks following equation (1), it suffices to show that if $f \in L^1(b\Omega) \cap \mathrm{CR}(b\Omega \backslash X)$ then $f \in \mathrm{CR}(b\Omega)$, that is, that

$$\int_{b\Omega} f \bar{\partial} \phi = 0 \tag{6}$$

for all smooth $(n, n-2)$ forms ϕ on $b\Omega$, given that it holds for ϕ vanishing in a neighborhood of X. Fix a basis for the $(n, n-2)$ forms on $\mathbf{C}^n$; if a form g has coefficients with respect to this basis which are differentiable in a neighborhood of a set Y, write $\|g\|_{1,Y}$ for the supremum of the coefficients and their first derivatives on Y. We claim that it suffices to show that (6) holds for all ϕ vanishing on X. Let ϕ be an $(n, n-2)$ form supported in a neighborhood of X. By the Range–Siu theorem [8], since M is totally real, there exists a form ϕ_h with coefficients holomorphic in a neighborhood of X such that $\|\phi - \phi_h\|_{1,M} < \delta$, where δ is a small constant to be determined. By the Oka–Weil theorem (see [4]), since X is polynomially convex, there exists an $(n, n-2)$ form ϕ_P with polynomial coefficients such that $\|\phi_h - \phi_P\|_{1,X} < \delta$. Let $\tilde{\phi} = \phi - \phi_P$. By Whitney's extension theorem (see [13, p. 322]) there exists a form ψ with coefficients differentiable on $\mathbf{C}^n$ such that $\psi = \tilde{\phi}$ on X and $\|\psi\|_{1,\mathbf{C}^n} < C\|\tilde{\phi}\|_{1,X}$, where C is a constant depending only on X. Therefore, given $\epsilon > 0$, we can choose $\delta > 0$ such that

$$\left| \int_{b\Omega} f \bar{\partial} \phi - \int_{b\Omega} f \bar{\partial}(\tilde{\phi} - \psi) \right| = \left| \int_{b\Omega} f \bar{\partial} \tilde{\phi} - \int_{b\Omega} f \bar{\partial}(\tilde{\phi} - \psi) \right|$$
$$= \left| \int_{b\Omega} f \bar{\partial} \psi \right| < \epsilon.$$

Since $\psi - \tilde{\phi} = 0$ on X, this establishes the claim, and henceforth we may assume that $\phi = 0$ on X. Now let $X_n = \{z \in b\Omega : \mathrm{dist}(z, X) < 1/n\}$. Since M is a proper submanifold of $b\Omega$, we have $\lim_{n\to\infty} \sigma(X_n) = 0$. Choose forms χ_n with $\chi_n = 1$ on $b\Omega \backslash X_n$, $\chi_n = 0$ on X_{2n}, and $\|\chi_n\|_{1,b\Omega} = O(n)$. Since χ_n is supported outside X, we have

$$0 = \int_{b\Omega} f \bar{\partial}(\chi_n \phi) = \int_{b\Omega} f \chi_n \bar{\partial} \phi + \int_{b\Omega} f \phi \, \bar{\partial} \chi_n. \tag{7}$$

The first integral on the right of (7) tends to $\int_{b\Omega} f\bar{\partial}\phi$ as $n\to\infty$. Because ϕ vanishes on X and $|\bar{\partial}\chi_n| = O(n)$, the second integral on the right of (7) is $O(\int_{X_n}|f|\,d\sigma)$, which also tends to zero as $n\to\infty$. The proof is complete. □

EXAMPLE 1. Let Ω be a domain in $\mathbf{C}^2$ with strictly pseudoconvex boundary $b\Omega$ of class C^2, and suppose that $M\subset b\Omega$ is diffeomorphic to a disk. Jöricke [5] has proved that if M is totally real then M is removable for continuous CR functions and hence is $O(\bar{\Omega})$-convex. If $\bar{\Omega}$ is polynomially convex then M is polynomially convex, and hence by Theorem 1 is removable for L^1 CR functions. Recently Forstnerič and Stout [3] have shown that under the same hypotheses on Ω, if $M\subset b\Omega$ is diffeomorphic to a disk and M has only finitely many complex tangents, each of hyperbolic type, then M is polynomially convex. Note that the proof of Theorem 1 breaks down in this case, since we cannot have C^1 approximation of arbitrary smooth functions on M by holomorphic functions if M is not totally real. We do not know if such a disk is removable for L^1 CR functions.

Our next result, which uses the estimates established in Section 2, allows us to prove removability for another class of polynomially convex sets, including some too large to be imbedded in a disk (see Example 2).

We assume in what follows that the domain Ω is *strongly starlike*—that is, for $r<1$, $\Omega_r=\{z\in\mathbf{C}^n\colon z=rz' \text{ for some } z'\in b\Omega\}$ is a compact subset of Ω. In what follows, we use the domains Ω_r, $0<r<1$, as approximating domains (previously denoted Ω_ϵ).

THEOREM 2. *Let Ω be a smoothly bounded strongly starlike domain in $\mathbf{C}^n$, with strictly pseudoconvex boundary $b\Omega$ of class C^2. Let X be a closed subset of $b\Omega$, and assume that there exists a smooth family Ψ of analytic disks in Ω with parameter space $V\subset\mathbf{C}^{n-1}$, and a set $K\subset V$ of $2n-2$ Lebesgue measure zero, such that $X\subset\Psi(K,b\Delta)$. If F is a $(b\Omega\setminus X)$-local H^p extension of $f\in L^p(b\Omega)$ to Ω, then $F\in H^p(\Omega)$.*

Proof. We may assume that $K=\{c\in V\colon \Psi(c,\zeta)\in X \text{ for some } \zeta\in b\Delta\}$. Choose a sequence of compact subsets K_j of V with $K_{j+1}\subset\operatorname{int}(K_j)$ and $K=\bigcap_{j=1}^{\infty}K_j$. Given $f\in L^p(b\Omega)$, we assume there exists $F\in H(\Omega)$ such that, for some neighborhood W of each point in $b\Omega\setminus X$, (2) and (3) hold; that is,

$$\sup_{r<1}\int_{b\Omega_r\cap W}|F(z)|^p\,d\sigma_r(z)<\infty; \tag{8}$$

$$\lim_{r\to1}\int_{b\Omega\cap W}|F(rz)-f(z)|^p\,d\sigma(z)=0. \tag{9}$$

We must show that $F\in H^p(\Omega)$. Write

$$\int_{b\Omega_r}|F(z)|^p\,d\sigma_r(z)=\int_{b\Omega_r\setminus\Psi_{K_1}}|F(z)|^p\,d\sigma_r(z)+\int_{b\Omega_r\cap\Psi_{K_1}}|F(z)|^p\,d\sigma_r(z). \tag{10}$$

We estimate the first integral on the right of (10). Since $X \cap \{b\Omega \setminus \Psi_{K_1}\} = \emptyset$, we can choose a finite collection $W_1, \ldots, W_N$ of open sets so that for all r sufficiently close to 1,

$$b\Omega_r \setminus \Psi_{K_1} \subset \bigcup_{j=1}^{N} W_j \quad \text{and} \quad \sup_{r<1} \int_{W_j \cap b\Omega_r} |F(z)|^p \, d\sigma_r(z) < \infty,$$

which implies that

$$\sup_{r<1} \int_{b\Omega_r \setminus \Psi_{K_1}} |F(z)|^p \, d\sigma_r(z) < \infty.$$

It remains to estimate

$$I_r = \int_{b\Omega_r \cap \Psi_{K_1}} |F(z)|^p \, d\sigma_r(z).$$

Since the $(2n-2)$-dimensional Lebesgue measure of K is zero,

$$\lim_{n \to \infty} \sigma_r(\Psi_{K_n} \cap b\Omega_r) = 0;$$

hence, for fixed $r < 1$,

$$I_r = \lim_{n \to \infty} \int_{b\Omega_r \cap \Psi_{K_1 \setminus K_n}} |F(z)|^p \, d\sigma_r(z).$$

Thus it suffices to show that

$$I_{n,r} = \int_{b\Omega_r \cap \Psi_{K_1 \setminus K_n}} |F(z)|^p \, d\sigma_r(z)$$

is bounded independently of n and r. Fix $R < 1$. By Lemma 2, since $F(Rz)$ is holomorphic in a neighborhood of $\bar{\Omega}$, there exists $C > 0$ independent of R such that

$$\int_{b\Omega_r \cap \Psi_{K_1 \setminus K_n}} |F(Rz)|^p \, d\sigma_r(z) \leq C \int_{b\Omega \cap \Psi_{K_1 \setminus K_n}} |F(Rz)|^p \, d\sigma_r(z).$$

As $R \uparrow 1$, for fixed r, $F(Rz)$ converges uniformly on $b\Omega_r$ to $F(z)$, while (again by the local H^p properties (8) and (9)) $F(Rz)$ converges in L^p to $f(z)$ on $b\Omega \cap \Psi_{K_1 \setminus K_n}$. Thus

$$I_{n,r} = \int_{b\Omega_r \cap \Psi_{K_1 \setminus K_n}} |F(z)|^p \, d\sigma_r(z) \leq C \int_{b\Omega \cap \Psi_{K_1 \setminus K_n}} |f(z)|^p \, d\sigma(z)$$
$$\leq C \int_{b\Omega} |f(z)|^p \, d\sigma(z).$$

The proof is complete. □

Combining Theorem 2 with Lemma 2, we obtain the following corollary.

COROLLARY 1. *Let X be a closed polynomially convex subset of $b\Omega$, with Ω and X satisfying the hypotheses of Theorem 2. Then X is removable for L^p CR functions, $1 \leq p < \infty$.*

EXAMPLE 2. Take $n=2$, and let π be the projection $(z,w)\to z$. Let X be a closed polynomially convex subset of bB such that $X\cap\{w=0\}=\emptyset$ and $\pi(X)$ has 2-dimensional Lebesgue measure zero. Then we can choose $\delta>0$ such that $X\cap\{|w|\le\delta\}=\emptyset$. Consider the map $\Psi(\lambda,\zeta)=(\lambda,\zeta\sqrt{1-|\lambda|^2})$ with parameter space $V=\{\lambda\in\mathbf{C}:|\lambda|<\sqrt{1-\delta^2}\}$. Then Ψ is a smooth family of analytic disks in B, and $X\subset\Psi(\pi(X),b\Delta)$. By Corollary 1, X is removable for L^1 CR functions. Although such sets X must have $\sigma(X)=0$, they can be quite large in the sense of Hausdorff dimension. Fix α, $0<\alpha<1$. We construct $X\subset bB$ such that X is polynomially convex, $\pi(X)$ has 2-dimensional measure zero, and X has Hausdorff dimension 3α. The construction is based on one in [11] for constructing polynomially convex sets of large σ-measure. Let C_n denote the nth stage in the construction of the generalized Cantor set of Hausdorff dimension α; that is, $C_0=[0,1]$ and $C_n=\bigcup_{j=1}^{2^n}C_{n,j}$, where each $C_{n,j}$ is an interval of length $\xi_n=2^{-n/\alpha}$ and C_{n+1} is formed from C_n by removing from each $C_{n,j}$ the open middle interval of length $\xi_n-2\xi_{n+1}$. If $C=\bigcap_{n=1}^{\infty}C_n$ then C has Hausdorff dimension α and 1-dimensional Lebesgue measure zero (see [2, Example 4.5]), as does $\frac{1}{2}C=\{x:2x\in C\}$. The Hausdorff dimension of the Cartesian product $(\frac{1}{2}C)^3=\frac{1}{2}C\times\frac{1}{2}C\times\frac{1}{2}C$ is 3α (in general, the Hausdorff dimension of a product is at least the sum of the Hausdorff dimensions of the factors; in this case, it can be shown that equality holds. See [2, pp. 94–95, esp. Example 7.6]). Let

$$X_n=\{(z,w)\in bB: z=x+iy,\ w=u+iv,\ (x,y,u)\in(\tfrac{1}{2}C_n)^3\},$$

and set $X=\bigcap_{n=1}^{\infty}X_n$. Then, since X is the image of $(\frac{1}{2}C)^3$ under a diffeomorphism of $[0,1/2]^3$ with a subset of bB, and Hausdorff dimension is invariant under diffeomorphisms (see [2, Cor. 2.4]), X has Hausdorff dimension 3α. Moreover, $\pi(X)=(\frac{1}{2}C)^2$ has 2-dimensional Lebesgue measure zero. It remains to show that X is polynomially convex. Write each X_n as the union of 2^{3n} disjoint "cubes"

$$X_{n,\beta}=\{(z,w)\in bB:(x,y,u)\in C_{n,\beta_1}\times C_{n,\beta_2}\times C_{n,\beta_3}\},$$

where $\beta=(\beta_1,\beta_2,\beta_3)$ and $1\le\beta_j\le 2^n$. First we claim that $\hat{X}_n=\bigcup_\beta\hat{X}_{n,\beta}$. To see this, use the fact that if $K_1,\dots,K_n$ are compact subsets of $\mathbf{C}^n$, $K=\bigcup_{j=1}^{N}K_j$, π denotes the projection of $\mathbf{C}^n$ onto some fixed coordinate plane, and $\pi(K_j)$ are disjoint polynomially convex subsets of $\mathbf{C}$, then $\hat{K}=\bigcup_{j=1}^{N}\hat{K}_j$. (This follows from Lemma 29.21 of [12].) From this it follows that for fixed $\beta_3=\beta_3^0$, $\bigcup_{\beta_3=\beta_3^0}X_{n,\beta}=\bigcup_{\beta_3=\beta_3^0}\hat{X}_{n,\beta}$ since the projections to the z-plane of the $X_{n,\beta}$ for fixed β_3 are pairwise disjoint squares. The claim is proved by then noting that the projections of $\bigcup_{\beta_3=\beta_3^0}X_{n,\beta}$ for distinct β_3^0 to the w-plane are again disjoint, polynomially convex sets. Next we note that given $\delta>0$, for sufficiently large n, each $X_{n,\beta}$ (and thus each $\hat{X}_{n,\beta}$) is contained in the intersection of bB with a ball of radius δ. Thus $\hat{X}\setminus X=\hat{X}\cap B$ is empty; that is, X is polynomially convex.

REMARKS. A reasonable conjecture, for which we have no proof as yet, would be the following: If Ω is strictly pseudoconvex with C^2 boundary $b\Omega$, and X is an $O(\bar{\Omega})$-convex subset of $b\Omega$ with $\sigma(X)=0$, then X is removable for L^1 CR functions.

We also note that the theory of L^2 removability (say) is quite different than the L^1 theory. For example, if B is the unit ball in $\mathbf{C}^2$, then $X=\{(z,w): z=0\}$ is not removable for L^1 CR functions ($w^{-1}\in L^1(bB)$) but *is* removable for L^2 CR functions: every $f\in L^2(bB)\cap \mathrm{CR}(bB\setminus X)$ has an extension F to $B\setminus\{w=0\}$, which (by estimating on the analytic disks $\{w=\text{constant}\}$) is L^2 with respect to volume measure on B. By a classical result, F extends holomorphically (and as an H^2 function) to B. It would be interesting to give some conditions on the size of $\hat{X}_\Omega$ which would guarantee that X is removable for L^2 CR functions.

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