On the Dirichlet Problem for the Complex Monge–Ampère Operator

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1. Introduction

Let D be a bounded domain in $\mathbf{C}^n$. Given $f \in C(D)$ with $f \geq 0$ and given $\phi \in C(\partial D)$, we study the nonlinear Dirichlet problem:

$$\begin{aligned} &u \text{ is plurisubharmonic (psh) in } D, \text{ i.e., } u \in P(D), \\ &(dd^c u)^n = f^n \, dV \text{ in } D, \quad \text{and} \\ &u = \phi \text{ on } \partial D \end{aligned} \tag{1.1}$$

where $(dd^c(\cdot))^n$ is the complex Monge–Ampère operator studied extensively by Bedford and Taylor. For D strictly pseudoconvex, existence and uniqueness of the solution u were shown in [BT1]. The same result holds more generally for the class of B-regular domains introduced by Sibony [Si] (for the definition of B-regular, see Section 2). For further results when $f \in L^2(D)$ we refer the reader to [CP].

In Section 2 we outline an iterative balayage-type procedure for constructing u which uses only classical potential theory in R^{2n}. The idea is motivated by the fact that for u in $P(D) \cap C^2(D)$,

$$\left[\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)\right]^{1/n} = \frac{1}{n} \inf\{\Delta_a u : a \in A\},$$

where

$$A = \{a \in GL(n, \mathbf{C}) : a \text{ is positive definite and Hermitian with } \det a = 1\} \tag{1.2}$$

and

$$\Delta_a u = \sum a_{ij} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = a-\text{Laplacian of } u. \tag{1.3}$$

Our construction may be considered as a potential-theoretic interpretation of Gaveau's approach to (1.1) in [G1]. For a different approach to the homogeneous equation ($f \equiv 0$), see Poletsky [Po] and Bedford [Be]. We should also call attention to Bremermann's work [Br].

In Section 3 we study (1.1) for the bidisc $\mathbf{U}$ in $\mathbf{C}^2$. This domain is not B-regular. However, the homogeneous Monge–Ampère equation for $\mathbf{U}$ was

Received February 3, 1992. Revision received March 22, 1993.
Michigan Math. J. 40 (1993).

previously studied by Sadullaev [Sa]. For the general case, we use a modification of Gaveau's Kähler control method (cf. [G1]) to construct a plurisubharmonic u satisfying (1.1) for certain allowable ϕ in $C(\partial \mathbf{U})$. This technique enables us to solve (1.1) for certain unbounded f in $C(\mathbf{U}) \cap L^1(\mathbf{U})$.

This research was conducted during the spring/summer of 1991 while the authors were visiting Indiana University. We would like to thank the department of mathematics for its hospitality. Special thanks are due to Eric Bedford for many valuable conversations.

2. A Potential-Theoretic Approach to (1.1)

We first introduce some notation which will be used throughout. Given a in A and a bounded domain Ω in $\mathbf{C}^n$, we let $g_\Omega^a(\cdot, z') \geq 0$ be the Green function with respect to $\Delta_a \equiv \sum a_{ij}(\partial^2/\partial z_i \partial \bar{z}_j)$ for Ω with pole at z' in Ω, and we let $h_\Omega^a(\cdot, \xi)$ be the Poisson kernel for $\partial\Omega$ where $\xi \in \partial\Omega$. Thus, given $f \in C(\Omega)$ with $f \geq 0$ and given $\phi \in C(\partial\Omega)$, we have that

$$U_a(z) \equiv -\int_\Omega g_\Omega^a(z, z') f(z')\, dV(z') + \int_{\partial\Omega} h_\Omega^a(z, \xi) \phi(\xi)\, d\sigma(\xi)$$
$$\equiv (G_\Omega^a f)(z) + (H_\Omega^a \phi)(z) \tag{2.1}$$

is the solution of the a-Dirichlet problem

$$\Delta_a U_a = f \text{ in } \Omega \quad \text{and} \quad U_a = \phi \text{ on } \partial\Omega$$

if $\partial\Omega$ is regular for the a-Dirichlet problem.

REMARK 2.1. We can replace f by a positive Borel measure μ on Ω such that $G_\Omega^a \mu(z) \equiv \int_\Omega g_\Omega^a(z, z')\, d\mu(z')$ converges for z in Ω. Then $\Delta_a U_a = \mu$ as measures. If ϕ is only required to be upper semicontinuous (usc) on $\partial\Omega$, we can choose a sequence $\{\phi_j\}$ in $C(\partial\Omega)$ with $\phi_j \searrow \phi$ on $\partial\Omega$. Then $H_\Omega^a \phi_j \searrow H_\Omega^a \phi$ and $\overline{\lim}_{z \to \xi} U_a(z) = \phi(\xi)$ for $\xi \in \partial\Omega$.

For $a = I = n \times n$ identity matrix, we write $\Delta_a = \Delta$, $g_\Omega^a = g_\Omega$, and so on. Then $U(z) \equiv (G_\Omega f)(z) + (H_\Omega \phi)(z)$ is the solution of the usual Dirichlet problem

$$\Delta U = f \text{ in } \Omega \quad \text{and} \quad U = \phi \text{ on } \partial\Omega \tag{2.2}$$

if $\partial\Omega$ is regular. Recall that if $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$ on $\mathbf{C}^n$, then $dd^c u = 2i\partial\bar{\partial} u$. Thus, if $u \in C^2(\Omega)$,

$$(dd^c u)^n = \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n \text{ times}} = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right] dV.$$

We have the following relationships between the a-Laplacian operators Δ_a for a in A and $(dd^c u)^n$ for u in $P(\Omega) \cap C^2(\Omega)$.

PROPOSITION 2.2 [G1]. *Let $u \in P(\Omega) \cap C^2(\Omega)$ and let $a \in A$. Then*

$$\left[\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)\right]^{1/n} \leq \frac{1}{n} \Delta_a u \text{ in } \Omega \tag{2.3}$$

and

$$\left[\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)\right]^{1/n} = \inf\left\{\frac{1}{n}\Delta_a u : a \in A\right\}. \tag{2.4}$$

Proof. For each positive semidefinite Hermitian matrix b,

$$\inf\{\text{trace}(ab) : a \in A\} = n(\det b)^{1/n} \tag{2.5}$$

[G1, Lemma 1]. Apply this to $b = (\partial^2 u/\partial z_i \partial \bar{z}_j)$. □

To relate our candidate for a solution to (1.1) with the upper envelopes constructed in [BT1], we need to modify Proposition 2.2 for locally bounded u.

COROLLARY 2.3. *Let $u \in P(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and let $\partial^2 u/\partial z_i \partial \bar{z}_j = u_{i\bar{j}}\,dV + ds_{i\bar{j}}$ be the Lebesgue decomposition of the Borel measure $\partial^2 u/\partial z_i \partial \bar{z}_j$. Define*

$$\Phi(u) = c_n g\, dV \quad \text{where } c_n = 4(n!)^{1/n} \text{ and } g = (\det u_{i\bar{j}})^{1/n}. \tag{2.6}$$

Then

(i) $g = \inf\{(1/n)\sum a_{ij} u_{i\bar{j}} : a \in A\}$, *and*

(ii) $c_n g \le f$, *where $(dd^c u)^n = f^n\, dV + ds$ is the Lebesgue decomposition of $(dd^c u)^n$.*

Proof. This is essentially a restatement of Theorem 5.8 in [BT1]. Property (i) follows from (2.5). □

We mention the following useful criterion for determining whether a locally integrable function is, up to regularization, plurisubharmonic.

PROPOSITION 2.4. *Let $u \in L^1_{\text{loc}}(\Omega)$ and suppose that $\Delta_a u \ge 0$ for each a in A; that is, $\Delta_a u$ is a positive measure. Then $u^*(z) \equiv \overline{\lim}_{\zeta \to z} u(\zeta)$ is plurisubharmonic in Ω.*

With these preliminaries, we are ready to construct a solution u for (1.1). For now, we assume only that D is a bounded domain in $\mathbf{C}^n$ which is regular for the usual Dirichlet problem (2.2), where $f \in C(D)$ with $f \ge 0$ and $\phi \in C(\partial D)$ are given. Let

$$u_0(z) = (G_D f)(z) + (H_D \phi)(z)$$

be the solution to (2.2) with $\Omega = D$. This will be our 0th approximation to a solution u of (1.1). Clearly $u_0 \ge u$ if u exists with equality precisely when u is pluriharmonic in D. Note that $f \equiv 0$ in this (trivial!) case.

Given z in D, we define

$$\begin{aligned} u_1(z) &\equiv \inf\{(G_B^a f)(z) + (H_B^a u_0)(z) : a \in A,\ B = B(z,r) \subset D\} \\ &\equiv \inf_{a,B}[(G_B^a f)(z) + (H_B^a u_0)(z)], \end{aligned}$$

where $B(z,r) = \{\xi \in \mathbf{C}^n : |\xi - z| < r\}$. This will be our first approximation to u. Note the following properties.

(1) u_1 is usc in D. For since $f \in C(D)$, each function $(G_B^a f)(z)+(H_B^a u_0)(z)$ is continuous in $B=B(z,r)$. Fixing z in D and given $\epsilon>0$, we can find a and B with $u_1(z)+\epsilon \geq (G_B^a f)(z)+(H_B^a u_0)(z)$. By the continuity of f and u_0, for z' in D sufficiently close to z we can translate $B=B(z,r)$ to $B'=B(z',r)$ and conclude that

$$(G_{B'}^a f)(z')+(H_{B'}^a u_0)(z')<(G_B^a f)(z)+(H_B^a u_0)(z)+\epsilon.$$

By the definition of $u_1(z')$ we thus obtain $u_1(z')<u_1(z)+2\epsilon$. This implies that u_1 is usc. Note that we really only required u_0 to be usc in the proof.

(2) $u_1(z) \leq u_0(z)$ for all z in D. For if we take $a=I$ and z in D, by the continuity of f and the harmonicity of u_0 we have

$$\lim_{r \downarrow 0^+}(G_{B(z,r)} f)(z)=0 \quad \text{and} \quad (H_{B(z,r)} u_0)(z)=u_0(z).$$

Hence

$$u_1(z) \leq \lim_{r \downarrow 0^+}[(G_{B(z,r)} f)(z)+(H_{B(z,r)} u_0)(z)]=u_0(z).$$

(3) If $u_1(z)=\inf_{a,B}[(G_B^a f)(z)+(H_B^a u_1)(z)]$, then $u_1 \in P(D)$. This follows from the next proposition.

PROPOSITION 2.5. *Let w be usc in D. If there is an f in $L_{\mathrm{loc}}^1(D)$ with $f \geq 0$ in D and $w(z)=\inf_{a,B}[(G_B^a f)(z)+(H_B^a w)(z)]$, then $w \in P(D)$. Furthermore, for each a in A, $\sum a_{ij} w_{i\bar{j}} \geq f$ a.e. in D.*

Proof. To show $w \in P(D)$, by Proposition 2.4 it suffices to show that for each a in A we have $\Delta_a w \geq 0$ in D. Since $f \geq 0$, for each pair a and B we have $G_B^a f \leq 0$ in B. Thus

$$w(z) \leq \inf_{a,B}(H_B^a w)(z) \leq (H_B^a w)(z), \tag{2.7}$$

so that w is a-subharmonic in B. Since $\Delta_a w \geq 0$ in B for each ball $B=B(z,r)$ in D, $\Delta_a w \geq 0$ in D.

For the second part of the proposition we need a lemma about Green potentials of Borel measures. For μ a Borel measure in D, we let μ_B denote the restriction of μ to $B \subset D$.

LEMMA 2.6. *Let μ be a Borel measure in D and let $\mu=g\,dV+\nu_S$ be the Lebesgue decomposition of μ. If there exists an a in A with $G_B^a \mu_B \leq 0$ in B for each ball $B=B(z,r) \subset D$, then $g \geq 0$ a.e. in D.*

Proof. This is Theorem 5 in [G1]. □

We now finish with the proof of Proposition 2.5. For each a in A and $B=B(z,r) \subset D$,

$$w(z) \leq (G_B^a f)(z)+(H_B^a w)(z) \quad \text{for } z \text{ in } B. \tag{2.8}$$

On the other hand, by the Riesz decomposition theorem

$$w(z) = (G_B^a(\Delta_a w))(z) + (H_B^a w)(z) \quad \text{for } z \text{ in } B.$$

Thus $G_B^a(\Delta_a w) \le G_B^a f$ in B, so that $\sum a_{ij} w_{i\bar{j}} \ge f$ a.e. in D by Lemma 2.6. □

Thus u_1 is a better approximation to the solution u of (1.1) than is u_0. We'll see in what follows that if u_1 satisfies (3) then $(dd^c u_1)^n = ((c_n/n)f)^n \, dV \equiv (\tilde{f})^n \, dV$ in D. If not, we proceed to "push down" u_1. Since u_1 is usc in D, by Remark 2.1 we can define

$$u_2(z) \equiv \inf_{a,B} [(G_B^a f)(z) + (H_B^a u_1)(z)].$$

In analogy with properties (1)–(3) of u_1, we have the following.

(1′) u_2 is usc in D. As remarked in the proof of (1), only the upper semicontinuity of u_0 was used to obtain the upper semicontinuity of u_1.

(2′) $u_2(z) \le u_1(z)$ for all z in D. For if we take $a = I$ and z in D, by the upper semicontinuity of u_1 we have $\overline{\lim}_{r\downarrow 0^+} (H_{B(z,r)} u_1)(z) \le u_1(z)$. Also, $G_{B(z,r)} f \le 0$, so that $u_2(z) \le \overline{\lim}_{r\downarrow 0^+} (H_{B(z,r)} u_1)(z) \le u_1(z)$.

(3′) If $u_2(z) = \inf_{a,B}[(G_B^a f)(z) + (H_B^a u_2)(z)]$, then $u_2 \in P(D)$. This follows from Proposition 2.5.

Continuing this process recursively, having constructed u_{n-1} we define

$$u_n(z) \equiv \inf_{a,B} [(G_B^a f)(z) + (H_B^a u_{n-1})(z)]. \tag{2.9}$$

The functions $\{u_n\}$ are usc in D and form a decreasing sequence. Since D is regular for the standard Dirichlet problem,

$$\lim_{z\to\xi} u_0(z) = \phi(\xi) \quad \text{for all } \xi \text{ in } \partial D$$

and

$$\overline{\lim_{z\to\xi}}\, u_n(z) \le \phi(\xi) \quad \text{for all } \xi \text{ in } \partial D, \quad n = 0, 1, 2, \ldots. \tag{2.10}$$

We are now ready for the main result of this section.

THEOREM 2.7. *Let D be a bounded domain in $\mathbf{C}^n$ which is regular for the standard Dirichlet problem. Let $f \in C(D) \cap L^\infty(D)$ with $f \ge 0$, and let $\phi \in C(\partial D)$. With $\{u_n\}$ defined in* (2.9), *let*

$$v(z) \equiv \lim_{n\to+\infty} u_n(z) \quad \textit{for all } z \textit{ in } D. \tag{2.11}$$

Then

(i) *$v \in P(D) \cap L^\infty_{\mathrm{loc}}(D)$, and for each a in A, $\sum a_{ij} v_{i\bar{j}} \ge f$ a.e. in D.*

(ii) *$v(z) = \sup\{w(z) : w \in P(D) \cap L^\infty_{\mathrm{loc}}(D),\ \Phi(w) \ge (c_n/n) f \, dV \equiv \tilde{f} \, dV$ and $\overline{\lim}_{z\to\xi} w(z) \le \phi(\xi)$ for all ξ in $\partial D\}$.*

Proof. First note that v is usc in D, since each u_n is usc in D and the sequence $\{u_n\}$ is decreasing. Thus, to prove (i) it suffices, by Proposition 2.5, to show

$$v(z) = \inf_{a,B} [(G_B^a f)(z) + (H_B^a v)(z)] \quad \text{for all } z \text{ in } D. \tag{2.12}$$

To prove (2.12), note that for each a in A and each ball $B=B(z,r)\subset D$,

$$v(z)\le (G_B^a f)(z)+(H_B^a u_n)(z) \quad \text{for } n=0,1,2,\ldots$$

by (2.9). By the monotone convergence theorem,

$$\lim_{n\to+\infty} (H_B^a u_n)(z)=(H_B^a v)(z) \quad \text{for } z \text{ in } B.$$

Hence

$$v(z)\le (G_B^a f)(z)+(H_B^a v)(z)$$

so that

$$v(z)\le \inf_{a,B}[(G_B^a f)(z)+(H_B^a v)(z)].$$

For the reverse inequality, note that for each n,

$$\begin{aligned} u_n(z) &= \inf_{a,B}[(G_B^a f)(z)+(H_B^a u_{n-1})(z)] \\ &\ge \inf_{a,B}[(G_B^a f)(z)+(H_B^a v)(z)] \end{aligned}$$

by (2.9) and the fact that $v\le u_{n-1}$. Thus

$$v(z)=\lim_{n\to+\infty} u_n(z)\ge \inf_{a,B}[(G_B^a f)(z)+(H_B^a v)(z)].$$

Note that $f\in C(D)\cap L^\infty(D)$ implies that $v\in L^\infty_{\text{loc}}(D)$.

To prove (ii), following Bedford and Taylor, we let

$$\mathcal{B}(\phi,\tilde f)\equiv\{w\in P(D)\cap L^\infty_{\text{loc}}(D): \Phi(w)\ge \tilde f\, dV \text{ and } \overline{\lim}_{z\to\xi} w(z)\le\phi(\xi) \text{ on } \partial D\}. \tag{2.13}$$

We show that for each w in $\mathcal{B}(\phi,\tilde f)$, $w\le v$ in D. Fix $w\in\mathcal{B}(\phi,\tilde f)$. By Corollary 2.3(i), since $\Delta_a w\ge\sum a_{ij}w_{i\bar j}$ as measures, $\Phi(w)\ge\tilde f\,dV$ implies $\Delta_a w\ge f$ as measures for each a in A. Thus for each ball $B=B(z,r)\subset D$,

$$G_B^a f\ge G_B^a(\Delta_a w) \text{ in } B.$$

Hence

$$w=G_B^a(\Delta_a w)+H_B^a w\le G_B^a f+H_B^a w \text{ in } B. \tag{2.14}$$

Clearly $w\le u_0$ in D, since $\overline{\lim}_{z\to\xi} w(z)\le u_0(\xi)$ for each ξ in ∂D and u_0 is harmonic in D. Thus

$$w\le G_B^a f+H_B^a u_0 \text{ in } B.$$

This inequality holds for each a in A and each ball $B=B(z,r)\subset D$. Hence

$$w(z)\le \inf_{a,B}[(G_B^a f)(z)+(H_B^a u_0)(z)]=u_1(z) \quad \text{for all } z \text{ in } D.$$

Using (2.14) and $w\le u_1$, we obtain

$$w\le G_B^a f+H_B^a u_1 \text{ in } B,$$

which yields $w\le u_2$ in D. By induction, it follows that $w(z)\le u_n(z)$ for $n=1,2,\ldots$ and for all z in D. Hence $w(z)\le v(z)$ in D. Since w was an arbitrary element of $\mathcal{B}(\phi,\tilde f)$,

$$\sup\{w(z): w \in \mathcal{B}(\phi, \tilde{f})\} \le v(z) \quad \text{for all } z \text{ in } D.$$

On the other hand, since $\sum a_{ij} v_{i\bar{j}} \ge f$ a.e., from Corollary 2.3(i) we have $\Phi(v) \ge \tilde{f}\, dV$. From (2.10),

$$\overline{\lim_{z \to \xi}}\, v(z) \le \phi(\xi) \quad \text{for all } \xi \text{ in } \partial D, \tag{2.15}$$

so that $v \in \mathcal{B}(\phi, \tilde{f})$ and equality holds in (ii). □

REMARK. The theorem is true for general bounded domains. We only require our initial function u_0 to be continuous and superharmonic with respect to Δ in D, and to satisfy $\overline{\lim}_{z \to \xi} u_0(z) \le \phi(\xi)$ for all ξ in ∂D.

We next show that our v coincides with an upper envelope defined using the complex Monge–Ampère operator. Let

$$\mathfrak{F}(\phi, \tilde{f}) = \{w \in P(D) \cap L^\infty_{\text{loc}}(D) : (dd^c w)^n \ge \tilde{f}^n\, dV \text{ in } D \text{ and } \overline{\lim}_{z \to \xi} w(z) \le \phi(\xi) \text{ for all } \xi \text{ in } \partial D\}.$$

THEOREM 2.8. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$. Let $f \in C(D) \cap L^\infty(D)$ with $f \ge 0$, and let $\phi \in C(\partial D)$. Set*

$$v(z) = \sup\{w(z): w \in \mathcal{B}(\phi, \tilde{f})\}$$

as in Theorem 2.7, and let

$$U(z) = \sup\{w(z): w \in \mathfrak{F}(\phi, \tilde{f})\}.$$

Then $v = U$ in D. Furthermore, $v \in P(D) \cap L^\infty_{\text{loc}}(D)$ and v satisfies $(dd^c v)^n = \tilde{f}^n\, dV$ in D, $\Phi(v) = \tilde{f}\, dV$ in D, and $\overline{\lim}_{z \to \xi} v(z) \le \phi(\xi)$ for all ξ in ∂D.

Proof. Write $D = \bigcup D_m$, $D_m \subset D_{m+1}$, where the D_m are strictly pseudoconvex domains with C^2 boundary. Using the Dirichlet data $f^{(m)} \equiv f|_{D_m}$ on D_m and $\phi^{(m)} \equiv (H_D \phi)|_{\partial D_m}$, the restriction to ∂D_m of the harmonic extension of ϕ to D, we denote the corresponding envelope functions in D_m by $v^{(m)}$ and $U^{(m)}$. By results of [BT1], $v^{(m)} = U^{(m)}$ in D_m. Moreover, by Theorem 6.2 of [BT1], $v^{(m)} \in P(D_m) \cap C(\bar{D}_m)$ and $v^{(m)}$ satisfies

$$(dd^c v^{(m)})^n = (\tilde{f}^{(m)})^n\, dV = \tilde{f}^n\, dV \quad \text{in } D_m,$$
$$\Phi(v^{(m)}) = \tilde{f}^{(m)}\, dV = \tilde{f}\, dV \quad \text{in } D_m, \quad \text{and}$$
$$v^{(m)} = \phi^{(m)} = H_D \phi \quad \text{on } \partial D_m.$$

(1) $v^{(m)} \ge v^{(m+1)}$ in D_m. Since $v^{(m+1)} \le H_D \phi$ in D_{m+1}, $v^{(m+1)} \le \phi^{(m)}$ on ∂D_m. Also $(dd^c v^{(m)})^n = (dd^c v^{(m+1)})^n = \tilde{f}^n\, dV$ in D_m, so that (1) follows by the domination principle [BT2, Cor. 4.5].

Thus, for each z in D, $z \in D_m$ for $m \ge m(z)$ and

$$\lim_{m \to +\infty} v^{(m)}(z) = \inf_{m \ge m(z)} v^{(m)}(z) \equiv \tilde{u}(z)$$

defines a function $\tilde{u}$ in $P(D) \cap L^\infty_{\text{loc}}(D)$. This function satisfies $(dd^c \tilde{u})^n = \tilde{f}^n\, dV$ and $\Phi(\tilde{u}) = \tilde{f}\, dV$ in D, since these relations hold on any ball B in D. Since $\phi^{(m)} = (H_D \phi)|_{\partial D_m}$, we clearly have $\overline{\lim}_{z \to \xi} \tilde{u}(z) \le \phi(\xi)$ for all ξ in

∂D. Thus it remains to show that $\tilde{u}=v=U$. Note that $\tilde{u}$, v, and U are elements of $L^\infty_{\text{loc}}(D)$ by the assumptions that $f\in L^\infty(D)$ and D is bounded. For example, $\max_{\xi\in\partial D}|\phi(\xi)|\geq U(z)\geq A|z|^2-B$ for sufficiently large constants A and B.

(2) $\tilde{u}\leq v$ and $\tilde{u}\leq U$ in D. This follows from the previous paragraph, which shows that $\tilde{u}\in\mathcal{B}(\phi,\tilde{f})\cap\mathcal{F}(\phi,\tilde{f})$.

(3) $\tilde{u}\geq v$ in D. Since $D_m\subset D$, $G_{D_m}f^{(m)}\geq G_Df$ on D_m. By definition, $\phi^{(m)}=H_D\phi$ on D_m. Thus, for each m, $u_0^{(m)}\geq u_0$ in D_m. Using the definitions of $u_1^{(m)}$ and u_1 and the fact that $D_m\subset D$, we have $u_1^{(m)}\geq u_1$ in D_m. By induction, $u_n^{(m)}\geq u_n$ in D_m for $n=0,1,2,\dots$. Hence $v^{(m)}\geq v$ in D_m for each m, and we obtain (3).

(4) $\tilde{u}\geq U$ in D. Let $w\in\mathcal{F}(\phi,\tilde{f})$. We show that for each m, $w\leq v^{(m)}$ in D_m, which proves (4). Since $w\in\mathcal{F}(\phi,\tilde{f})$, $\overline{\lim}_{z\to\xi}w(z)\leq\phi(\xi)$ on ∂D. Thus $w\leq H_D\phi$ in D and hence in D_m. Furthermore, $(dd^cw)^n\geq\tilde{f}^n\,dV$ in D and hence in D_m. Since $v^{(m)}$ satisfies $v^{(m)}=H_D\phi$ on ∂D_m and $(dd^cv^{(m)})^n=\tilde{f}^n\,dV$ on D_m, by the domination principle of Bedford and Taylor, $w\leq v^{(m)}$ in D_m. □

Thus our potential-theoretic approach (v) to the Dirichlet problem (1.1) coincides with the Perron–Bremermann upper envelope (U). The stumbling block to solvability of (1.1) is the boundary behavior of U. For a bounded domain D in $\mathbf{C}^n$, we get solvability if the domain is B-regular. Recall that a bounded domain D is *hyperconvex* if D admits a bounded psh exhaustion function—that is, if there exists a function ρ in $P(D)$ with $D=\{z:\rho(z)<0\}$ and $D_c\equiv\{z\in D:\rho(z)<c\}\subset\subset D$ for all $c<0$.

DEFINITION 2.9. Let D be a bounded hyperconvex domain. D is *B-regular* if for each ξ in ∂D there exists $\psi\in P(D)\cap C(\bar{D})$ with $\psi(\xi)=0$ and $\psi<0$ on $\bar{D}-\{\xi\}$. Equivalently, for each ϕ in $C(\partial D)$ there exists $u\in P(D)\cap C(\bar{D})$ with $u=\phi$ on ∂D (cf. [Si, Thm. 2.3]).

COROLLARY 2.10. *Let D be a B-regular domain with smooth boundary. Then $v=U$ satisfies* (1.1) *for a given $\tilde{f}\in C(D)\cap L^\infty(D)$, with $\tilde{f}\geq 0$ and a given $\phi\in C(\partial D)$.*

Proof. It suffices to show that $U=\phi$ on ∂D, that is, that $\lim_{z\to\xi}U(z)$ exists and equals $\phi(\xi)$ for all ξ in ∂D. From Theorem 2.4 in [Si], given $0<\eta<1$, there is a defining function r of D such that $\rho=-(-r)^\eta$ is a bounded psh exhaustion function which is strictly psh in D and satisfies

$$\sum\frac{\partial^2\rho}{\partial z_i\partial\bar{z}_j}(z)t_i\bar{t}_j\geq m|t|^2$$

for all z in D and all t in $\mathbf{C}^n$ for some $m>0$. Given $\tilde{f}$ and ϕ, from Definition 2.9 we can find $w\in P(D)\cap C(\bar{D})$ with $w=\phi$ on ∂D. Then, for sufficiently large $C>0$, the function

$$\tilde{w}\equiv w+C\rho\in\mathcal{F}(\phi,\tilde{f}).$$

For ξ in ∂D we have

$$\phi(\xi) = \lim_{z\to\xi} \tilde{w}(z) \le \overline{\lim_{z\to\xi}}\, U(z) \le \phi(\xi),$$

so that $\lim_{z\to\xi} U(z)$ exists and equals $\phi(\xi)$. □

3. Dirichlet Problem for the Bidisc

Let $\mathbf{U} \equiv \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| < 1 \text{ and } |z_2| < 1\}$ be the open unit bidisc in $\mathbf{C}^2$, and let $\partial\mathbf{U}$ denote the topological boundary of $\mathbf{U}$. Let

$$T \equiv \{(z_1, z_2) : |z_1| = |z_2| = 1\}$$

be the distinguished boundary of $\mathbf{U}$. We are interested in Dirichlet-type problems of the form (1.1) for $D = \mathbf{U}$. Following Bremermann [Br], it is perhaps more natural to consider ϕ specified only on T. We first discuss some results in this direction. Given $f \in C(\mathbf{U})$ with $f \ge 0$ and given $\phi \in C(T)$, we consider the Bremermann Dirichlet problem

$$(dd^c u)^2 = f^2\, dV \text{ in } \mathbf{U} \quad \text{and} \quad u = \phi \text{ on } T. \tag{3.1}$$

Problem (3.1) need not admit unique solutions. For example, if we let $f \equiv 0$ and $\phi \equiv 1$ then $u(z_1, z_2) = |z_1|^{2j}|z_2|^{2k}$ satisfies (3.1) for any nonnegative integers j and k. However, $u(z_1, z_2) \equiv 1$ $(j = k = 0)$ clearly gives the largest solution. Gaveau [G2] has shown that under certain hypotheses on f, such as f having compact support in $\mathbf{U}$, there exists a largest solution u_m to (3.1). Thus if u is any other solution to (3.1), $u \le u_m$. This solution u_m is harmonic on each complex disc in $\partial\mathbf{U}$. We will outline Gaveau's method shortly.

In general, if a solution u to (3.1) is continuous in $\bar{\mathbf{U}}$, then u is subharmonic on each disc in $\partial\mathbf{U}$. For example, if $|z_2^0| = 1$, the subharmonic functions $u(z_1, rz_2^0) \equiv v_r(z_1)$ converge uniformly as $r \to 1$ to $u(z_1, z_2^0) \equiv v_1(z_1)$ on $|z_1| < 1$. Thus, if we try to specify boundary values ϕ on all of $\partial\mathbf{U}$, a necessary condition for the existence of a solution u to

$$(dd^c u)^2 = f^2\, dV \text{ in } \mathbf{U} \quad \text{and} \quad u = \phi \text{ on } \partial\mathbf{U}, \tag{3.2}$$

for a given f in $C(\mathbf{U})$ with $f \ge 0$ and a given ϕ in $C(\partial\mathbf{U})$, is that ϕ be subharmonic on each complex disc in $\partial\mathbf{U}$. In the notation of Sadullaev [Sa], we require that $\phi = \hat{\phi}$, where

$$\hat{\phi}(z) = \sup\{\psi(z) : \psi \in C(\partial\mathbf{U}),\ \psi \le \phi, \text{ and } \psi \text{ is subharmonic on each disc in } \partial\mathbf{U}\}. \tag{3.3}$$

In [Sa], Sadullaev shows that if $\phi = \hat{\phi}$ and $f \equiv 0$ (the homogeneous case), then a solution u to (3.2) exists and is unique. In the previous example, if $f \equiv 0$ and $\phi \equiv 1$ on $\partial\mathbf{U}$, then clearly $u \equiv 1$ is a solution to (3.2). We generalize Sadullaev's result to the nonhomogeneous case.

THEOREM 3.1. *Let $f \in C(\mathbf{U})$ satisfy*

$$|f(z_1, z_2)| \le \frac{c}{(1-|z_1|^2)^\beta (1-|z_2|^2)^\beta} \tag{3.4}$$

for (z_1, z_2) *in* $\mathbf{U}$ *and for constants* $c>0$ *and* $0<\beta<1$. *Let* $\phi=\hat{\phi}\in C(\partial\mathbf{U})$. *Then there exists a unique solution* u *to* (3.2) *and* $u\in P(\mathbf{U})\cap C(\bar{\mathbf{U}})$.

REMARK 3.2. Condition (3.4) implies that we can solve (3.2) even if f is mildly unbounded. Note that if f satisfies (3.4) then $f\in L^1(\mathbf{U})$.

Before we prove the theorem, we give a brief sketch of Gaveau's probabilistic approach to (1.1) and the modifications necessary to get a solution to (3.1). Let $C(\mathbf{W},\mathbf{V})$ denote the continuous $\mathbf{V}$-valued functions on $\mathbf{W}$. We consider the space H of non-anticipating Kähler controls $\sigma=(\sigma_{ij})$ where $\sigma=\sigma(s,w)$ is a positive Hermitian matrix-valued function on $\mathbf{C}^n$ for each $(s,w)\in\mathbf{R}^+\times C(\mathbf{R}^+,\mathbf{C}^n)$, $\det(\sigma\sigma^*)\geq 1$, and $\sigma(\cdot,w)$ is continuous for each w in $C(\mathbf{R}^+,\mathbf{C}^n)\equiv\Omega$. We refer the reader to [Du] or [Kr] for definitions of any unfamiliar terms (e.g., non-anticipating). These will not be essential for understanding the ideas involved.

If we let $b=(b_1,\dots,b_n)$ denote the standard Brownian motion on $\mathbf{C}^n$, for each σ in H we can consider the stochastic process

$$X_t^{\sigma,z}(w)=(X_t^{(\sigma,z)_1},\dots,X_t^{(\sigma,z)_n})$$

given by the stochastic integrals

$$X_t^{(\sigma,z)_j}=z_j+\int_0^t\sum_k\sigma_{jk}(s,w)\,db_k(s),\quad j=1,\dots,n. \tag{3.5}$$

We will omit the subscript j in using vector notation. Given a bounded domain D in $\mathbf{C}^n$, f in $C(D)$ with $f\geq 0$, and ϕ in $C(\partial D)$, we set

$$w_\sigma(z)\equiv E\left[-\int_0^\tau f(X_s^{\sigma,z})\,ds+\phi(X_\tau^{\sigma,z})\right]\quad\text{for } z \text{ in } D, \tag{3.6}$$

where $\tau=\tau_{\partial D}$ is the first hitting time of $X_t^{\sigma,z}$ on ∂D; we consider the lower envelope

$$u(z)\equiv\inf\{w_\sigma(z):\sigma\in H\}. \tag{3.7}$$

In [G1], Gaveau shows this u satisfies (1.1) for $\tilde{f}$ when D is strictly pseudoconvex by showing that:

(1) u is continuous in $\bar{D}$;
(2) $u\in P(D)$; and
(3) $u(z)=U_c(z)\equiv\sup\{w(z):w\in B_c(\phi,\tilde{f})\}$, where $B_c(\phi,\tilde{f})=\mathcal{B}(\phi,\tilde{f})\cap C(D)$.

The proof of continuity of u in D essentially follows from the fact that each $w_\sigma(z)$ is continuous with the same modulus of continuity. This follows from properties of b and continuity of $\sigma(\cdot,w)$ for each w in Ω. Continuity up to ∂D requires the existence of a strictly psh defining function in a neighborhood of $\bar{D}$. For our purposes, continuity up to $\partial\mathbf{U}$ of our proposed solution u in Theorem 3.1 is most difficult; the rest will follow in a fashion similar to that of Gaveau.

Gaveau's approach to (3.1) was to define a certain class of stochastic processes $X_t^\sigma = (X_t^{\sigma_1}, X_t^{\sigma_2})$ in $\mathbf{U}$ all starting at the origin; that is, $X_t^\sigma = X_t^{\sigma,0}$ in (3.5). Here we define

$$\tilde{w}_\sigma(z) \equiv E\left[-\int_0^\tau (f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds + (\phi\circ g_z)(X_\tau^\sigma)\right], \tag{3.8}$$

where

$$g_z(\xi) \equiv (g_{z_1}(\xi_1), g_{z_2}(\xi_2)) \equiv \left(\frac{z_1+\xi_1}{1+\bar{z}_1\xi_1}, \frac{z_2+\xi_2}{1+\bar{z}_2\xi_2}\right).$$

Note $g_z(0) = z$. Here $\tau = \tau_{\partial\mathbf{U}}$ is the first hitting time of X_t^σ at $\partial\mathbf{U}$ and

$$|J_z(\xi)| = \left(\frac{1-|z_1|^2}{|1+\bar{z}_1\xi_1|^2}\right)\left(\frac{1-|z_2|^2}{|1+\bar{z}_2\xi_2|^2}\right)$$

is the Jacobian determinant of $g_z(\xi)$. Then

$$u_m(z) \equiv \inf\{\tilde{w}_\sigma(z) : \sigma\in H,\ \tau=\tau_T\} \tag{3.9}$$

gives the largest solution to (3.1). Note that the automorphism group Aut($\mathbf{U}$) of the bidisc is transitive, so for each σ in H there exists $\tilde{\sigma}$ in H with $\tilde{w}_\sigma = w_{\tilde{\sigma}}$.

Thus (3.9) is essentially equivalent to (3.7) except for the fact that we require $\tau=\tau_T$ in (3.9). This gives an idea why (3.9) yields the *largest* solution to (3.1): given $\phi\in C(T)$, there exist many continuous extensions $\tilde{\phi}$ in $C(\partial\mathbf{U})$ with $\tilde{\phi}=\phi$ on T and $\hat{\tilde{\phi}}=\tilde{\phi}$. Gaveau's u_m in (3.9) corresponds to $\tilde{\phi}$, which is *harmonic* on each disc in $\partial\mathbf{U}$.

In Theorem 3.1, we modify (3.9). Our solution u will be given by

$$u(z) \equiv \inf\{\tilde{w}_\sigma(z) : \sigma\in H\}, \tag{3.10}$$

so that we allow X_t^σ to exit $\mathbf{U}$ through any point in $\partial\mathbf{U}$. We show that u in (3.10) satisfies (3.2).

For the convenience of the reader, and also to indicate the relationship between the probabilistic approach in this section and the potential-theoretic discussion in Section 2, we sketch the proofs that u defined in (3.7) is psh in D and $(dd^c u)^n \geq \tilde{f}^n\,dV$ in D. The proofs for u in (3.10) all require a minor modification.

LEMMA 3.3 (Principle of Bellman). *Let w_σ and u be as in* (3.6) *and* (3.7). *Suppose that $u\in C(\bar{D})$. Let D' be a subdomain of D. Then, for each $t>0$,*

$$u(z) = \inf_{\sigma\in H} E\left[-\int_0^{\tau\wedge t} f(X_s^{\sigma,z})\,ds + u(X_{\tau\wedge t}^{\sigma,z})\right] \tag{3.11}$$

for z in D', where $\tau=\tau_{\partial D'}$ and $\tau\wedge t = \min(\tau,t)$.

Assuming the lemma, which we will not prove here, the plurisubharmonicity of u is established as follows. Fix $D'=B$ and let $t\to+\infty$ in (3.11). Since $f\geq 0$ and $\tau\wedge t\to\tau$, we obtain

$$u(z) \leq \inf_{\sigma\in H} E[u(X_{\tau_{\partial B}}^{\sigma,z})] \quad \text{for } z \text{ in } B.$$

If we take $\sigma\sigma^* = a$, a constant matrix in A, then this is essentially the statement that $\Delta_a u \geq 0$ in B, that is, u is a-subharmonic (cf. (2.7) in Proposition 2.5). Since this is true for each a in A and B in D, by Proposition 2.4 $u \in P(D)$. The proof that $(dd^c u)^n \geq \tilde{f}^n \, dV$ also follows from the lemma. Indeed, again fixing $D' = B$, fixing $\sigma\sigma^* = a$ in A, and letting $t \to +\infty$, we obtain

$$u(z) \leq E\left[-\int_0^{\tau_{\partial B}} f(X_s^{\sigma,z})\, ds + u(X_{\tau_{\partial B}}^{\sigma,z})\right]$$
$$= (G_a^B f)(z) + (H_a^B u)(z) \quad \text{for } z \text{ in } B$$

since $\sigma\sigma^* = a$. Thus $\sum a_{ij} u_{i\bar{j}} \geq f$ a.e. by Proposition 2.5 (cf. (2.8)). Since $u \in P(D) \cap C(\bar{D})$, it follows from Corollary 2.3 that $\Phi(u) \geq \tilde{f}\, dV$. Again from Corollary 2.3(ii), $(dd^c u)^n \geq \tilde{f}^n \, dV$ in D.

REMARK 3.4. As mentioned in [G1], the continuity of u in D is *not* essential in Lemma 3.3. Indeed, the upper semicontinuity of u and the regularity at ∂D, in the sense that $\lim_{z \to \xi} u(z) = \phi(\xi)$ for each ξ in ∂D, are all that is required for the conclusion. This fact will be used in the proof of Theorem 3.1.

To show that $(dd^c u)^n = \tilde{f}^n \, dV$ in D, we prove more generally that the upper envelopes $v(z) = \sup\{w(z) : w \in \mathcal{B}(\phi, \tilde{f})\}$ and $U(z) = \sup\{w(z) : w \in \mathfrak{F}(\phi, \tilde{f})\}$ agree with $u(z)$ when D is pseudoconvex and bounded. We first state a version of Itô's formula which we need.

LEMMA 3.5 (Itô's formula in $\mathbf{C}^n$). *Let $X_t^{\sigma,z}$ be a stochastic process associated with a non-anticipating Kähler control $\sigma = (\sigma_{ij})$, and let $a = \sigma\sigma^*$. Let $g \in C_0^2(\mathbf{C}^n, \mathbf{R})$. Then for each $t > 0$,*

$$g(X_t^{\sigma,z}) = g(z) + \sum_{i=1}^n \left[\int_0^t \sum_{j=1}^n \sigma_{ij} \frac{\partial g}{\partial z_j}(X_s^{\sigma,z})\, db_i(s) + \int_0^t \sum_{j=1}^n \bar{\sigma}_{ij} \frac{\partial g}{\partial \bar{z}_j}(X_s^{\sigma,z})\, d\bar{b}_i(s)\right]$$
$$+ \int_0^t \sum a_{ij} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j}(X_s^{\sigma,z})\, ds.$$

If we set $t = \tau = \tau_{\partial D}$ and take expectations, we obtain

$$g(z) = E\left[-\int_0^\tau \sum a_{ij} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j}(X_s^{\sigma,z})\, ds + g(X_\tau^{\sigma,z})\right]. \tag{3.12}$$

THEOREM 3.6. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$. Let $f \in C(D) \cap L^\infty(D)$ with $f \geq 0$, and let $\phi \in C(\partial D)$. Let $u(z) = \inf\{w_\sigma(z) : \sigma \in H\}$ be defined as in (3.7), and let $v(z) = \sup\{w(z) : w \in \mathcal{B}(\phi, \tilde{f})\}$ and $U(z) = \sup\{w(z) : w \in \mathfrak{F}(\phi, \tilde{f})\}$. Then $v(z) = U(z) = u(z)$ for all z in D. In particular, $u(z)$ satisfies $(dd^c u)^n = \tilde{f}^n \, dV$ in D and $\overline{\lim}_{z \to \xi} u(z) \leq \phi(\xi)$ for each ξ in ∂D.*

Proof. As in the proof of Theorem 2.8, we write $D = \bigcup D_m$ with $D_m \subset D_{m+1}$ and each D_m being a strictly pseudoconvex domain with C^2 boundary. Let

$v^{(m)}$, $U^{(m)}$, and $u^{(m)}$ be the envelope functions in D_m corresponding to the Dirichlet data $f^{(m)} \equiv f|_{D_m}$ and $\phi^{(m)} \equiv (H_D\phi)|_{\partial D_m}$. From [BT1] and [G1] it follows that $v^{(m)} = U^{(m)} = u^{(m)}$ in D_m, $u^{(m)} \in P(D_m) \cap C(\bar{D}_m)$, $(dd^c u^{(m)})^n = \tilde{f}^n \, dV$ in D_m, $\Phi(u^{(m)}) = \tilde{f}\, dV$, and $u^{(m)} = \phi^{(m)} = H_D\phi$ on ∂D_m. Again, let $\tilde{u} = \lim_{m\to+\infty} u^{(m)}$. Clearly $u \le u^{(m)}$ in D_m so that $u \le \tilde{u}$ in D. Thus it suffices to prove that $\tilde{u} \le u$ in D.

Note that $\Phi(u^{(m)}) = \tilde{f}\, dV$ in D_m, $\Phi(\tilde{u}) = \tilde{f}\, dV$ in D, and, if χ_ϵ is a standard smoothing kernel,

$$\Phi(\tilde{u} * \chi_\epsilon) \ge \Phi(\tilde{u}) * \chi_\epsilon = \tilde{f}\, dV * \chi_\epsilon$$

[BT1, Thm. 5.7].

We introduce the temporary notation $\Phi(w) = \Phi_w \, dV$; thus $\Phi_{\tilde{u}*\chi_\epsilon} \ge \tilde{f} * \chi_\epsilon$. Fix a control σ and let $a = \sigma\sigma^*$. Given ϵ and χ_ϵ we choose m and D_m so that $\tilde{u}_\epsilon \equiv \tilde{u} * \chi_\epsilon$ is defined in D_m. We can apply (3.12) to the process $X_t^{\sigma,z}$ and the function $g = \tilde{u}_\epsilon$ for z in D_m on the domain D_m, so that $\tau = \tau_{\partial D_m} \equiv \tau_m$, to obtain

$$\begin{aligned} \tilde{u}_\epsilon(z) &= E\left[-\int_0^{\tau_m} \sum a_{ij} \frac{\partial^2 \tilde{u}_\epsilon}{\partial z_i \partial \bar{z}_j}(X_s^{\sigma,z})\, ds + \tilde{u}_\epsilon(X_{\tau_m}^{\sigma,z})\right] \\ &\le E\left[-\int_0^{\tau_m} \frac{n}{c_n} \Phi_{\tilde{u}_\epsilon}(X_s^{\sigma,z})\, ds + \tilde{u}_\epsilon(X_{\tau_m}^{\sigma,z})\right] \end{aligned}$$

by Corollary 2.3(i). Since $\Phi_{\tilde{u}_\epsilon} \ge \tilde{f} * \chi_\epsilon$,

$$\tilde{u}_\epsilon(z) \le E\left[-\int_0^{\tau_m} (f*\chi_\epsilon)(X_s^{\sigma,z})\, ds + \tilde{u}_\epsilon(X_{\tau_m}^{\sigma,z})\right].$$

Since $f * \chi_\epsilon \to f$ and $\tilde{u}_\epsilon \to \tilde{u}$ in D, letting $\epsilon \downarrow 0$ yields

$$\tilde{u}(z) \le E\left[-\int_0^{\tau_m} f(X_s^{\sigma,z})\, ds + \tilde{u}(X_{\tau_m}^{\sigma,z})\right] \quad \text{for each } \sigma \text{ in } H. \tag{3.13}$$

Thus

$$\tilde{u}(z) \le \inf_{\sigma\in H} E\left[-\int_0^{\tau_m} f(X_s^{\sigma,z})\, ds + \tilde{u}(X_{\tau_m}^{\sigma,z})\right] \quad \text{for } z \text{ in } D_m.$$

We want to let $m \uparrow +\infty$ in the above inequality. To be precise, fix z in D. Then $z \in D_m$ for $m \ge m(z)$. Fix one such domain D_m. Given $\epsilon > 0$, choose $\sigma_1 = \sigma_1(\epsilon, z)$ in H so that

$$u(z) = \inf\{w_\sigma(z) : \sigma \in H\} \ge w_{\sigma_1}(z) - \epsilon.$$

In other words,

$$u(z) + \epsilon \ge E\left[-\int_0^{\tau} f(X_s^{\sigma_1,z})\, ds + \phi(X_\tau^{\sigma_1,z})\right], \tag{3.14}$$

where $\tau = \tau_{\partial D}$. From (3.13), for this choice of σ_1,

$$\tilde{u}(z) \le E\left[-\int_0^{\tau_m} f(X_s^{\sigma_1,z})\, ds + \tilde{u}(X_{\tau_m}^{\sigma_1,z})\right] \tag{3.13'}$$

for $m \ge m(z)$.

Now for each path w in $C(\mathbf{R}^+, \mathbf{C}^n)$, by continuity of w, $\tau_m = \tau_m(w) \to \tau = \tau(w)$ and $X_{\tau_m}^{\sigma_1, z}(w) \to X_\tau^{\sigma_1, z}(w)$ as $m \to +\infty$. Letting $m \to +\infty$ in (3.13′),

$$\begin{aligned}\tilde{u}(z) &\le \varlimsup_{m\to+\infty} E\left[-\int_0^{\tau_m} f(X_s^{\sigma_1,z})\,ds + \tilde{u}(X_{\tau_m}^{\sigma_1,z})\right] \\ &= E\left[-\int_0^{\tau} f(X_s^{\sigma_1,z})\,ds + \varlimsup_{m\to+\infty} E[\tilde{u}(X_{\tau_m}^{\sigma_1,z})]\right] \\ &\le E\left[-\int_0^{\tau} f(X_s^{\sigma_1,z})\,ds + E\left[\varlimsup_{m\to+\infty} \tilde{u}(X_{\tau_m}^{\sigma_1,z})\right]\right].\end{aligned}$$

In the last inequality we have used Fatou's lemma. This is valid because by subtracting a constant, we may assume $\tilde{u} \le 0$. Since $X_{\tau_m}^{\sigma_1,z} \to X_\tau^{\sigma_1,z} \in \partial D$ as $m \to +\infty$, and we know from Theorem 2.8 that $\varlimsup_{z\to\xi} \tilde{u}(z) \le \phi(\xi)$ for all ξ in ∂D, we see from (3.14) that

$$\begin{aligned}\tilde{u}(z) &\le E\left[-\int_0^{\tau} f(X_s^{\sigma_1,z})\,ds\right] + E[\phi(X_\tau^{\sigma_1,z})] \\ &= E\left[-\int_0^{\tau} f(X_s^{\sigma_1,z})\,ds + \phi(X_\tau^{\sigma_1,z})\right] \le u(z) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\tilde{u}(z) \le u(z)$. □

REMARK 3.7. In Theorem 3.6, as well as in Theorem 2.8, we required our Dirichlet data f to be in $L^\infty(D)$. This was only used to ensure that the envelopes U, v, and $\tilde{u}$ belonged to $L^\infty_{\text{loc}}(D)$. If we know a priori that $\mathcal{B}(\phi, \tilde{f})$ or $\mathcal{F}(\phi, \tilde{f})$ is nonempty, then: U, v, and $\tilde{u}$ belong to $L^\infty_{\text{loc}}(D)$; $U = v = \tilde{u} = u$ in D with $(dd^c u)^n = \tilde{f}^n\, dv$; and $\varlimsup_{z\to\xi} u(z) \le \phi(\xi)$ for each ξ in ∂D; so that the conclusions of the theorems are still valid. This fact will also be used in the proof of Theorem 3.1.

Proof of Theorem 3.1. We can write $\mathbf{U} = \bigcup \mathbf{U}_m$, where $\mathbf{U}_m \subset \mathbf{U}_{m+1}$ and each $\mathbf{U}_m$ is strictly pseudoconvex. Then

$$\begin{aligned}u(z) &= \lim_{m\to+\infty} \left[\inf_{\sigma\in H} E\left[-\int_0^{\tau_m} (f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds + (\phi\circ g_z)(X_{\tau_m}^\sigma)\right]\right] \\ &\equiv \lim_{m\to+\infty} v_m(z),\end{aligned}$$

where $\tau_m = \tau_{\partial\mathbf{U}_m}$. By Gaveau's work we have $v_m \in C(\mathbf{U}_m)$. Since $v_{m+1} \le v_m$ in D_m, u is usc in $\mathbf{U}$. We next verify the boundary regularity of u.

Fix ξ in $\partial\mathbf{U}$ and assume for simplicity that $\phi(\xi) = 0$. If we write $\tau = \tau_{\partial\mathbf{U}}$ and $\tilde{w}_\sigma(z) = w_\sigma^1(z) + w_\sigma^2(z)$, where

$$w_\sigma^1(z) = E\left[-\int_0^{\tau} (f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds\right]$$

and

$$w_\sigma^2(z) = E[(\phi\circ g_z)(X_\tau^\sigma)],$$

then we will first show that

$$\lim_{z \to \xi} w_\sigma^1(z) = 0 \quad \text{for all } \sigma \in H. \tag{3.15}$$

Equation (3.15) follows from the estimate of f in (3.4) if we prove that

$$E\left[\int_0^\tau \frac{1}{[1-|g_{z_1}(X_s^{\sigma_1})|^2]^\beta [1-|g_{z_2}(X_s^{\sigma_2})|^2]^\beta} \frac{1-|z_1|^2}{|1+\bar{z}_1 X_s^{\sigma_1}|^2} \frac{1-|z_2|^2}{|1+\bar{z}_2 X_s^{\sigma_2}|^2}\, ds\right] \tag{3.16}$$

tends to 0 as $z \to \xi$. Using the elementary identity

$$|1+\alpha\beta|^2 = |\alpha+\beta|^2 + (1-|\alpha|^2)(1-|\beta|^2), \tag{3.17}$$

which is valid for any complex numbers α and β, and using the definition of g_z in (3.8), the integrand in (3.16) becomes

$$\frac{(1-|z_1|^2)(1-|z_2|^2)}{(1-|X_s^{\sigma_1}|^2)^\beta (1-|X_s^{\sigma_2}|^2)^\beta |1+\bar{z}_1 X_s^{\sigma_1}|^{2(1-\beta)} |1+\bar{z}_2 X_s^{\sigma_2}|^{2(1-\beta)}}. \tag{3.18}$$

To estimate (3.16), we need the following lemma.

LEMMA 3.8. *For each β satisfying $\frac{1}{2} < \beta < 1$ and each (z_1, z_2) in* **U**, *the functions*

$$q_\beta(w_1, w_2) \equiv -\left[\left(\frac{1-|w_1|^2}{|1+\bar{z}_1 w_1|^2}\right)\left(\frac{1-|w_2|^2}{|1+\bar{z}_2 w_2|^2}\right)\right]^{1-\beta}$$

are plurisubharmonic in **U** *and satisfy*

$$\begin{aligned}(dd^c q_\beta)^2 \geq (1-\beta)^2(2\beta-1)|1+\bar{z}_1 w_1|^{4(\beta-1)} |1+\bar{z}_2 w_2|^{4(\beta-1)} \\ \times (1-|w_1|^2)^{-2\beta}(1-|w_2|^2)^{-2\beta}\, dV.\end{aligned} \tag{3.19}$$

For each $\sigma \in H$ and each t satisfying $0 \leq t \leq \tau = \tau_{\partial \mathbf{U}}$,

$$E[q_\beta(X_t^\sigma)] = q_\beta(0,0) + E\left[\int_0^t \Delta_{\sigma\sigma^*} q_\beta(X_s^\sigma)\, ds\right]. \tag{3.20}$$

REMARK. Note that $q_\beta(w) = -|J_w(z)|^{1-\beta}$ (cf. (3.8)). Thus q_β is really an auxillary function introduced to show how, in a vague sense, the behavior of $J_z(X_s^\sigma)$ as $z \to \xi$ compensates for the behavior of $(f \circ g_z)(X_s^\sigma)$ as $z \to \xi$.

Proof of Lemma 3.8. The inequality (3.19) follows from direct computation and use of (3.17). Formula (3.20) is a consequence of Itô's formula (3.12) applied to the function $q_\beta(X_s^\sigma) = q_\beta(X_s^{\sigma_1}, X_s^{\sigma_2})$. □

Returning to the proof of (3.16), from (3.20) and Corollary 2.3 we obtain

$$\begin{aligned}E[q_\beta(X_\tau^\sigma) - q_\beta(0,0)] &= E\left[\int_0^\tau \Delta_{\sigma\sigma^*} q_\beta(X_s^\sigma)\, ds\right] \\ &\geq 2E\left[\int_0^\tau \left[\det\left(\frac{\partial^2 q_\beta}{\partial Z_i \partial \bar{Z}_j}(X_s^\sigma)\right)\right]^{1/2} ds.\right.\end{aligned}$$

Here we are writing $q_\beta = q_\beta(Z_1, Z_2)$. By (3.19) and (3.18) the integrand in (3.14) is majorized by

$$\frac{[(1-|z_1|^2)(1-|z_2|^2)]^{1-\beta}}{(1-\beta)\sqrt{2\beta-1}}\left[\det\left(\frac{\partial^2 q_\beta}{\partial Z_i \partial \bar{Z}_j}(X_s^\sigma)\right)\right]^{1/2}. \tag{3.21}$$

Since $X_\tau^\sigma = (X_\tau^{\sigma_1}, X_\tau^{\sigma_2}) \in \partial\mathbf{U}$, either $|X_\tau^{\sigma_1}|=1$ or $|X_\tau^{\sigma_2}|=1$ (or both). Thus, for each (z_1, z_2) in $\mathbf{U}$,

$$q_\beta(X_\tau^\sigma) = -\left[\left(\frac{1-|X_\tau^{\sigma_1}|^2}{|1+\bar{z}_1 X_\tau^{\sigma_1}|^2}\right)\left(\frac{1-|X_\tau^{\sigma_2}|^2}{|1+\bar{z}_2 X_\tau^{\sigma_2}|^2}\right)\right]^{1-\beta} = 0.$$

In addition, $q_\beta(0,0)=1$, so that

$$[(1-|z_1|^2)(1-|z_2|^2)]^{1-\beta} E[q_\beta(X_\tau^\sigma) - q_\beta(0,0)] \to 0$$

as $(z_1, z_2) \to \xi$ in $\partial\mathbf{U}$. Hence

$$\frac{[(1-|z_1|^2)(1-|z_2|^2)]^{1-\beta}}{(1-\beta)\sqrt{2\beta-1}} E\left[\int_0^\tau \left[\det\left(\frac{\partial^2 q_\beta}{\partial Z_i \partial \bar{Z}_j}(X_s^\sigma)\right)\right]^{1/2} ds\right] \to 0$$

as $(z_1, z_2) \to \xi$ in $\partial\mathbf{U}$ for $\frac{1}{2} < \beta < 1$. This yields (3.16).

We now show that

$$\inf_{\sigma \in H} w_\sigma^2(z) \to 0 \quad \text{as } z \to \xi. \tag{3.22}$$

We first claim that

$$\varliminf_{z\to\xi} w_\sigma^2(z) \equiv \varliminf_{z\to\xi} E[(\phi \circ g_z)(X_\tau^\sigma)] \geq 0 \quad \text{for each } \sigma \text{ in } H. \tag{3.23}$$

To see this, fix σ in H. Since $X_\tau^\sigma \in \partial\mathbf{U}$, $g_z(X_\tau^\sigma) \in \partial\mathbf{U}$. Fix ξ in $\partial\mathbf{U}$. We may assume that $\xi = (\xi_1, \xi_2) = (e^{i\theta}, \xi_2)$ with $|\xi_2| \leq 1$. Then

$$\lim_{z\to\xi} g_z(X_\tau^\sigma) = \left(e^{i\theta}, \frac{\xi_2 + X_\tau^{\sigma_2}}{1+\bar{\xi}_2 X_\tau^{\sigma_2}}\right) = (e^{i\theta}, g_{\xi_2}(X_\tau^{\sigma_2})). \tag{3.24}$$

From the definition of X_t^σ in terms of σ and b (3.5), it follows that

$$E\left[\phi\left(e^{i\theta}, \frac{\xi_2 + X_\tau^{\sigma_2}}{1+\bar{\xi}_2 X_\tau^{\sigma_2}}\right)\right] = \int_0^1 \left[\frac{1}{2\pi}\int_0^{2\pi} \phi\left(e^{i\theta}, \frac{\xi_2 + re^{i\alpha}}{1+\bar{\xi}_2 re^{i\alpha}}\right) d\alpha\right] d\mu(r)$$

for some probability measure $d\mu = d\mu(r)$ on $[0,1]$. Thus, by subharmonicity of $\phi(e^{i\theta}, \cdot)$,

$$E\left[\phi\left(e^{i\theta}, \frac{\xi_2 + X_\tau^{\sigma_2}}{1+\bar{\xi}_2 X_\tau^{\sigma_2}}\right)\right] \geq \phi(e^{i\theta}, \xi_2). \tag{3.25}$$

From (3.24) and the continuity of ϕ on $\partial\mathbf{U}$,

$$\lim_{z\to\xi} \phi(g_z(X_\tau^\sigma)) = \phi(e^{i\theta}, g_{\xi_2}(X_\tau^{\sigma_2})).$$

Since ϕ is bounded, we can apply the bounded convergence theorem to conclude that

$$\lim_{z\to\xi} E[(\phi \circ g_z)(X_\tau^\sigma)] = E[\phi(e^{i\theta}, g_{\xi_2}(X_\tau^{\sigma_2}))]. \tag{3.26}$$

Thus

$$
\begin{aligned}
&E[(\phi\circ g_z)(X_\tau^\sigma)-\phi(\xi)] \\
&\quad = E[(\phi\circ g_z)(X_\tau^\sigma)-\phi(e^{i\theta},\xi_2)] \\
&\quad = E[(\phi\circ g_z)(X_\tau^\sigma)-\phi(e^{i\theta},g_{\xi_2}(X_\tau^{\sigma_2}))]+E[\phi(e^{i\theta},g_{\xi_2}(X_\tau^{\sigma_2}))-\phi(e^{i\theta},\xi_2)]\geq 0
\end{aligned}
$$

by (3.25) and (3.26). This gives (3.23). To complete the proof of boundary regularity it suffices to show that for each fixed ξ in $\partial\mathbf{U}$ and each $\epsilon>0$, there exists a σ in H so that

$$\overline{\lim_{z\to\xi}}\, E[(\phi\circ g_z)(X_\tau^\sigma)]<\epsilon. \tag{3.27}$$

We see that we need to construct σ so that the inequality (3.25) is very nearly an equality. Equivalently, we must find a σ in H so that $d\mu(r)$ approximates a unit mass at $r=0$. Thus we require that $E[|X_\tau^{\sigma_2}|]$ should be small. If we define σ^δ by specifying the matrix entries

$$\sigma_{11}^\delta=1/\delta,\quad \sigma_{21}^\delta=\sigma_{12}^\delta=0,\quad \text{and}\quad \sigma_{22}^\delta=\delta,$$

then it can be shown that the corresponding measures μ^δ converge to the unit mass at $r=0$. Given $\epsilon>0$, we can then choose $\delta>0$ sufficiently small so that (3.27) holds for $\sigma=\sigma^\delta$.

We can now apply the argument following Bellman's principle (Lemma 3.3) to conclude from Remark 3.4 that $u\in P(\mathbf{U})$ and $(dd^cu)^2\geq\tilde f^2\,dV$ in $\mathbf{U}$.

Next, if we recall the proof of the statement

$$\lim_{z\to\xi} w_\sigma^1(z)=0\quad \text{for all } \sigma \text{ in } H, \tag{3.15}$$

we see that we actually proved that this limit is uniform in z and σ. Precisely, given $\epsilon>0$, there exists an m_0 such that for each $m>m_0$, $|w_\sigma^1(z)|<\epsilon$ for z in $\mathbf{U}-\mathbf{U}_m$. Thus

$$
\begin{aligned}
&\inf_{\sigma\in H} w_\sigma^1\equiv u_1\in L^\infty_{\mathrm{loc}}(\mathbf{U}) \quad \text{and}\\
&\inf_{\sigma\in H} w_\sigma^2\equiv u_2\in L^\infty_{\mathrm{loc}}(\mathbf{U}) \quad \text{since } \phi\in C(\partial\mathbf{U}).
\end{aligned}
$$

Therefore $u\in L^\infty_{\mathrm{loc}}(\mathbf{U})$. We conclude that $u\in\mathfrak{F}(\phi,\tilde f)$. In particular, $\mathfrak{F}(\phi,\tilde f)\neq\emptyset$. From Remark 3.7, $U=v=\tilde u=u$ in $\mathbf{U}$ and $(dd^cu)^2=\tilde f^2\,dV$ in $\mathbf{U}$. Thus u satisfies (3.2).

It remains to prove that $u\in C(\bar{\mathbf{U}})$. The uniqueness of the solution u will then follow from the comparison theorems of Bedford and Taylor in [BT1] and [BT2]. To verify continuity of u on $\bar{\mathbf{U}}$ we use the facts (already proved) that u satisfies

(1) $u\in P(\mathbf{U})\cap L^\infty_{\mathrm{loc}}(\mathbf{U})$;
(2) $(dd^cu)^2=\tilde f^2\,dV$ in $\mathbf{U}$; and
(3) $u=\phi$ on $\partial\mathbf{U}$.

This shows that u solves the Dirichlet problem (3.2) with the continuous boundary values ϕ; thus the proof below that $u\in C(\mathbf{U})$ is a J. B. Walsh-type theorem for the bidisc $\mathbf{U}$. It is a version of Theorem 6.2 of [BT1] showing that solvability of (3.2) plus *boundary* regularity of the solution yields *inner* regularity of the solution.

We continue to write $\mathbf{U}=\bigcup \mathbf{U}_m$, with $\mathbf{U}_m \subset \mathbf{U}_{m+1}$ and each $\mathbf{U}_m$ being a strictly pseudoconvex domain. Recall that $u(z)=\inf\{\tilde{w}_\sigma(z): \sigma \in H\}$. We fix m and fix σ in H and write

$$\tilde{w}_\sigma(z)=E\left[-\int_0^{\tau_m}(f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds+(\phi\circ g_z)(X_\tau^\sigma)\right]$$
$$+E\left[-\int_{\tau_m}^{\tau}(f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds\right]. \tag{3.28}$$

We first show that

$$\inf_{\sigma\in H} E\left[-\int_{\tau_m}^{\tau}(f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds\right]\to 0 \tag{3.29}$$

as $m\to+\infty$ locally uniformly for z in $\mathbf{U}$. Let $z\in\mathbf{U}_m$, and define

$$w_\sigma^m(z)=E\left[-\int_{\tau_m}^{\tau}(f\circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\,ds\right].$$

Then, by renormalizing, we can find a σ' in H such that

$$w_\sigma^m(z)=E\left[-\int_{\tau_m}^{\tau} f(X_s^{\sigma',z})\,ds\right] \tag{3.30}$$

(cf. (3.6) and (3.8)). Note that the process $X_s^{\sigma',z}$ starts at z. In other words, by setting $s=0$ we obtain $X_0^{\sigma',z}=z$. Taking conditional expectations in (3.30) and using the strong Markov property for $X_s^{\sigma',z}$ (cf. [G2]), we obtain

$$w_\sigma^m(z)=E\left[-\int_0^{\tau} f(X_s^{\sigma',z'})\,ds\right]\equiv w_{\sigma'}^m(z')$$

for some z' in $\partial\mathbf{U}_m$. Again, after renormalizing,

$$w_\sigma^m(z)=w_{\sigma'}^m(z')=E\left[-\int_0^{\tau} f(X_s^{\sigma',z'})\,ds\right]$$
$$=E\left[-\int_0^{\tau}(f\circ g_{z'})(X_s^{\sigma''})|J_{z'}(X_s^{\sigma''})|\,ds\right] \tag{3.31}$$

for some σ'' in H. But

$$w(z')\equiv \inf_{\sigma''\in H}\left\{E\left[-\int_0^{\tau}(f\circ g_{z'})(X_s^{\sigma''})|J_{z'}(X_s^{\sigma''})|\,ds\right]\right\}$$

satisfies $(dd^c w)^n=f^n\,dV$ in $\mathbf{U}$ and $\lim_{z'\to\xi} w(z')=0$ for each ξ in $\partial\mathbf{U}$. To be precise, we have shown in the proof of (3.15) that given $\epsilon>0$, there exists an m_0 such that for each $m'>m_0$, $|w(z')|<\epsilon$ for all z' in $\mathbf{U}-\mathbf{U}_{m'}$. Hence $\sup_{z'\in\partial\mathbf{U}_m} w(z')\to 0$ as $m\to\infty$. From (3.31) we conclude that

$$\inf_{\sigma\in H} w_\sigma^m(z)\le w(z')\le \sup_{z'\in\partial\mathbf{U}_m} w(z') \quad \text{for all } z \text{ in } \mathbf{U}_m.$$

Thus $\inf_{\sigma\in H} w_\sigma^m(z)\to 0$ as $m\to+\infty$ locally uniformly in $\mathbf{U}$. This is (3.29).

We now show that there exists a constant $M < +\infty$ such that

$$E[\tau_m(\sigma)] < M \quad \text{for all } \sigma \text{ in } H. \tag{3.32}$$

To prove (3.32), it suffices to prove that

$$E[\tau_m(\sigma, z)] < M \quad \text{for all } \sigma \text{ in } H, \tag{3.33}$$

where $X_t^{\sigma,z} = z + \int_0^t \sigma\, db$ (see (3.5)). By renormalization, for each σ in H there exists a $\hat{\sigma}$ in H with $\tilde{w}_\sigma = w_{\hat{\sigma}}$ (cf. (3.6) and (3.8)). Fix σ in H and let $a = \sigma\sigma^*$. Then $\det a \geq 1$. By the arithmetic-geometric mean inequality, $\operatorname{tr}(a) \equiv \sum a_{ii} \geq 2$. Let $g(z) = |z|^2$ and apply Itô's formula (3.12) with g and τ_m to obtain

$$\begin{aligned} 2 > |g(z) - g(X_{\tau_m}^{\sigma,z})| &\geq \left| E \int_0^{\tau_m} \sum a_{ij} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j}(X_s^{\sigma,z})\, ds \right| \\ &\geq 2E[\tau_m(\sigma, z)] \end{aligned}$$

for any z in $\mathbf{U}$. This gives (3.33) with $M = 1$.

Fix z_0 in $\mathbf{U}$ and fix a neighborhood $V \subset \mathbf{U}$ of z_0. Given $\epsilon > 0$, by (3.29) we can find an m large so that

$$\left| \inf_{\sigma \in H} E\left[-\int_{\tau_m}^{\tau} (f \circ g_z)(X_s^\sigma) |J_z(X_s^\sigma)|\, ds \right] \right| < \epsilon \quad \text{for all } z \text{ in } \bar{V}. \tag{3.34}$$

By the uniform continuity of ϕ on $\partial\mathbf{U}$ and the continuity of g_z in z, there exists a $\delta > 0$ such that

$$|\phi \circ g_{z_0}(\xi) - \phi \circ g_z(\xi)| < \epsilon \text{ for all } \xi \text{ in } \partial\mathbf{U} \quad \text{if } |z - z_0| < \delta. \tag{3.35}$$

We next choose a (perhaps) smaller neighborhood V' of z_0 with $V' \subset V \cap \{z : |z - z_0| < \delta\}$ such that

$$|(f \circ g_z)(\eta)|J_z(\eta)| - (f \circ g_{z_0})(\eta)|J_{z_0}(\eta)|| < \epsilon \tag{3.36}$$

if $\eta \in \overline{\bigcup_{z \in V'} g_z(D_m)}$. For this m, if $z \in V'$,

$$\begin{aligned} &\left| \inf_{\sigma \in H} E\left[-\int_0^{\tau_m} (f \circ g_z)(X_s^\sigma)|J_z(X_s^\sigma)|\, ds + (\phi \circ g_z)(X_\tau^\sigma) \right] \right. \\ &\qquad \left. - \inf_{\sigma \in H} E\left[-\int_0^{\tau_m} (f \circ g_{z_0})(X_s^\sigma)|J_{z_0}(X_s^\sigma)|\, ds + (\phi \circ g_{z_0})(X_\tau^\sigma) \right] \right| \\ &\leq \sup_{\sigma \in H} E\left[\int_0^{\tau_m} \left| f \circ g_z(X_s^\sigma)|J_z(X_s^\sigma)| - f \circ g_{z_0}(X_s^\sigma)|J_{z_0}(X_s^\sigma)| \right| ds \right. \\ &\qquad \left. + |(\phi \circ g_z)(X_\tau^\sigma) - (\phi \circ g_{z_0})(X_\tau^\sigma)| \right] \\ &\leq \epsilon M + \epsilon \end{aligned}$$

from (3.35) and (3.36). Combined with (3.34), we have shown using (3.28) that given $\epsilon > 0$, there exists a neighborhood V' of z_0 such that

$$|\inf_{\sigma \in H} \tilde{w}_\sigma(z) - \inf_{\sigma \in H} \tilde{w}_\sigma(z_0)| < (M + 2)\epsilon \quad \text{for all } z \text{ in } V'.$$

Thus $u(z) = \inf_{\sigma \in H} \tilde{w}_\sigma(z)$ is continuous at z_0. □

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