Generalized Feynman Integrals via Conditional Feynman Integrals

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1. Introduction

In various Feynman integration theories, the integrand F of the Feynman integral is a functional of the standard Wiener (i.e., Brownian) process. In this paper we first define a Feynman integral for functionals of more general stochastic processes. Then, for various classes of functionals, we express this generalized Feynman integral as an integral operator whose kernel involves the conditional Feynman integral.

In defining various analytic Feynman integrals of F, one usually starts, for $\lambda > 0$, with the Wiener integral

$$\int_{C_0[0,T]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(T)+\xi)m(dx),$$

and then extends analytically in λ to the right-half complex plane. In this paper, our starting point is the Wiener integral

$$\int_{C_0[0,T]} F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)m(dx), \tag{1.1}$$

where Z is the Gaussian process

$$Z(x,t) = \int_0^t h(s)\,dx(s) \tag{1.2}$$

with h in $L_2[0,T]$, and where $\int_0^T h(s)\,dx(s)$ denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral.

A very important class of functionals in quantum mechanics are functionals on Wiener space $C_0[0,T]$ of the form

$$F(x) = \exp\left\{\int_0^T \theta(s,x(s))\,ds\right\} \quad \text{and} \quad G(x) = F(x)\psi(x(T)) \tag{1.3}$$

for appropriate functions θ and ψ. Functionals like these have appeared in many papers, including [1–4; 7; 8; 14; 15; 19], involving various Feynman integration theories. In particular, Cameron and Storvick [8] obtain a formula

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for the analytic but scalar-valued Feynman integral of various functionals of the form (1.3). It is clear (simply let $h(t)\equiv 1$ on $[0,T]$) that functionals of the form (1.3) are contained in the class of functionals of the form

$$H(x)=\exp\left\{\int_0^T \theta(t,Z(x,t))\,dt\right\} \quad\text{and}\quad G(x)=H(x)\psi(Z(x,T)). \tag{1.4}$$

In this paper we use the theory of the conditional Feynman integral introduced in [11; 12] to obtain formulas for the analytic operator-valued Feynman integral of various classes of functionals, including those of the form (1.4). Then as special cases we obtain formulas for the analytic but scalar-valued Feynman integral of various functionals.

REMARK. It is not hard to verify that the results of this paper can be extended to ν-dimensional Wiener space $C_0^\nu[0,T]$ for $\nu=2,3,\dots$. We decided to work with $\nu=1$ for notational simplicity. Papers [5–9; 11; 12; 15; 18; 19; 24] involve general ν, while papers [1–4; 10; 14; 16; 17; 25] involve $\nu=1$.

2. Definitions and Preliminaries

Let $C[0,T]$ denote the space of $\mathbf{R}$-valued continuous functions on $[0,T]$, and let $C_0[0,T]$ denote Wiener space—that is, the space of all functions $x(t)$ in $C[0,T]$ with $x(0)=0$. We denote the Wiener integral of a Wiener measurable function F by

$$E[F]=\int_{C_0[0,T]} F(x)m(dx)$$

whenever the integral exists.

Let h be an element of $L_2[0,T]$ with $\|h\|>0$, let $Z(x,t)$ be given by (1.2), and let

$$a(t)=\int_0^t h^2(s)\,ds. \tag{2.1}$$

Then Z is a Gaussian process with mean zero and covariance

$$E[Z(x,s)Z(x,t)]=a(\min\{s,t\}).$$

In addition, by [23, Thm. 21.1], $Z(\cdot,t)$ is stochastically continuous in t on $[0,T]$.

Next we state the definitions of the conditional Wiener integral [22; 25] and the conditional Feynman integral [11; 12].

DEFINITION 1. Let X be an $\mathbf{R}$-valued Wiener measurable function on $C_0[0,T]$ whose probability distribution function P_X is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}$. Let F be a $\mathbf{C}$-valued Wiener integrable function on $C_0[0,T]$. Then the conditional Wiener integral of F given X, denoted by $E(F\mid X)(\eta)$, is a Lebesgue measurable function of η, unique up to null sets in $\mathbf{R}$, satisfying the equation

$$\int_{X^{-1}(B)} F(x)m(dx) = \int_B E(F|X)(\eta)P_X(d\eta)$$

for all Borel sets B in $\mathbf{R}$.

DEFINITION 2. Let $\mathbf{C}$, $\mathbf{C}_+$ and $\mathbf{C}_+^{\sim}$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let $F: C[0,T] \to \mathbf{C}$ be such that for each $\lambda > 0$,

$$\int_{C_0[0,T]} |F(\lambda^{-1/2}Z(x,\cdot)+\xi)|m(dx) < \infty$$

for a.e. $\xi \in \mathbf{R}$. Let $X: C[0,T] \to \mathbf{R}$ be such that for each $\lambda > 0$ and a.e. $\xi \in \mathbf{R}$, $X(\lambda^{-1/2}x+\xi)$ is a Wiener measurable function of x on $C_0[0,T]$; that is, for a.e. $\xi \in \mathbf{R}$, $Y(x) = X(\lambda^{-1/2}x+\xi)$ is scale-invariant measurable on $C_0[0,T]$. For $\lambda > 0$ and $\xi \in \mathbf{R}$, let

$$J_\lambda(\xi,\eta) = E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\,|\,X(\lambda^{-1/2}x+\xi))(\eta)$$

denote the conditional Wiener integral of

$$F(\lambda^{-1/2}Z(x,\cdot)+\xi) \quad \text{given} \quad X(\lambda^{-1/2}x+\xi).$$

If for a.e. $\eta \in \mathbf{R}$ there exists a function $J_\lambda^*(\xi,\eta)$, analytic in λ on $\mathbf{C}_+$ such that $J_\lambda^*(\xi,\eta) = J_\lambda(\xi,\eta)$ for all $\lambda > 0$, then $J_\lambda^*(\xi,\cdot)$ is defined to be the conditional analytic Wiener integral of F given X with parameter λ and we write

$$E^{\mathrm{anw}_\lambda}(F|X)(\xi,\eta) = J_\lambda^*(\xi,\eta).$$

If for fixed real $q \neq 0$ the limit

$$\lim_{\lambda \to -iq} E^{\mathrm{anw}_\lambda}(F|X)(\xi,\eta)$$

exists for a.e. $\eta \in \mathbf{R}$, where $\lambda \to -iq$ through $\mathbf{C}_+$, we will denote the value of this limit by $E^{\mathrm{anf}_q}(F|X)(\xi,\cdot)$ and call it the conditional analytic Feynman integral of F given X with parameter q.

Next let $M(\mathbf{R})$ denote the space of all $\mathbf{C}$-valued countably additive Borel measures on $\mathbf{R}$, and let

$$K(\mathbf{R}) = \{\psi_1+\psi_2 : \psi_1 \in L_1(\mathbf{R}) \text{ and } \psi_2 \in \hat{M}(\mathbf{R})\}, \tag{2.3}$$

where $\hat{M}(\mathbf{R})$ is the space of Fourier transforms of measures from $M(\mathbf{R})$.

Using (1.1) as our starting point, we now state the definition of the (generalized) analytic operator-valued Feynman integral as an element of $\mathcal{L}(K(\mathbf{R}), L_\infty(\mathbf{R}))$.

DEFINITION 3. Let h be an element of $L_2[0,T]$ with $\|h\| > 0$, and let $Z(x,t)$ be given by (1.2). For each $\lambda > 0$, $\psi \in K(\mathbf{R})$, and $\xi \in \mathbf{R}$, assume that $F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)$ is Wiener integrable with respect to x on $C_0[0,T]$, and let

$$(I_\lambda(F)\psi)(\xi)=\int_{C_0[0,T]} F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)m(dx). \tag{2.4}$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbf{R})$ as a function of ξ and if the correspondence $\psi\to I_\lambda(F)\psi$ gives an element of $\mathcal{L}(K(\mathbf{R}),L_\infty(\mathbf{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose there exists an $\mathcal{L}$-valued function which is analytic in λ on $\mathbf{C}_+$ and agrees with $I_\lambda(F)$ on $(0,+\infty)$; then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{\text{an}}(F)$ and is called the analytic operator-valued Wiener integral of F associated with λ. For $\lambda=-iq\in\tilde{\mathbf{C}}_+$, suppose there exists an operator $J_q^{\text{an}}(F)$ in $\mathcal{L}(K(\mathbf{R}),L_\infty(\mathbf{R}))$ such that for every ψ in $K(\mathbf{R})$,

$$\|J_q^{\text{an}}(F)\psi-I_\lambda^{\text{an}}(F)\psi\|_\infty\to 0 \tag{2.5}$$

as $\lambda\to -iq$ through $\mathbf{C}_+$; then $J_q^{\text{an}}(F)$ is called the (generalized) analytic operator-valued Feynman integral of F with parameter q.

REMARK. Note that if $h(t)\equiv 1$ on $[0,T]$ then this definition agrees with previous definitions of the analytic operator-valued Feynman integral [1-4; 14; 15].

The following well-known integration formula,

$$\int_{-\infty}^{\infty}\exp\left\{\frac{-b\eta^2}{2}+i\eta\xi\right\}d\eta=\left(\frac{2\pi}{b}\right)^{1/2}\exp\left\{-\frac{\xi^2}{2b}\right\},\quad \operatorname{Re} b>0, \tag{2.6}$$

and a formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals [22],

$$\begin{aligned}E(F(Z(x,\cdot)+\xi)\,|\,Z(x,T)+\xi=\eta)\\ =E\left[F\left(Z(x,\cdot)+\xi-\frac{a(\cdot)}{a(T)}Z(x,T)+\frac{a(\cdot)}{a(T)}(\eta-\xi)\right)\right],\end{aligned} \tag{2.7}$$

are used several times in this paper.

3. Formulas for $J_q^{\text{an}}(F)$ for F in S

The Banach algebra S of functionals on $C_0[0,T]$, each of which is a type of stochastic Fourier transform of a bounded $\mathbf{C}$-valued Borel measure, was introduced in [5] by Cameron and Storvick. Further work on S shows that it contains many classes of functionals of interest in Feynman integration theory [6; 9; 17-20]. In this section we show that $J_q^{\text{an}}(F)$ is in $\mathcal{L}(K(\mathbf{R}),L_\infty(\mathbf{R}))$ for each F in S.

The Banach algebra S consists of functions on $C_0[0,T]$ expressible in the form

$$F(x)=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(s)\,dx(s)\right\}d\sigma(v) \tag{3.1}$$

for s a.e. $x\in C_0[0,T]$ (i.e., except on a scale-invariant null set), where $\sigma\in M(L_2[0,T])$, the space of $\mathbf{C}$-valued, countably additive Borel measures on $L_2[0,T]$.

The following lemma, which follows quite easily from the definition of the PWZ stochastic integral, plays a key role in the proof of Theorem 1.

LEMMA 1. *For each $v \in L_2[0,T]$ and each $h \in L_\infty[0,T]$,*

$$\int_0^T v(s)\,dZ(x,s) \equiv \int_0^T v(s)\,d\left[\int_0^s h(u)\,dx(u)\right] = \int_0^T v(s)h(s)\,dx(s) \tag{3.2}$$

for s a.e. $x \in C_0[0,T]$.

LEMMA 2. *Let $F \in S$ be given by* (3.1) *and let h be in $L_\infty[0,T]$ with $\|h\| > 0$. Then $G: C_0[0,T] \to \mathbf{C}$ given by*

$$G(x) = F(Z(x,\cdot))$$

belongs to the Banach algebra S.

Proof. Let $\Phi: L_2[0,T] \to L_2[0,T]$ be given by $\Phi(v)(s) = v(s)h(s)$. Φ is easily seen to be continuous and so is Borel measurable. Hence $\mu \equiv \sigma \circ \Phi^{-1}$ is in $M(L_2[0,T])$. In addition, for each $\rho > 0$, using the change-of-variables theorem [13, p. 163] and Lemma 1, we see that the following are equal for a.e. x in $C_0[0,T]$:

$$\begin{aligned}
&\int_{L_2[0,T]} \exp\left\{i\rho \int_0^T u(s)\,dx(s)\right\} d\mu(u) \\
&\quad = \int_{L_2[0,T]} \exp\left\{i\rho \int_0^T u(s)\,dx(s)\right\} d(\sigma \circ \Phi^{-1}(u)) \\
&\quad = \int_{L_2[0,T]} \exp\left\{i\rho \int_0^T \Phi(v)(s)\,dx(s)\right\} d\sigma(v) \\
&\quad = \int_{L_2[0,T]} \exp\left\{i\rho \int_0^T v(s)h(s)\,dx(s)\right\} d\sigma(v) \\
&\quad = \int_{L_2[0,T]} \exp\left\{i\rho \int_0^T v(s)\,d\left[\int_0^s h(\tau)\,dx(\tau)\right]\right\} d\sigma(v) \\
&\quad = \int_{L_2[0,T]} \exp\left\{i\rho \int_0^T v(s)\,d[Z(x,s)]\right\} d\sigma(v) \\
&\quad = F(\rho Z(x,\cdot)) = G(\rho x).
\end{aligned}$$

□

In Theorem 1, we condition by the function $X: C[0,T] \to \mathbf{R}$ given by

$$X(y) = \int_0^T h(s)\,dy(s) + y(0). \tag{3.3}$$

Thus, for all $(\lambda, \xi) \in (0,\infty) \times \mathbf{R}$, we have that

$$\begin{aligned}
X(\lambda^{-1/2}x + \xi) &= \lambda^{-1/2} \int_0^T h(s)\,dx(s) + \xi \\
&= \lambda^{-1/2} Z(x,T) + \xi
\end{aligned} \tag{3.4}$$

for a.e. $x \in C_0[0,T]$.

THEOREM 1. *Let $F\in S$ be given by* (3.1), *let $h\in L_\infty[0,T]$, and let X be given by* (3.3). *Then, for all $(\xi,\eta)\in\mathbf{R}\times\mathbf{R}$, we have*

$$E^{\mathrm{anw}_\lambda}(F\,|\,X)(\xi,\eta)$$
$$=\int_{L_2[0,T]}\exp\left\{\frac{i(\eta-\xi)(v,h^2)}{a(T)}-\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\}d\sigma(v) \tag{3.5}$$

for all $\lambda\in\mathbf{C}_+$, and

$$E^{\mathrm{anf}_q}(F\,|\,X)(\xi,\eta)$$
$$=\int_{L_2[0,T]}\exp\left\{\frac{i(\eta-\xi)(v,h^2)}{a(T)}-\frac{i}{2q}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\}d\sigma(v) \tag{3.6}$$

for all real $q\neq 0$, where $(v,h^2)=\int_0^T v(s)h^2(s)\,ds$.

Proof. In view of Lemma 1 and the definition of S, we have that

$$F(\lambda^{-1/2}Z(x,\cdot)+\xi)=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(s)\,d[\lambda^{-1/2}Z(x,s)+\xi]\right\}d\sigma(v)$$

for s a.e. $x\in C_0[0,T]$. Next—using the Fubini theorem, (2.5), and (3.2)—we obtain, for all $(\lambda,\xi,\eta)\in(0,\infty)\times\mathbf{R}\times\mathbf{R}$, the formula

$$E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\,|\,X(\lambda^{-1/2}x+\xi))(\eta)$$
$$=E\left(\int_{L_2[0,T]}\exp\left\{i\int_0^T v(s)\,d[\lambda^{-1/2}Z(x,s)+\xi]\right\}d\sigma(v)\,\Big|\,\lambda^{-1/2}Z(x,T)=\eta-\xi\right)$$
$$=\int_{L_2[0,T]}E\left[\exp\left\{i\int_0^T v(s)\,d\left[\lambda^{-1/2}\left(Z(x,s)-\frac{a(s)}{a(T)}Z(x,T)\right)+\frac{a(s)}{a(T)}(\eta-\xi)\right]\right\}\right]d\sigma(v)$$
$$=\int_{L_2[0,T]}E\left[\exp\left\{i\lambda^{-1/2}\int_0^T vh\,dx-\frac{i\lambda^{-1/2}}{a(T)}\int_0^T h\,dx(v,h^2)+\frac{i(\eta-\xi)}{a(T)}(v,h^2)\right\}\right]d\sigma(v)$$
$$=\int_{L_2[0,T]}\exp\left\{\frac{i(\eta-\xi)(v,h^2)}{a(T)}\right\}E\left[\exp\left\{i\lambda^{-1/2}\int_0^T\left[vh-\frac{(v,h^2)h}{a(T)}\right]dx\right\}\right]d\sigma(v)$$
$$=\int_{L_2[0,T]}\exp\left\{\frac{i(\eta-\xi)(v,h^2)}{a(T)}-\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\}d\sigma(v).$$

But the last expression above is an analytic function of λ throughout $\mathbf{C}_+$, and is a continuous function of λ on $\tilde{\mathbf{C}}_+$ since σ is a finite Borel measure. Thus equations (3.5) and (3.6) are established. □

In our next theorem we obtain a formula for $J_q^{\mathrm{an}}(F)\psi_1$ for $F\in S$ and ψ_1 in $L_1(\mathbf{R})$. Note that by choosing $\psi_2\equiv 0$ we can also think of $\psi_1=\psi_1+\psi_2$ as an element of $K(\mathbf{R})$.

THEOREM 2. *Let F, h, and X be as in Theorem 1. Then, for all real $q \neq 0$ and all $\psi_1 \in L_1(\mathbf{R})$, $J_q^{\text{an}}(F)\psi_1$ is in $L_\infty(\mathbf{R})$ and for all $\xi \in \mathbf{R}$ is given by*

$$(J_q^{\text{an}}(F)\psi_1)(\xi) = \int_{-\infty}^{\infty} E^{\text{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2} \exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}\psi_1(\eta)\,d\eta \tag{3.7}$$

with $E^{\text{anf}_q}(F|X)(\xi,\eta)$ given by (3.6).

Proof. For each $(\lambda,\xi) \in (0,\infty)\times\mathbf{R}$ we have that

$$\begin{aligned}
&(I_\lambda(F)\psi_1)(\xi)\\
&\quad = E[F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi_1(\lambda^{-1/2}Z(x,T)+\xi)]\\
&\quad = \int_{-\infty}^{\infty} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi_1(\lambda^{-1/2}Z(x,T)+\xi)\,|\,X(\lambda^{-1/2}x+\xi))(\eta)\\
&\qquad\qquad \cdot\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\pi a(T)}(\eta-\xi)^2\right\}d\eta\\
&\quad = \int_{-\infty}^{\infty} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi_1(\lambda^{-1/2}Z(x,T)+\xi)\,|\,\lambda^{-1/2}Z(x,T)+\xi=\eta)\\
&\qquad\qquad \cdot\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\pi a(T)}(\eta-\xi)^2\right\}d\eta\\
&\quad = \int_{-\infty}^{\infty} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\,|\,\lambda^{-1/2}Z(x,T)+\xi=\eta)\\
&\qquad\qquad \cdot\psi_1(\eta)\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\}d\eta.
\end{aligned}$$

But, by Theorem 1 and Morera's theorem, we see that the above expression is analytic in λ throughout $\mathbf{C}_+$. It is also continuous in λ on $\tilde{\mathbf{C}}_+$ since $E^{\text{anw}_\lambda}(F|X)(\eta)$ is bounded and ψ_1 is in $L_1(\mathbf{R})$. Thus $J_q^{\text{an}}(F)\psi_1$ is in $L_\infty(\mathbf{R})$ and is given by (3.7). □

In Theorem 3 we need a summation procedure since ψ_2 need not be in $L_1(\mathbf{R})$.

DEFINITION 4. Let

$$\int_{\mathbf{R}}^{-} F(\eta)\,d\eta \equiv \lim_{A\to+\infty}\int_{\mathbf{R}} f(\eta)\exp\left\{-\frac{\eta^2}{2A}\right\}d\eta \tag{3.8}$$

whenever the expression on the right-hand side exists.

THEOREM 3. *Let F, h, and X be as in Theorem 1. Let $\psi_2 \in \hat{M}(\mathbf{R})$ be given by*

$$\psi_2(\eta) = \int_{-\infty}^{\infty} \exp\{iu\eta\}\,d\phi(\mu) \tag{3.9}$$

with $\phi \in M(\mathbf{R})$. Then, for all real $q \neq 0$, $J_q^{\text{an}}(F)\psi_2$ is in $L_\infty(\mathbf{R})$ and for $\xi \in \mathbf{R}$ is given by

$$\begin{aligned}
&(J_q^{\text{an}}(F)\psi_2)(\xi)\\
&\quad = \int_{\mathbf{R}}^{-} E^{\text{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}\psi_2(\eta)\,d\eta,
\end{aligned} \tag{3.10}$$

with $E^{\mathrm{anf}_q}(F|X)(\xi,\eta)$ *given by* (3.6). *In addition, we have the alternative expression*

$$(J_q^{\mathrm{an}}(F)\psi_2)(\xi) = \int_{L_2[0,T]} \exp\left\{-\frac{i}{2q}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\} \cdot\int_{-\infty}^{\infty} \exp\left\{iu\xi-\frac{i}{2qa(T)}[(v,h^2)+ua(T)]^2\right\} d\phi(u)\,d\sigma(v). \quad (3.11)$$

Proof. We will first obtain the alternative expression (3.11). Proceeding as in the proof of Theorem 2, and then using (3.5) and (3.9), we obtain that for all $(\lambda,\xi)\in(0,\infty)\times\mathbf{R}$,

$$\begin{aligned}(I_\lambda(F)\psi_2)(\xi) &= \int_{-\infty}^{\infty} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\,|\,\lambda^{-1/2}Z(x,T)+\xi=\eta) \cdot\psi_2(\eta)\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta \\ &= \int_{-\infty}^{\infty}\int_{L_2[0,T]} \exp\left\{\frac{i(\eta-\xi)(v,h^2)}{a(T)} -\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\} d\sigma(v) \\ &\quad\cdot\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\}\int_{-\infty}^{\infty}\exp\{iu\eta\}\,d\phi(u)\,d\eta.\end{aligned}$$

We next use the Fubini theorem and then carry out the integration with respect to η, using (2.6), to obtain

$$(I_\lambda(F)\psi_2)(\xi) = \int_{L_2[0,T]} \exp\left\{-\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\} \cdot\int_{-\infty}^{\infty}\exp\left\{iu\xi-\frac{1}{2\lambda a(T)}[(v,h^2)+ua(T)]^2\right\} d\phi(u)\,d\sigma(v).$$

But clearly the right-hand side of the above expression is analytic in λ on $\mathbf{C}_+$ and continuous for λ in $\tilde{\mathbf{C}}_+$, which establishes (3.11).

Equation (3.10) follows from the calculations below and equation (3.11). Note that in the third equality we first use the Fubini theorem and then carry out the integration with respect to η using (2.6):

$$\begin{aligned}&\int_{\mathbf{R}}^{-} E^{\mathrm{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}\psi_2(\eta)\,d\eta \\ &= \lim_{A\to+\infty}\int_{-\infty}^{\infty} E^{\mathrm{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}-\frac{\eta^2}{2A}\right\}\psi_2(\eta)\,d\eta \\ &= \lim_{A\to+\infty}\int_{-\infty}^{\infty}\int_{L_2[0,T]}\exp\left\{\frac{i(v,h^2)(\eta-\xi)}{a(T)}\right.\end{aligned}$$

$$-\frac{i}{2q}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\Bigg\} d\sigma(v)$$

$$\cdot\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}-\frac{\eta^2}{2A}\right\}\int_{-\infty}^{\infty}\exp\{iu\eta\}\,d\phi(u)\,d\eta$$

$$=\lim_{A\to+\infty}\int_{L_2[0,T]}\exp\left\{-\frac{i}{2q}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\}$$

$$\cdot\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\left(\frac{2\pi A a(T)}{a(T)-Aiq}\right)^{1/2}$$

$$\cdot\int_{-\infty}^{\infty} exp\left\{iu\xi-\frac{\xi^2}{2A}-\frac{Aa(T)}{2[a(T)-iqA]}\left[u+\frac{(v,h^2)}{a(T)}+\frac{\xi i}{A}\right]^2\right\} d\phi(u)\,d\sigma(v)$$

$$=\int_{L_2[0,T]}\exp\left\{-\frac{i}{2q}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\right\}$$

$$\cdot\int_{-\infty}^{\infty}\exp\left\{iu\xi-\frac{i}{2qa(T)}[(v,h^2)+ua(T)]^2\right\} d\phi(u)\,d\sigma(v)$$

$$=(J_q^{\mathrm{an}}(F)\psi_2)(\xi).$$ □

The following theorem is an immediate consequence of Theorems 2 and 3.

THEOREM 4. *Let F, h, and X be as in Theorem 1. Then, for all real $q\neq 0$, $J_q^{\mathrm{an}}(F)$ exists as an element of $\mathcal{L}(K(\mathbf{R}),L_\infty(\mathbf{R}))$ and, for each $\psi=\psi_1+\psi_2$ in $K(\mathbf{R})$,*

$$(J_q^{\mathrm{an}}(F)\psi)(\xi)=(J_q^{\mathrm{an}}(F)\psi_1)(\xi)+(J_q^{\mathrm{an}}(F)\psi_2)(\xi) \tag{3.12}$$

for all $\xi\in\mathbf{R}$, where $J_q^{\mathrm{an}}(F)\psi_1$ is given by (3.7) *and $(J_q^{\mathrm{an}}(F)\psi_2$ is given by either* (3.10) *or* (3.11).

The following corollary follows easily from either equation (3.10) or equation (3.11).

COROLLARY 1. *Let F and h be as in Theorem 1 and let $\psi=\psi_1+\psi_2$ with $\psi_1\equiv 0$ and $\psi_2\equiv 1$. Then for all real $q\neq 0$ and all $\xi\in\mathbf{R}$,*

$$(J_q^{\mathrm{an}}(F)\psi)(\xi)=\int_{L_2[0,T]}\exp\left\{-\frac{i}{2q}\|hv\|^2\right\}d\sigma(v).$$

THEOREM 5. *Let $\theta(\cdot,\cdot)$: $[0,T]\times\mathbf{R}\to\mathbf{C}$ be given by*

$$\theta(t,u)=\int_{-\infty}^{\infty}\exp\{iu\eta\}\,d\sigma_t(\eta), \tag{3.13}$$

where $\{\sigma_t: 0\le t\le T\}$ is a family from $M(\mathbf{R})$ with $\|\sigma_t\|\in L_1[0,T]$ and, for each $B\in\mathcal{B}(\mathbf{R})$, $\sigma_t(B)$ is a Borel measurable function of t. Then

$$G_1(x)=\int_0^T\theta(t,Z(x,t))\,dt \quad and \quad G_2(x)=\exp\left\{\int_0^T\theta(t,Z(x,t))\,dt\right\}$$

are members of S.

Proof. In [17; 19] it was shown that $\int_0^T \theta(t,x(t))\,dt$ and $\exp\{\int_0^T \theta(t,x(t))\,dt\}$ were members of the Banach algebra S. Hence, by Lemma 2, both G_1 and G_2 are members of S. □

REMARK. Throughout the rest of this section we only need require that h be in $L_2[0,T]$ with $\|h\|>0$, rather than requiring that $h\in L_\infty[0,T]$.

THEOREM 6. *Let θ be given by* (3.13) *and let*

$$F(x)=\exp\left\{\int_0^T \theta(s,x(s))\,ds\right\}. \tag{3.14}$$

Let $h\in L_2[0,T]$ and let X be given by (3.3). *Then, for all $(\xi,\eta,\lambda)\in\mathbf{R}\times\mathbf{R}\times\mathbf{C}_+$, we have that*

$$\begin{aligned}
&E^{\mathrm{anw}_\lambda}(F\mid X)(\xi,\eta)\\
&=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}v_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{v_ja(s_j)}{a(T)}\right\}\\
&\quad\cdot\exp\left\{-\frac{1}{2\lambda}\left(\sum_{k=1}^{n}[a(s_k)-a(s_{k-1})]\left[\sum_{m=k}^{n}v_m-\sum_{m=1}^{n}\frac{v_ma(s_m)}{a(T)}\right]^2\right.\right.\\
&\qquad\left.\left.+\frac{[a(T)-a(s_n)]}{a^2(T)}\left(\sum_{m=1}^{n}v_ma(s_m)\right)^2\right)\right\}d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s},
\end{aligned}\tag{3.15}$$

where $\Delta_n(T)=\{\vec{s}=(s_1,\dots,s_n): 0=s_0<s_1<\cdots<s_n<T\}$.

Furthermore, for all real $q\neq 0$,

$$\begin{aligned}
&E^{\mathrm{anf}_q}(F\mid X)(\xi,\eta)\\
&=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}v_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{v_ja(s_j)}{a(T)}\right\}\\
&\quad\cdot\exp\left\{-\frac{i}{2q}\left(\sum_{k=1}^{n}[a(s_k)-a(s_{k-1})]\left[\sum_{m=k}^{n}v_m-\sum_{m=1}^{n}\frac{v_ma(s_m)}{a(T)}\right]^2\right.\right.\\
&\qquad\left.\left.+\frac{[a(T)-a(s_n)]}{a^2(T)}\left(\sum_{m=1}^{n}v_ma(s_m)\right)^2\right)\right\}d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s}.
\end{aligned}\tag{3.16}$$

Proof. Using the Fubini theorem, (2.7), a well-known Wiener integration formula, and (3.13), we see that for all $(\xi,\eta,\lambda)\in\mathbf{R}\times\mathbf{R}\times(0,\infty)$ we have

$$\begin{aligned}
J_\lambda(\xi,\eta)&=E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\mid X(\lambda^{-1/2}x+\xi))(\eta)\\
&=E\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^T\theta(s,\lambda^{-1/2}Z(x,s)+\xi)\,ds\right)^n\middle|\lambda^{-1/2}Z(x,T)+\xi=\eta\right)\\
&=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}E\left[\prod_{j=1}^{n}\theta(s_j,\lambda^{-1/2}Z(x,s_j)+\xi\right.\\
&\qquad\left.-\frac{a(s_j)}{a(T)}(\lambda^{-1/2}Z(x,T)+\xi)+\frac{a(s_j)}{a(T)}\eta)\right]d\vec{s}
\end{aligned}$$

$$=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)} E\Bigg[\prod_{j=1}^{n}\theta\Bigg(s_j,\xi+\lambda^{-1/2}\sum_{k=1}^{j}\text{ask}^{1/2}\int_{s_{k-1}}^{s_k}\frac{h\,dx}{\text{ask}^{1/2}}$$
$$-\frac{a(s_j)\lambda^{-1/2}}{a(T)}\sum_{k=1}^{n+1}\text{ask}^{1/2}\int_{s_{k-1}}^{s_k}\frac{h\,dx}{\text{ask}^{1/2}}$$
$$+\frac{a(s_j)}{a(T)}(\eta-\xi)\Bigg)\Bigg]\,d\vec{s}$$

$$=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^{n+1}}(2\pi)^{-(n+1)/2}\exp\{-\tfrac{1}{2}(u_1^2+\cdots+u_{n+1}^2)\}$$
$$\cdot\prod_{j=1}^{n}\theta\Bigg(s_j,\xi+\lambda^{-1/2}\sum_{k=1}^{j}\text{ask}^{1/2}u_k+\frac{a(s_j)}{a(T)}(\eta-\xi)$$
$$-\frac{a(s_j)\lambda^{-1/2}}{a(T)}\sum_{k=1}^{n+1}\text{ask}^{1/2}u_k\Bigg)\,du_1\cdots du_{n+1}\,d\vec{s}$$

$$=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}(2\pi)^{-(n+1)/2}\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}v_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{a(s_j)}{a(T)}\right\}$$
$$\cdot\int_{\mathbf{R}^{n+1}}\exp\Bigg\{-\frac{1}{2}\sum_{j=1}^{n}u_j^2$$
$$+i\lambda^{-1/2}\sum_{j=1}^{n}v_j\Bigg[\sum_{k=1}^{j}\text{ask}^{1/2}u_k-\frac{a(s_j)}{a(T)}\sum_{k=1}^{n+1}\text{ask}^{1/2}u_k\Bigg]\Bigg\}$$
$$du_1\cdots du_{n+1}\,d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s},$$

where for notational convenience "ask" denotes $a(s_k)-a(s_{k-1})$.

Next, we carry out the integrations with respect to $u_1,\ldots,u_{n+1}$ using (2.6) and obtain

$$J_\lambda(\xi,\eta)=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}v_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{v_j a(s_j)}{a(T)}\right\}$$
$$\cdot\exp\Bigg\{-\frac{1}{2\lambda}\Bigg(\sum_{k=1}^{n}\text{ask}\Bigg[\sum_{m=k}^{n}v_m-\sum_{m=1}^{n}\frac{v_m a(s_m)}{a(T)}\Bigg]^2$$
$$+\frac{[a(T)-a(s_n)]}{a^2(T)}\Bigg(\sum_{m=1}^{n}v_m a(s_m)\Bigg)^2\Bigg)\Bigg\}$$
$$d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s}.$$

But the right-hand side above is an analytic function of λ throughout $\mathbf{C}_+$, and is a continuous function of λ on $\tilde{\mathbf{C}}_+$ since each σ_{s_j} is a finite Borel measure with $\|\sigma_{s_j}\|\in L_1[0,T]$. Thus equations (3.15) and (3.16) are established. □

Theorem 7 follows from Theorem 6 in much the same way as Theorems 2 and 3 followed from Theorem 1, so we omit the proof.

THEOREM 7. *Let F, h, and X be as in Theorem 6. Then, for all real $q\neq 0$, $J_q^{\text{an}}(F)$ exists as an element of $\mathcal{L}(K(\mathbf{R}),L_\infty(\mathbf{R}))$ and for all $\psi=\psi_1+\psi_2$ in $K(\mathbf{R})$ is given by*

$$(J_q^{\text{an}}(F)\psi)(\xi)=(J_q^{\text{an}}(F)\psi_1)(\xi)+(J_q^{\text{an}}(F)\psi_2)(\xi) \tag{3.17}$$

for all $\xi\in\mathbf{R}$, *where*

$$(J_q^{\text{an}}(F)\psi_1)(\xi)=\int_{-\infty}^{\infty} E^{\text{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}\psi_1(\eta)\,d\eta \tag{3.18}$$

and

$$(J_q^{\text{an}}(F)\psi_2)(\xi)=\int_{\mathbf{R}}^{-} E^{\text{anf}_q}(F|X)(\xi,\eta)\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}\psi_2(\eta)\,d\eta, \tag{3.19}$$

with $E^{\text{anf}_q}(F|X)(\xi,\eta)$ *given by* (3.16). *In addition, we have an alternative expression for* $J_q^{\text{an}}(F)\psi_2$; *namely,*

$$\begin{aligned}
&(J_q^{\text{an}}(F)\psi_2)(\xi)\\
&=\int_{-\infty}^{\infty}\exp\left\{iu\xi-\frac{ia(T)}{2q}u^2\right\}d\phi(u)\\
&\quad+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^{n+1}}\exp\left\{i\xi\sum_{j=1}^{n}v_j-\frac{i}{2q}\left(\sum_{k=1}^{n}[a(s_k)-a(s_{k-1})]\right.\right.\\
&\qquad\qquad\cdot\left[\sum_{m=k}^{n}v_m-\sum_{m=1}^{n}\frac{v_m a(s_m)}{a(T)}\right]^2\\
&\qquad\qquad\left.+\frac{[a(T)-a(s_n)]}{a^2(T)}\left(\sum_{m=1}^{n}v_m a(s_m)\right)^2\right)\\
&\qquad\left.+iu\xi-\frac{ia(T)}{2q}\left(u+\sum_{j=1}^{n}\frac{v_j a(s_j)}{a(T)}\right)^2\right\}\\
&\qquad\qquad d\sigma(u)\,d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s}.
\end{aligned}\tag{3.20}$$

4. Analytic Feynman Integrals

In this section we use the results obtained in Section 3 to obtain formulas for the analytic but scalar-valued Feynman integral of various functionals, including those of the form (1.4).

DEFINITION 5. For each $\lambda>0$ assume that $G(\lambda^{-1/2}Z(x,\cdot))$ is Wiener integrable with respect to x on $C_0[0,T]$, and let

$$J(\lambda)=\int_{C_0[0,T]}G(\lambda^{-1/2}Z(x,\cdot))m(dx).$$

If there exists a function $J^*(\lambda)$ analytic on $\mathbf{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the (generalized) analytic Wiener integral of G over $C_0[0,T]$ with parameter λ, and for λ in $\mathbf{C}_+$ we write

$$\int_{C_0[0,T]}^{\text{anw}_\lambda}G(Z(x,\cdot))m(dx)=J^*(\lambda).$$

Let $\lambda = -iq \in \tilde{\mathbf{C}}_+$ be given, and let G be a functional whose analytic Wiener integral exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the (generalized) analytic Feynman integral of G over $C_0[0,T]$ with parameter q, and we write

$$\int_{C_0[0,T]}^{\text{anf}_q} G(Z(x,\cdot))m(dx) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbf{C}_+}} \int_{C_0[0,T]}^{\text{anw}_\lambda} G(Z(x,\cdot))m(dx).$$

The following theorem follows immediately by letting $\xi = 0$ in Theorem 2 and 3.

THEOREM 8. *Let $F \in S$ be given by* (3.1), $h \in L_\infty[0,T]$, *X be given by* (3.3), *and $\psi = \psi_1 + \psi_2$ be in $K(\mathbf{R})$. Let*

$$G(x) = F(x)\psi(x(T)).$$

Then, for all real $q \neq 0$, the analytic Feynman integral of G exists and is given by

$$\int_{C_0[0,T]}^{\text{anf}_q} G(Z(x,\cdot))m(dx) = (J_q^{\text{an}}(F)\psi)(0) = (J_q^{\text{an}}(F)\psi_1)(0) + (J_q^{\text{an}}(F)\psi_2)(0),$$

where $J_q^{\text{an}}(F)\psi_1$ is given by (3.7) *and $J_q^{\text{an}}(F)\psi_2$ is given by either* (3.10) *or* (3.11).

THEOREM 9. *Let F, h, and X be as in Theorem 6 and let $\psi = \psi_1 + \psi_2$ be in $K(\mathbf{R})$. Let*

$$G(x) = F(x)\psi(x(T)) = \exp\left\{\int_0^T \theta(s, x(s)\,ds\right\}\psi(x(T)).$$

Then, for all real $q \neq 0$, the analytic Feynman integral of G exists and is given by

$$\int_{C_0[0,T]}^{\text{anf}_q} G(Z(x,\cdot))m(dx) = (J_q^{\text{an}}(F)\psi_1)(0) + (J_q^{\text{an}}(F)\psi_2)(0),$$

where $J_q^{\text{an}}(F)\psi_1$ is given by (3.18) *and $J_q^{\text{an}}(F)\psi_2$ is given by* (3.19) *or* (3.20).

Our next four corollaries include the main results of [8] by Cameron and Storvick. The notation used in [8] is slightly different than ours. They work with general ν and we use $\nu = 1$. Where they use $[a,b]$ we use $[0,T]$; their ξ corresponds to our η; and the roles of F and G are interchanged.

COROLLARY 2. *Theorem 1 in* [8, p. 301].

Proof. In our Theorem 8 simply choose $\psi_2 \equiv 0$ and $h \equiv 1$. □

COROLLARY 3. *Theorem 2 in* [8, p. 304].

Proof. In our Theorem 8 simply choose $\psi_1 \equiv 0$ and $h \equiv 1$. □

COROLLARY 4. *Theorem 3 in* [8, p. 305].

Proof. Simply choose $h \equiv 1$ in Theorem 8. □

COROLLARY 5. *Theorem 4 in* [8, p. 306].

Proof. In our Theorem 9, choose $h \equiv 1$. □

REMARK. Let θ be given by (3.13). Then, by choosing $\psi_1 \equiv 0$ and $\psi_2 \equiv 1$, we can evaluate the analytic Feynman integral

$$\int_{C_0[0,T]}^{\text{anf}_q} \exp\left\{\int_0^T \theta(t, Z(x,t))\,dt\right\} m(dx)$$

for $h \in L_2[0,T]$. In fact,

$$\begin{aligned}\int_{C_0[0,T]}^{\text{anf}_q} & \exp\left\{\int_0^T \theta(t, Z(x,t))\,dt\right\} m(dx) \\ &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbf{R}^n} \exp\left\{-\frac{i}{2q}\left(\sum_{k=1}^{n} \text{ask}\left[\sum_{m=k}^{n} v_m - \sum_{m=1}^{n} \frac{v_m a(s_m)}{a(T)}\right]^2 \right.\right. \\ &\qquad \left. + \frac{a(T)-a(s_n)}{a^2(T)}\left(\sum_{m=1}^{n} v_m a(s_m)\right)^2\right) \\ &\qquad \left. - \frac{ia(T)}{2q}\left(\sum_{j=1}^{n} \frac{v_j a(s_j)}{a(T)}\right)^2\right\} d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\,d\vec{s}.\end{aligned}$$

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