A SCALAR TRANSPORT EQUATION, II

Z. A. Melzak

1. INTRODUCTION

This is the second paper in a series treating the formulation of a nonlinear integro-differential equation which occurs in a variety of physical problems, and discussing the existence and properties of its solutions. The first paper [2] was concerned with the equation

$$(1)\qquad \begin{aligned} \frac{\partial f(x, t)}{\partial t} &= \frac{1}{2}\int_0^x f(y, t) f(x-y, t)\phi(y, x-y)\, dy - f(x, t)\int_0^\infty f(y, t)\phi(x, y)\, dy \\ &\quad + \int_x^\infty f(y, t)\psi(y, x)\, dy - \frac{f(x, t)}{x}\int_0^x y\psi(x, y)\, dy \qquad (x, t \geq 0), \end{aligned}$$

where $f(x, 0)$, $\phi(x, y)$ and $\psi(x, y)$ are given and $f(x, t)$ is to be determined. It was shown that under certain hypotheses equation (1) possesses a unique solution $f(x, t)$.

The present paper will treat a more general equation in which $\phi(x, y, t)$ and $\psi(x, y, t)$ replace $\phi(x, y)$ and $\psi(x, y)$, respectively. The following is the main result obtained.

THEOREM 1. *Let* $f(x, y)$, $\phi(x, y, t)$ *and* $\psi(x, y, t)$ *be functions which satisfy the following hypotheses:*

(H_1) $f(x, 0)$ *is nonnegative, bounded, continuous and integrable, and*

$$\int_0^\infty x f(x, 0)\, dx < \infty ;$$

(H_2) $\phi(x, y, t)$ *is nonnegative and bounded,*

$\phi(x, y, t) = \phi(y, x, t)$,

$\phi(x, y, t)$ *is continuous with respect to* x, y, *and* t, *and continuity in* t *is uniform with respect to* x *and* y;

(H_3) $\psi(x, y, t)$ *is nonnegative and bounded,*

$$\int_0^x \psi(x, y, t)\, dy < E - 1 < \infty,$$

$\psi(x, y, t)$ *is continuous with respect to* x, y *and* t, *and continuity in* t *is uniform with respect to* x *and* y; *also, the function* $\frac{1}{x}\int_0^x y\psi(x, y, t)\, dy$ *is bounded*

Received January 17, 1957.

from above by 1, *it is continuous in* x *and* t, *and its continuity in* t *is uniform with respect to* x.

Then the equation

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2}\int_0^x f(y, t) f(x - y, t) \phi(y, x - y, t)\, dy - f(x, t)\int_0^\infty f(y, t)\phi(x, y, t)\, dy$$
$$(2) \qquad + \int_0^\infty f(y, t)\psi(y, x, t)\, dy - \frac{f(x, t)}{x}\int_0^x y\psi(x, y, t)\, dy$$

possesses a solution f(x, t) *which is valid for* $x, t \geq 0$. *This solution is continuous, nonnegative, continuously differentiable in* t *for each* x, *and integrable in* x *for each* t. *It is the only solution of equation* (2) *which is continuous and integrable in* x *for all* t *and which also assumes the prescribed initial value* f(x, 0) *for* t = 0.

To dispense with needless repetitions, an acquaintance with [2] is assumed throughout. The proof of Theorem 1 follows closely the idea of the Cauchy-Peano existence proof in the theory of ordinary differential equations; in this connection, see [1]. The interval [0, T] $(0 < T < \infty)$ is subdivided into a number of segments; over each segment, $\phi(x, y, t)$ and $\psi(x, y, t)$ are approximated by functions independent of t; and then the resultant system of integro-differential equations is solved by means of Theorem 1 of [2]. The notion of an approximate solution is introduced next. The number of subdivisions of [0, T] is allowed to tend to infinity while their meshes tend to zero. A sequence of approximate solutions is thus obtained, and it is shown to be a uniformly convergent Cauchy sequence. In this way a limit function is obtained without the usual recourse to a theorem of the Ascoli-Arzelà type. The limit function, which possesses all the properties required in Theorem 1, is then shown to be an actual solution of equation (2).

2. PRELIMINARY ESTIMATES

In the terminology of [2], equation (1) is written as

$$\frac{\partial f(x, t)}{\partial t} = [f(x, t), f(x, t)] + Lf(x, t),$$

where [f, f] and Lf are suitably defined operators. In order to stress the dependence on ϕ and ψ, and for reasons of brevity and convenience, this terminology will be modified. The part of the right-hand side of (2) consisting of the first two integrals will be denoted by $[f(x, t), f(x, t), \phi(x, y, t)]$, and the remainder will be written as $(f(x, t), \psi(x, y, t))$. The relation

$$[f + g, f + g, \phi] = [f, f, \phi] + [g, g, \phi] + 2[f, g, \phi]$$

defines a trilinear operator $[f, g, \phi]$, where f and g are any two functions of the class $L^1(0, \infty)$ in x, and where $\phi = \phi(x, y, t)$ satisfies the hypothesis (H_2) but is otherwise arbitrary. Similarly, (f, ψ) is a bilinear operator in which f is a member of $L^1(0, \infty)$ in x and ψ is any function satisfying the hypothesis (H_3). The basic estimates (4) to (9) of [2] remain valid for $[f, g, \phi]$ and (f, ψ) if one writes

$$A = \text{l.u.b. } \phi(x, y, t), \qquad C = \text{l.u.b. } \psi(x, y, t).$$

Let also

$$\text{l.u.b. } f(x, 0) = B, \qquad \int_0^\infty f(x, 0)\, dx = N.$$

Let T be a fixed positive number, and let $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$. Consider the system of equations

$$(3) \quad \frac{\partial f_k(x, t)}{\partial t} = [f_k(x, t), f_k(x, t), \phi(x, y, t_k)] + (f_k(x, t), \psi(x, y, t_k)) \qquad (t_k \leq t \leq t_{k+1}),$$

where

$$f_0(x, 0) = f(x, 0), \qquad f_k(x, t_{k+1}) = f_{k+1}(x, t_{k+1}) \quad (k = 0, 1, 2, \cdots, n - 1).$$

The hypotheses (H_1), (H_2) and (H_3) are assumed to apply to $f(x, 0)$, $\phi(x, y, t)$ and $\psi(x, y, t)$. By means of Theorem 1 of [2] and an induction on k, one shows that each equation of (3) possesses a solution which is continuous, nonnegative, analytic in t and integrable in x. Let

$$(4) \quad \begin{cases} N_k = \int_0^\infty f_k(x, t_k)\, dx, & N_0 = N, \\ B_k = \text{l.u.b. } f_k(x, t_k), & B_0 = B, \\ A_k = \text{l.u.b. } \phi(x, y, t_k) \leq A, & \\ C_k = \text{l.u.b. } \psi(x, y, t_k) \leq C, & \\ E_k = 1 + \text{l.u.b.} \int_0^x \psi(x, y, t_k)\, dy \leq E. & \end{cases}$$

By the estimates (36) of [2], the N_k satisfy the condition

$$N_{k+1} \leq N_k \left(1 + \frac{\exp (E_k - 1)(t_{k+1} - t_k) - 1}{E_k - 1} \right);$$

therefore

$$N_{k+1} \leq N \prod_{j=0}^{k} \left(1 + \frac{\exp (E_j - 1)(t_{j+1} - t_j) - 1}{E_j - 1} \right),$$

and consequently

$$\log N_{k+1}/N \leq \sum_{j=0}^{k} \frac{\exp(E_j - 1)(t_{j+1} - t_j) - 1}{E_j - 1}.$$

On account of the last inequality in (4), one obtains

$$\log N_{k+1}/N \leq \sum_{j=0}^{k} \frac{\exp(E - 1)(t_{j+1} - t_j) - 1}{E - 1}.$$

It is easily verified that, for $\alpha, \beta \geq 0$,

$$e^{\alpha} - 1 + e^{\beta} - 1 \leq e^{\alpha+\beta} - 1,$$

and repeated application shows that if $\alpha_j > 0$, then

$$\sum_{j=0}^{k} (e^{\alpha_j} - 1) \leq \exp\left(\sum_{j=0}^{k} \alpha_j \right) - 1.$$

Applying the last inequality, one finds that

$$\log N_{k+1}/N \leq \frac{1}{E - 1} \exp\left((E - 1) \sum_{j=0}^{k} (t_{j+1} - t_j) - 1 \right) = \frac{\exp(E - 1)t_{k+1} - 1}{E - 1}.$$

Let now $U_k = N \exp \dfrac{\exp(E - 1)t_{k+1} - 1}{E - 1}$; then

(5) $$N_k \leq U_k \leq U_{k+1} \leq U_n \leq N \exp \frac{\exp(E - 1)T - 1}{E - 1}.$$

A lower estimate for B_k will now be obtained. By (39) of [2],

$$B_{k+1} \geq B_k \exp\{-(t_{k+1} - t_k)(1 + A_k U_k)\} \geq B_k \exp\{-(t_{k+1} - t_k)(1 + AU_k)\}.$$

Therefore

$$B_{k+1} \geq B \exp\{-\sum_{j=0}^{k} (t_{j+1} - t_j)(1 + AU_j)\} \geq B \exp\{-(1 + AU_n) \sum_{j=0}^{k} (t_{j+1} - t_j)\}$$

$$= B \exp\{-(1 + AU_n)t_{k+1}\},$$

and consequently

(6) $$B_{k+1} \geq B \exp\{-(1 + AU_n)T\}.$$

Let $m_k = 3A_k N_k/2 + \max(E_k, C_k N_k/B_k)$; then, as shown in [2], $f_k(x, t)$ is represented on the interval $t_k \leq t < t_k + 1/m_k$ by a power series in t, and certain important estimates hold there. By the estimates (4), (5) and (6) there exists a constant $M = M(T)$ such that $m_k < M$. It follows that if $t_{k+1} - t_k < 1/M$, then the estimates (17) of [2] apply to $f_k(x, t)$ on its entire domain of definition, with B_k, N_k, f_k and M replacing B, N, f and m. One obtains therefore

$$B_{k+1} \leq \frac{B_k}{1 - M(t_{k+1} - t_k)}.$$

If it is assumed that $t_{k+1} - t_k \leq 1/2M$, then

$$\frac{1}{1 - M(t_{k+1} - t_k)} \leq 1 + 2M(t_{k+1} - t_k),$$

and the following upper bound holds for B_{k+1}:

$$B_{k+1} \leq B \prod_{j=0}^{k} [1 + 2M(t_{j+1} - t_j)] \leq B \exp\left\{2M \sum_{j=0}^{k} (t_{j+1} - t_j)\right\} \leq B\, e^{2MT}. \tag{7}$$

Further, by (17) of [2],

$$\int_0^\infty f_k(x, t)\, dx \leq \frac{N_k}{1 - M(t - t_k)}, \tag{8}$$

$$\operatorname*{l.u.b.}_{x}\, f_k(x, t) \leq \frac{B_k}{1 - M(t - t_k)}. \tag{9}$$

By the hypothesis (H_1), $\int_0^\infty x f(x, 0)\, dx = D < \infty$. Since

$$xf_0(x, t) - xf_0(x, 0) = \int_0^t \{x[f_0(x, t), f_0(x, t), \phi(x, y, 0)] + x(f_0(x, t), \psi(x, y, 0))\}\, dt,$$

integration of both sides with respect to x from 0 to ∞ shows that the right-hand side vanishes. Therefore

$$\int_0^\infty x f_0(x, t)\, dx = \int_0^\infty x f_0(x, 0)\, dx = D,$$

and induction on k shows that

$$\int_0^\infty x f_k(x, t)\, dx = D. \tag{10}$$

Let $\psi_1(x, y)$ and $\psi_2(x, y)$ be two functions satisfying the relevant parts of the hypothesis (H_3), and let $f(x)$ be a nonnegative function in $L^1(0, \infty)$, such that the integral $\int_0^\infty x f(x)\, dx$ is finite. Then

$$\int_0^\infty |(f, \psi_1 - \psi_2)|\, dx \leq \int_0^\infty \int_x^\infty f(y)\,|\psi_1(y, x) - \psi_2(y, x)|\, dy\, dx$$

$$+ \int_0^\infty \int_0^x \frac{y\, f(x)}{x}\,|\psi_1(x, y) - \psi_2(x, y)|\, dy\, dx,$$

and

$$\int_0^\infty \int_x^\infty f(y)\,|\psi_1(y, x) - \psi_2(y, x)|\, dy\, dx = \int_0^\infty f(x) \int_0^x |\psi_1(x, y) - \psi_2(x, y)|\, dy\, dx;$$

therefore

$$\int_0^\infty |(f, \psi_1 - \psi_2)|\, dx \leq \frac{3}{2}\, \text{l.u.b.}\, |\psi_1(x, y) - \psi_2(x, y)| \int_0^\infty x f(x)\, dx. \tag{11}$$

3. THE MAIN LEMMA

LEMMA 1. *Let* T *be a fixed number* $(0 < T < \infty)$, *and let* $0 = t_0 < t_1 < \cdots t_n = T$ *be a subdivision of the interval* $[0, T]$, *such that* $t_{k+1} - t_k \leq 1/2M$. *Let* $\{f_{1k}(x, t)\}$ *and* $\{f_{2k}(x, t)\}$ *be two finite sequences of functions* $(k = 0, 1, \cdots, n - 1)$, *such that*

$$\frac{\partial f_{ik}(x, t)}{\partial t} = [f_{ik}(x, t), f_{ik}(x, t), \phi_{ik}(x, y)] + (f_{ik}(x, t), \psi_{ik}(x, y))$$

$$(i = 1, 2;\ t_k \leq t \leq t_{k+1};\ f_{i0}(x, 0) = f(x, 0);\ f_{ik+1}(x, t_{k+1}) = f_{ik}(x, t_{k+1})).$$

Let $f(x, 0)$, $\phi_{ik}(x, y)$ *and* $\psi_{ik}(x, y)$ *satisfy the hypotheses* (H_1), (H_2) *and* (H_3), *respectively. In addition, let*

$$|\phi_{1k}(x, y) - \phi_{2k}(x, y)| \leq \eta,$$

$$|\psi_{1k}(x, y) - \psi_{2k}(x, y)| \leq \eta,$$

$$\frac{1}{x}\int_0^x y\,|\psi_{ik}(x, y) - \psi_{2k}(x, y)|\, dy \leq \eta.$$

Then there exists a constant $K_2 = K_2(T)$ *such that*

$$|f_{1k}(x, t) - f_{2k}(x, t)| \leq K_2\eta \qquad (0 \leq t \leq T).$$

Proof. $K_2, K_3, \cdots$ will denote constants depending on T and on the constants of the problem. Let

$$\alpha_0(0) = 0, \qquad \alpha_k(t) = \int_0^\infty |f_{1k}(x, t) - f_{2k}(x, t)|\, dx,$$

$$\beta_0(0) = 0, \qquad \beta_k(t) = \text{l.u.b.}\, |f_{1k}(x, t) - f_{2,k}(x, t)| .$$

Recursion formulas will now be developed for the α_k. By the hypotheses of the lemma,

$$f_{1k}(x, t) = f_{1k}(x, t_k) + \int_{t_k}^{t} \{[f_{1k}, f_{1k}, \phi_{1k}] + (f_{1k}, \psi_{1k})\}\, dt,$$

$$f_{2k}(x, t) = f_{2k}(x, t_k) + \int_{t_k}^{t} \{[f_{2k}, f_{2k}, \phi_{2k}] + (f_{2k}, \psi_{2k})\}\, dt.$$

Subtracting and taking absolute values, one obtains

$$|f_{1k}(x, t) - f_{2k}(x, t)| \leq |f_{1k}(x, t_k) - f_{1k}(x, t_k)|$$
$$+ \int_{t_k}^{t} \{|[f_{1k}, f_{1k}, \phi_{1k}] - [f_{2k}, f_{2k}, \phi_{2k}]| + |(f_{1k}, \psi_{1k}) - (f_{2k}, \psi_{2k})|\}\, dt,$$

and this may be written in the form

$$\begin{aligned} &|f_{1k}(x, t) - f_{2k}(x, t)| \leq |f_{1k}(x, t_k) - f_{2k}(x, t_k)| \\ \text{(12)}\quad &+ \int_{t_k}^{t} \{|[f_{1k}, f_{1k}, \phi_{1k} - \phi_{2k}]| + |[f_{1k} + f_{2k}, f_{1k} - f_{2k}, \phi_{2k}]| + |(f_{1k}, \psi_{1k} - \psi_{2k})| \\ &+ |(f_{1k} - f_{2k}, \psi_{2k})|\}\, dt. \end{aligned}$$

Integrating with respect to x from zero to infinity and interchanging the order of the integrations (which is justified by the absolute convergence of the integrals), one obtains

$$\alpha_k(t) = \alpha_k(t_k) + \int_{t_k}^{t} \int_0^\infty (P + Q + R + S)\, dx\, dt, \tag{13}$$

where P, Q, R and S stand for the four absolute values, in their order of appearance, in the integrand of (12). The following estimates follow from (5), (7), (8) and (9), and from the assumption that $t_{k+1} - t_k \leq 1/2M$:

$$\int_0^\infty f_{1k}(x, t)\, dx \leq K_3, \qquad \text{l.u.b. } f_{ik}(x, t) \leq K_4 \quad (i = 1, 2). \tag{14}$$

By (14) above and by (6) of [2],

$$\int_0^\infty P\, dx \leq K_5 \eta, \tag{15}$$

$$\int_0^\infty Q\, dx \leq K_6\, \alpha_k(t). \tag{16}$$

Further, by (11) one gets

$$\int_0^\infty R\, dx \leq K_7 \eta. \tag{17}$$

Also,

$$\int_0^\infty S\, dx \leq \int_0^\infty \left\{ \int_x^\infty |f_{1k}(y, t) - f_{2k}(y, t)|\, \psi_{2k}(y, x)\, dy + \frac{|f_{1k}(x, t) - f_{2k}(x, t)|}{x} \cdot \int_0^x y\, \psi_{2k}(x, y)\, dy \right\} dx.$$

Proceeding as in the derivation of (11), one obtains

$$\int_0^\infty S\, dx \leq K_8 \int_0^\infty |f_{1k}(x, t) - f_{2k}(x, t)|\, dx = K_8\, \alpha_k(t). \tag{18}$$

If the estimates (15) to (18) are used in (13), one gets

$$\alpha_k(t) \leq \alpha_k(t_k) + \int_{t_k}^t (K_9 \eta + K_{10}\, \alpha_k(t))\, dt = \alpha_k(t_k) + K_9 \eta (t - t_k) + K_{10} \int_{t_k}^t \alpha_k(t)\, dt. \tag{19}$$

By the inequality in Problem 1, Ch. 1 of [1], this implies that

$$\alpha_k(t) \leq \alpha_k(t_k) + K_9\eta(t - t_k) + K_{10}\int_{t_k}^{t} \exp\{K_{10}(t - s)\}[\alpha_k(t_k) + K_9\eta(s - t_k)]\,ds;$$

evaluating the integral, one obtains

$$\begin{aligned}\alpha_k(t) &\leq \alpha_k(t_k) + K_9\eta(t - t_k) + \alpha_k(t_k)[\exp K_{10}(t - t_k) - 1] \\ &+ \frac{K_9\eta}{K_{10}}[\exp K_{10}(t - t_k) - K_{10}(t - t_k) - 1].\end{aligned} \tag{20}$$

Since $0 \leq t_{k+1} - t_k \leq 1/2M$, it follows that

$$0 \leq \exp K_{10}(t - t_k) - 1 \leq K_{10}(t - t_k) \exp K_{10}/2M,$$

$$0 \leq \exp K_{10}(t - t_k) - K_{10}(t - t_k) - 1 \leq K_{10}(t - t_k) \exp K_{10}/2M.$$

Therefore, by (20),

$$\alpha_k(t) \leq \alpha_k(t_k) + K_9\eta(t - t_k) + K_{10}(t - t_k)\alpha_k(t_k) \exp K_{10}/2M + K_9\eta(t - t_k) \exp K_{10}/2M,$$

that is,

$$\alpha_k(t) \leq \alpha_k(t_k)[1 + K_{11}(t - t_k)] + K_{12}\eta(t - t_k). \tag{21}$$

Let now $t = t_{k+1}$ in (21); then

$$\alpha_{k+1}(t_{k+1}) \leq \alpha_k(t_k)[1 + K_{11}(t_{k+1} - t_k)] + K_{12}\eta(t_{k+1} - t_k).$$

Introduce constants γ_k, given by

$$\gamma_{k+1} = \gamma_k[1 + K_{11}(t_{k+1} - t_k)] + K_{12}\eta(t_{k+1} - t_k), \quad \gamma_0 = \alpha_0(0) = 0.$$

Then clearly

$$\alpha_k(t_k) \leq \gamma_k. \tag{22}$$

Put $h_k(x) = a_k x + b_k$, $a_k = 1 + K_{11}(t_{k+1} - t_k)$, $b_k = K_{12}\eta(t_{k+1} - t_k)$; then

$$\gamma_{k+1} = h_k(h_{k-1}(\cdots h_0(0)\cdots)). \tag{23}$$

The right-hand side of (23), being an iterate of linear functions, is explicitly evaluable:

$$h_k(h_{k-1}(\cdots h_0(0)\cdots)) = \sum_{j=0}^{k}\left(b_j \prod_{\ell=j+1}^{k} a_\ell\right).$$

Therefore

$$\gamma_{k+1} = \sum_{j=0}^{k}\left\{K_{12}\eta(t_{j+1} - t_j) \prod_{\ell=j+1}^{k}[1 + K_{11}(t_{\ell+1} - t_\ell)]\right\}.$$

The following elementary estimate holds:

$$\prod_{\ell=j+1}^{k} [1 + K_{11}(t_{\ell+1} - t_\ell)] \leq \exp \sum_{\ell=0}^{k} K_{11}(t_{\ell+1} - t_\ell) = \exp K_{11} t_{k+1} \leq \exp K_{11} T,$$

and therefore

$$\gamma_{k+1} \leq \exp K_{11} T \sum_{j=0}^{k} K_{12}\eta(t_{j+1} - t_j) = K_{12}\eta t_{k+1} \exp K_{11} T = K_{13}\eta.$$

Now (22) implies that $\alpha_k(t_k) \leq K_{13}\eta$, and since $0 \leq t_{k+1} - t_k \leq 1/2M$, one gets from (21)

$$\alpha_k(t) \leq K_{14}\eta. \tag{24}$$

Similar results will be obtained for the $\beta_k(t)$. The following preliminary inequality is necessary: if $F(x, t)$ is a continuous, nonnegative function, then

$$\text{l.u.b.}_x \int_{u_1}^{u_2} F(x, t)\,dt \leq \int_{u_1}^{u_2} \text{l.u.b.}_x\, F(x, t)\,dt. \tag{25}$$

This is proved by approximating the integrals by finite sums and then passing to the limit.

Consider again the inequality (12). Taking the least upper bound over x and interchanging the order of this operation and the integration (which is allowed, by (25)), one obtains

$$\beta_k(t) \leq \beta_k(t_k) + \int_{t_k}^{t} (\text{l.u.b.}_x\, P + \text{l.u.b.}_x\, Q + \text{l.u.b.}_x\, R + \text{l.u.b.}_x\, S)\,dt, \tag{26}$$

where P, Q, R and S have the same meaning as before. The four terms in the integrand of (26) are estimated as follows. By (5) of [2] and by (14) above,

$$\text{l.u.b.}\, P \leq (3\eta/2)\, \text{l.u.b.}_x\, f_{1k}(x, t) \int_0^\infty f_{1k}(y, t)\,dy \leq K_{15}\eta, \tag{27}$$

and also

$$\text{l.u.b.}_x\, Q \leq (3A/2) \Big[\int_0^\infty \{f_{1k}(y, t) + f_{2k}(y, t)\}\,dy\ \text{l.u.b.}_y |f_{1k}(y, t) - f_{2k}(y, t)| \tag{28}$$

$$+ \int_0^\infty |f_{1k}(y, t) - f_{2k}(y, t)| \, dy \ \text{l.u.b.} \{ f_{1k}(y, t) + f_{2k}(y, t) \} \Big]$$

$$= K_{16} \beta_k(t) + K_{17} \alpha_k(t) .$$

Similarly, by (8) of [2] and by (14) above,

$$\text{(29)} \quad \underset{x}{\text{l.u.b.}} \, R \leq \eta \, \underset{x}{\text{l.u.b.}} \int_x^\infty f_{1k}(y, t) \, dy + \underset{x}{\text{l.u.b.}} \frac{f_{1k}(x, t)}{x} \int_0^x y \, |\psi_{1k}(x, y) - \psi_{2k}(x, y)| \, dy$$

$$\leq K_{18} \eta ,$$

and

$$\text{(30)} \quad \underset{x}{\text{l.u.b.}} \, S \leq C \, \underset{x}{\text{l.u.b.}} \int_x^\infty |f_{1k}(y, t) - f_{2k}(y, t)| \, dy$$

$$+ \underset{x}{\text{l.u.b.}} \frac{|f_{1k}(x, t) - f_{2k}(x, t)|}{x} \int_0^x y \, \psi_{2k}(x, y) \, dy \leq K_{19} \alpha_k(t) + \beta_k(t) .$$

If the estimates (27) to (30) are used in (26), one obtains

$$\text{(31)} \qquad \beta_k(t) \leq \beta_k(t_k) + \int_{t_k}^t [K_{20} \eta + K_{21} \alpha_k(t) + K_{22} \beta_k(t)] \, dt ;$$

this reduces to

$$\text{(32)} \qquad \beta_k(t) \leq \beta_k(t_k) + K_{23} \eta (t - t_k) + K_{24} \int_{t_k}^t \beta_k(t) \, dt .$$

This inequality is formally identical with the inequality (19); proceeding exactly as before, one obtains the same results for the β_k as for the α_k, that is, a formula analogous to (24):

$$\text{(33)} \qquad \beta_k(t) \leq K_2 \eta .$$

In view of the definition of $\beta_k(t)$, this completes the proof of the lemma.

4. THE SEQUENCE OF APPROXIMATE SOLUTIONS

Let T be a fixed number $(0 < T < \infty)$, and let $\{\eta_n\}$ $(n = 1, 2, \cdots)$ be a monotone decreasing sequence of positive numbers tending to zero as n tends to infinity. For each n, let $0 = t_{n1} < t_{n2} < \cdots < t_{nk(n)} = T$ be a subdivision of the interval $[0, t]$ which satisfies the following conditions:

$$(34)\qquad \begin{cases} |\phi(x, y, \tau_1) - \phi(x, y, \tau_2)| \leq \eta_n & (t_{nj} \leq \tau_2 \leq \tau_1 \leq t_{n,j+1}), \\ |\psi(x, y, \tau_1) - \psi(x, y, \tau_2)| \leq \eta_n & (j = 0, 1, \cdots, k(n) - 1), \\ \dfrac{1}{x}\displaystyle\int_0^x y|\psi(x, y, \tau_1) - \psi(x, y, \tau_2)|\,dy \leq \eta_n, & \\ t_{n,j+1} - t_{nj} \leq 1/2M. & \end{cases}$$

Since the functions $\phi(x, y, t)$, $\psi(x, y, t)$ and $\frac{1}{x}\int_0^x y\psi(x, y, t)\,dy$ are continuous in t and their continuity is uniform with respect to the other variables, it follows that for each positive η_n an appropriate finite partition can be found for which the conditions (34) are satisfied. For each n, construct a function $f_n(x, t)$ as follows. Consider a system of equations of the form (3), with $f_{nk}(x, t)$ and t_{nk} replacing $f_k(x, t)$ and t_k, respectively. Let $k = 0, 1, \cdots, k(n) - 1$; and put

$$(35)\qquad f_n(x, t) = f_{nk}(x, t) \qquad (t_{nk} \leq t \leq t_{n,k+1}).$$

The sequence $\{f_n(x, t)\}$ will be called the sequence of approximate solutions. Let $f_p(x, t)$ and $f_q(x, t)$ be two approximate solutions $(p \leq q)$, and let

$$0 = t_0 < t_1 < \cdots < t_M = T$$

be the partition of $[0, T]$ obtained by superimposing the two subdivisions corresponding to f_p and f_q. If one lets $\eta = \eta_p + \eta_q$, then all the hypotheses of Lemma 1 are satisfied, and therefore there exists a constant K, depending on T and on other constants of the problem, such that

$$(36)\qquad |f_p(x, t) - f_q(x, t)| \leq K(\eta_p + \eta_q).$$

This implies that the sequence $\{f_n(x, t)\}$ is a uniformly convergent Cauchy sequence. Therefore there exists a function $f(x, t)$ such that $f_n(x, t) \to f(x, t)$ uniformly for $x \geq 0$ and $0 \leq t \leq T$. This limit function shares with the functions of the sequence of approximate solutions the properties of continuity, nonnegativity, uniform boundedness from above, the constancy of the first moment and integrability in x. Once it is shown that $f(x, t)$ satisfies the equation (2), it will follow that $f(x, t)$ is also continuously differentiable in t.

It remains to be shown that $f(x, t)$ is an actual solution of (2). Let

$$\phi_n(x, y, t) = \phi(x, y, t_{nk}), \qquad \psi_n(x, y, t) = \psi(x, y, t_{nk}) \quad (t_{nk} \leq t \leq t_{n,k+1}).$$

Then

$$f_n(x, t) = f(x, 0) + \int_0^t \{[f_n, f_n, \phi_n] + (f_n, \psi_n)\} dt. \tag{37}$$

Since also $\int_0^\infty x f_n(x, t)\, dx = D$ for $n = 0, 1, \cdots$, and $f_n(x, t) \geq 0$, it follows that

$$\int_{x_1}^\infty f_n(x, t)\, dx \leq \frac{1}{x_1} \int_{x_1}^\infty x f_n(x, t)\, dx \leq D/x_1 \qquad (x_1 \geq 1),$$

which implies

$$\int_0^\infty f_n(x, t)\, dx \to \int_0^\infty f(x, t)\, dx \qquad \text{as } n \to \infty. \tag{38}$$

Proceeding now in the same way as in obtaining (12), one gets first

$$[f_n, f_n, \phi_n] - [f, f, \phi] = [f_n + f, f_n - f, \phi_n] + [f, f, \phi_n - \phi],$$

and then, by means of (4) in [2],

$$\begin{aligned} &\text{l.u.b.} \left| [f_n, f_n, \phi_n] - [f, f, \phi] \right| \\ &\leq K_{25} \int_0^\infty |f_n - f|\, dx + K_{26}\, \text{l.u.b.} |f_n - f| + K_{27}\, \text{l.u.b.} |\phi_n - \phi|. \end{aligned} \tag{39}$$

It follows that $[f_n, f_n, \phi_n] \to [f, f, \phi]$ as $n \to \infty$. In the same way it is proved that $f_n, \psi_n) \to (f, \psi)$ as $n \to \infty$. Therefore, by (37),

$$f(x, t) = f(x, 0) + \int_0^t \{[f, f, \phi] + (f, \psi)\}\, dt. \tag{40}$$

Differentiating both sides of (40) with respect to t, one shows that $f(x, t)$ satisfies the equation (2). Since the number T is arbitrary, a solution of (2) can be found which is valid on the whole interval $0 \leq t < \infty$. The conditions for uniqueness, mentioned in Theorem 1, are demonstrated exactly as in [2].

Acknowledgments. The help of the Office of Naval Research, Washington D.C., in the form of a Research Associateship in Mathematics, is gratefully acknowledged. The author also wishes to thank the Editor and the Referee for their criticisms and suggestions.

REFERENCES

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955.

2. Z. A. Melzak, *A scalar transport equation*, Trans. Amer. Math. Soc. 85 (1957), 547-560.

University of Michigan
and
McGill University