UNIQUENESS FOR THE ANALOGUE OF THE NEUMANN PROBLEM FOR MIXED EQUATIONS

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In this paper we shall consider a uniqueness problem for an equation of mixed type, that is, an equation which is partly elliptic and partly hyperbolic depending on the domain in question. Such problems were posed first by Tricomi; uniqueness has been proved in certain cases, for a boundary condition that corresponds to the Dirichlet problem, by Tricomi and many others. Here we shall consider the boundary value problem that corresponds to the Neumann problem. It arises in the study of transonic flow, and the proof of uniqueness in this case leads to a proof that continuous transonic flows past smooth profiles do not exist in general (see [4] and [5]).

Let ω be a solution of the equation

$$K(\sigma)\omega_{\theta\theta} + \omega_{\sigma\sigma} = 0, \tag{1}$$

where $K(\sigma) > 0$ for $\sigma > 0$, and $K(\sigma) < 0$ for $\sigma < 0$, in an open domain D (see Fig. 1). In $\sigma > 0$, D is bounded by an arc C_0 with continuous tangent which intersects the negative θ-axis at Π_1 and the positive θ-axis at Π_2. In $\sigma < 0$, D is bounded by two curves C_1 and C_2 issuing from Π_1 and Π_2, and by the two characteristics of (1), γ_1 and γ_2, that issue from a point Π_0 between Π_1 and Π_2 on the θ-axis. Here C_1 and C_2 satisfy the condition

$$K\left(\frac{d\sigma}{d\theta}\right)^2 + 1 \geq 0. \tag{2}$$

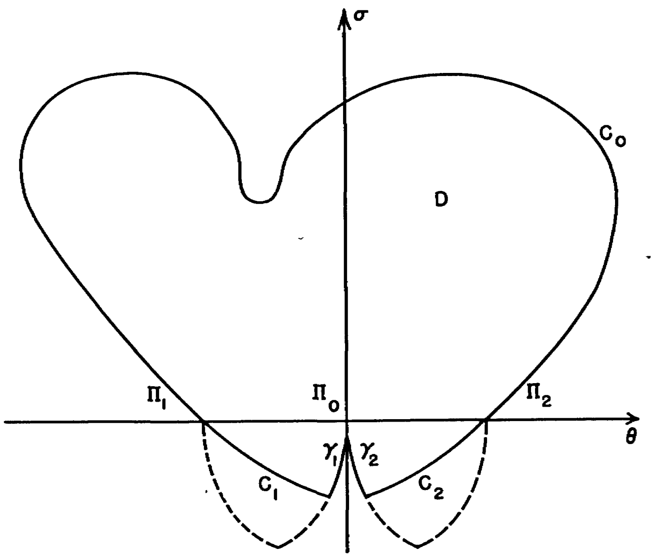

Figure 1

The analogue to the Neumann problem is to find a solution of (1) in D for which the oblique derivative $\omega_\sigma \frac{d\theta}{ds} - K\theta \frac{d\sigma}{ds}$ is prescribed as a function of arc length s on $C_0 + C_1 + C_2$.

This problem is called the analogue of the Neumann problem because if, for $\sigma \geq 0$, we introduce the variable

$$\mu = \int_0^\sigma \sqrt{K(\sigma)}\, d\sigma\ , \tag{3}$$

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equation (1) is reduced to the canonical form

$$\omega_{\theta\theta} + \omega_{\mu\mu} + \alpha u_{\mu} = 0, \tag{4}$$

where $\alpha = \frac{1}{2K^{3/2}}\frac{dK}{d\sigma}$ and the oblique derivative $\omega_\sigma \frac{d\theta}{ds} - K\omega_\theta \frac{d\sigma}{ds}$ becomes $-\sqrt{K}\frac{\partial\omega}{\partial n}$.

We shall show that, under certain continuity conditions and with some restriction on $C_0 + C_1 + C_2$, the solution of this problem is unique. We shall also show, by an explicit counter-example, that there are domains D for which uniqueness does not hold.

1. CONDITIONS FOR UNIQUENESS

Let D^+ be the subdomain of D with $\sigma > 0$. Let j be the angle that the tangent to the image C_0^* of C_0 in the (θ, μ)-plane makes with the θ-axis, that is,

$$\cos j = \frac{d\theta}{d\tau}, \qquad \sin j = \frac{d\mu}{d\tau},$$

where

$$d\tau = \sqrt{d\theta^2 + K d\sigma^2} = \sqrt{d\theta^2 + d\mu^2},$$

taken counterclockwise. In the body of this paper we consider only the case

$$0 \leq j \leq 2\pi. \tag{5}$$

We could also consider the case $-\pi \leq j \leq 3\pi$, but the method would then be extremely cumbersome. We assume that

$$\begin{aligned} j &= (2 - \alpha_1)\pi > \pi \quad \text{at } \Pi_1, \\ j &= \alpha_2\pi < \pi \quad \text{at } \Pi_2. \end{aligned} \tag{6}$$

It is for $j = \pi$ that we find a counter-example.

In the region $\sigma < 0$ we require that on going counterclockwise

$$\begin{aligned} d\sigma &\leq 0 \quad \text{on } C_1, \\ d\sigma &\geq 0 \quad \text{on } C_2. \end{aligned} \tag{7}$$

The function $K(\sigma)$ is required to have a bounded derivative which is nonnegative in some neighborhood of $\sigma = 0$.

To every solution ω of (1) there corresponds a function Ψ (see [6]), defined by the integral

$$\Psi = \int_{\Pi_1}^{(\theta,\sigma)} (K\omega_\theta^2 - \omega_\sigma^2)d\sigma - 2\omega_\theta\omega_\sigma d\theta. \tag{8}$$

which is independent of the path of integration, by (1). The solution ω is to satisfy the continuity conditions

$$\begin{aligned}&\omega \text{ is continuous in the closure of } D,\\ &\omega_\theta,\ \omega_\sigma \text{ are continuous in } D, \text{ and they are such that } \Psi \text{ is continuous in the closure of } D.\end{aligned} \tag{9}$$

In addition we require

$$K^{1/4}|\omega_\theta - i\omega_\mu| = O(r_i^{(\nu_i-1)/2}) \qquad \text{at } \Pi_i \ (i = 1, 2), \tag{10}$$

where $r_i = \sqrt{(\theta - \theta(\Pi_i))^2 + \mu^2}$ and $\frac{\alpha_i}{1 - \alpha_i} < \nu_i < \frac{1}{1 - \alpha_i}$.

THEOREM. *If* ω *satisfies the differential equation* (1) *and the continuity conditions* (9) *and* (10) *in* D, *and the boundary condition*

$$\omega_\sigma d\theta - K\omega_\theta d\sigma = 0 \qquad \text{on } C_0 + C_1 + C_2, \tag{11}$$

then $\omega \equiv$ constant *in* D.

2. PROOF OF THE UNIQUENESS THEOREM

We assume that ω satisfies the conditions of the theorem and is not a constant. We shall state five lemmas, to be proved later, from which we can prove the theorem by a simple geometric argument.

LEMMA 1. *If* $\omega_\theta = \omega_\sigma = 0$ *in some subdomain of* D^+ *or on an arc of* C_0, *and* ω *satisfies* (1) *and* (9), *then* $\omega_\theta = \omega_\sigma = 0$ *in* D.

LEMMA 2. Ψ *assumes its maximum and minimum, for any subdomain of* D^+, *on the boundary of that subdomain.*

LEMMA 3. *Let* X *be a point on the* θ*-axis. For* $\theta(\Pi_1) \leq \theta(X) \leq \theta(\Pi_0)$, *we have*

$$\Psi(X) < \Psi(\Pi_1) \qquad \text{or} \qquad \omega_\theta = \omega_\sigma = 0$$

on $\Pi_1 X$. *For* $\theta(\Pi_0) \leq \theta(X) \leq \theta(\Pi_2)$, *we have*

$$\Psi(X) < \Psi(\Pi_2) \qquad \textit{or} \qquad \omega_\theta = \omega_\sigma = 0$$

on $X\Pi_2$.

LEMMA 4. $\frac{d\Psi}{ds}\Big/\frac{d\sigma}{ds} < 0$ *on* C_0, *where* s *is arc length taken counterclockwise and* Ψ *cannot have a local minimum at any point on* C_0. *Near* Π_1, $\frac{d\Psi}{ds} < 0$, *and near* Π_2, $\frac{d\Psi}{ds} > 0$.

LEMMA 5. Ψ *does not have a local maximum at* Π_1 *or* Π_2.

The first is a well-known theorem for which we shall sketch a proof. Lemmas 2, 3, and 4 are easily proved by means of known theorems and Lemma 1. The main difficulty lies in proving Lemma 5.

Proof of the theorem. On every circular arc joining the line $\sigma = 0$ to C_0, with its center at Π_2 and with a sufficiently small radius, there exists by Lemma 5 at least one point at which $\Psi > \Psi(\Pi_1)$. By Lemmas 3 and 4 we see that the value of Ψ at the endpoints is not greater than at Π_1, and that it is actually less at the endpoint which lies on C_0. Therefore there exist at least two level lines of Ψ emanating from each of Π_1 and Π_2 into the closure of D^+. Consider four such lines. By Lemma 4 they do not coincide with C_0. Furthermore, it follows from Lemmas 1 and 2 that through every point in D^+ there passes a curve Ψ = constant. Therefore the four curves Ψ = constant which issue from Π_1 and Π_2 either intersect, or they end at the boundary of D^+, or they lie on $\sigma = 0$.

There are only four possibilities:

1) The two curves from Π_1 (or Π_2) form a loop. Let D_1 denote the domain bounded by these two curves. By Lemma 2, Ψ achieves its maximum and minimum at Π_1. This is impossible, since then $\Psi = \Psi(\Pi_1)$ in D_1 and by (8) and Lemma 1, $\omega \equiv$ constant in D.

2) At least one curve L_1 from Π_1 and another curve L_2 from Π_2 intersect. Consider the domain D_0 bounded by C_0, L_1 and L_2. By Lemmas 2 and 4, Ψ can achieve its minimum in D_0 only on $L_1 + L_2$ where $\Psi = \Psi(\Pi_1) = \Psi(\Pi_2)$; but this is impossible by Lemma 4, since Ψ is nonincreasing as s increases near Π_1.

3) At least one curve, say L_1, ends on C_0, $\sigma > 0$. For the domain bounded by C_0 and L_1 the argument of case 2) shows that this situation is impossible.

4) At least one curve L_1 from Π_1 and one curve L_2 from Π_2 issue into D^+ and end on the θ-axis. Let the points of intersection be X_1 and X_2. Then $\Psi(X_1) = \Psi(\Pi_1)$ and $\Psi(X_2) = \Psi(\Pi_2)$. By Lemma 3, either we have again case 1), or $\theta(X_1) > \theta(\Pi_0)$ and $\theta(X_2) \leq \theta(\Pi_0)$. Therefore L_1 and L_2 must intersect. This leads us back to case 2), and therefore it is impossible.

The theorem is proved.

3. PROOFS OF THE LEMMAS

Lemma 1. By the unique continuation theorem for a solution of an elliptic equation (see for example [1]), we have in both cases $\omega_\theta = \omega_\sigma = 0$ in D^+. Hence, by the continuity condition (9), $\omega_\theta = \omega_\sigma = 0$ on $\sigma = 0$. Then, from the differential equation (1), we have

$$0 = \iint_{D^-} \theta\omega_\theta (K\omega_{\theta\theta} + \omega_{\sigma\sigma}) d\theta\, d\sigma$$

$$= \int_{C_1+\gamma_1+\gamma_2+C_2} \theta(K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) + \iint_{D^-} \frac{1}{2}(-K\omega_\theta^2 + \omega_\sigma^2) d\theta\, d\sigma$$

$$= \int_{C_1+C_2} -K\theta\omega_\theta^2 \left(1 + K\left(\frac{d\sigma}{d\theta}\right)^2\right) d\sigma + \int_{\gamma_1} (-\theta) d\sigma \,(\sqrt{-K}\omega_\theta + \omega_\sigma)^2 + \int_{\gamma_2} (-\theta) d\sigma\,(\sqrt{-K}\omega_\theta - \omega_\sigma)^2 ,$$

by (11) and the equation for the characteristics, $\sqrt{-K} d\sigma \pm d\theta = 0$. Here D^- is the subdomain of D with $\sigma < 0$. The three line integrals are nonnegative, by (2) and (7). Therefore $\omega_\theta = \omega_\sigma = 0$ in D^-, and thus the lemma is proved.

Lemma 2. It is not difficult to show that, in D^+, Ψ satisfies the elliptic equation

$$\Psi_{\theta\theta} + \Psi_{\mu\mu} = \alpha\Psi_\mu + \beta\Psi_\theta \tag{12}$$

with coefficients

$$\alpha = -K'K^{-3/2}\Psi_\mu/\sqrt{\Psi_\mu^2 + \Psi_\theta^2}, \qquad \beta = -K'K^{-3/2}\Psi_\theta/\sqrt{\Psi_\mu^2 + \Psi_\theta^2}$$

which are bounded for $\sigma > 0$; that is, $\mu > 0$. Such an elliptic equation satisfies a maximum and minimum principle (see [1]).

Lemma 3. For $\theta(\Pi_1) < \theta(X) \leq 0$, it follows from (8) and (9) that

$$\Psi(X) = \Psi(\Pi_1) + \int_{C_1+\gamma_X} (K\omega_\theta^2 - \omega_\sigma^2)d\sigma - 2\omega_\theta\omega_\sigma\, d\theta,$$

where γ_X is the characteristic of positive slope $(-K)^{-1/2}$ through X, and where the integral along C_1 is taken as far as γ_X. By (11), the integral along C_1 is

$$\int_{C_1} -K\omega_\theta^2 \left(1 + K\left(\frac{d\sigma}{d\theta}\right)^2\right)d\sigma. \tag{13}$$

This integral is negative, by (2) and (7), unless $\omega_\theta = \omega_\sigma = 0$ on C_1. The integral along γ_X is

$$\int_{\gamma_X} -(\sqrt{-K}\omega_\theta + \omega_\sigma)^2\, d\sigma,$$

and it is negative unless $\sqrt{-K}\omega_\theta + \omega_\sigma = 0$ on γ_X. Therefore $\Psi(X) < \Psi(\Pi_1)$, unless $\omega_\theta = \omega_\sigma$ on C_1 and ω = constant on γ_X. In the latter case, by a well-known theorem on hyperbolic equations, $\omega \equiv$ constant in the domain bounded by γ_X, C_1 and the x-axis. This proves the lemma for $\Pi_1\Pi_0$; the result for $\Pi_0\Pi_2$ can be proved similarly.

Lemma 4. Along C_0 we have, from (8) and (11)

$$\frac{d\Psi}{ds} = -\left(K\omega_\theta^2 + \omega_\sigma^2\right)\frac{d\sigma}{ds}. \tag{14}$$

Therefore $\dfrac{d\Psi}{ds}\Big/\dfrac{d\sigma}{ds} \leq 0$.

For the second assertion, there are two cases to consider. First, if $d\sigma/ds$ changes sign on passing through a point P, then the point P can be a minimum for Ψ only if $d\sigma/ds$ changes from positive to negative. Therefore by (5) we have $j = \pi$. But then

$$\frac{\partial\Psi}{\partial n} = -\frac{\partial\Psi}{\partial\sigma} = -K\omega_\theta^2 \leq 0,$$

where $\partial/\partial n$ denotes differentiation in the direction of the inward normal. By a theorem of Hopf [3] for elliptic equations, $\partial\Psi/\partial n > 0$ at a point where Ψ has a minimum for some domain, unless $\Psi_\theta = \Psi_\sigma = 0$ in that domain. Therefore

$$\Psi_\theta = \Psi_\sigma = \omega_\theta = \omega_\sigma = 0$$

in some neighborhood of P, and by Lemma 1 also in D, which contradicts our assumption.

The second case occurs if $d\sigma/ds = 0$ or $\omega_\sigma = 0$ on an arc of C_0. However, by (14), if Ψ achieves a minimum at some point on this arc, then as we proceed counterclockwise on C_0, $d\Psi/ds$ is positive before the arc and negative afterwards. Hence $j = \pi$ at some point on the arc, and the same argument as above holds since Ψ achieves its minimum at all points on this arc.

The last statement of the lemma follows from (14) and Lemma 1.

Lemma 5. Suppose Ψ has a local maximum at Π_1. By Lemmas 3 and 4 and the continuity condition (9), we see that for δ sufficiently small there exists a simple curve B_δ on which $\Psi = \text{constant} = \Psi_\delta$, $0 < \sigma \leq \delta$, and which joins $\sigma = 0$ to C_0. B_δ, C_0 and the θ-axis bound a domain D_δ in which $K'(\sigma)$ is continuous, is positive for $\sigma > 0$ and vanishes for $\sigma = 0$ and $\Psi_\delta \leq \Psi(\theta, \sigma) \leq \Psi(\Pi_1)$.

Hence Ψ is nondecreasing in all directions pointing into D_δ from B_δ; in particular,

$$\Psi_\sigma \frac{d\theta}{ds} - K\Psi_\theta \frac{d\sigma}{ds} = \sqrt{K}\left(\Psi_\mu \frac{d\theta}{ds} - \Psi_\theta \frac{d\mu}{ds}\right) \geq 0, \tag{15}$$

going counterclockwise on B_δ.

Consider the area integral I over any domain R with boundary S,

$$I = \iint_R (B\omega_\theta + C\omega_\sigma)(K\omega_{\theta\theta} + \omega_{\sigma\sigma})d\theta\, d\sigma .$$

By (1), the integral I vanishes. We shall show that over D_δ it can be made positive by a proper choice of the functions B and C, and that Ψ does not have a local maximum at X_1. This contradiction proves the lemma.

If B, C, ω_θ, ω_σ are continuous in R, then

$$\begin{aligned} I = &\iint_R \left\{\frac{1}{2}\omega_\theta^2[-KB_\theta + (KC)_\sigma] + \omega_\theta\omega_\sigma[B_\sigma + KC_\theta] + \frac{1}{2}\omega_\sigma^2[B_\theta - C_\sigma]\right\}d\theta\, d\sigma \\ &+ \oint_S B[K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma] + C(K\omega_\theta^2\, d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2\, d\theta) \quad . \end{aligned} \tag{16}$$

Let

$$\sqrt{K}C - iB = e^{\frac{1}{2}\pi i}(\lambda - \lambda(\Pi_1))^{-\nu}, \tag{17}$$

where $\lambda = \theta + i\mu$ and ν, given by (10), satisfies

$$\frac{\alpha_1}{1 - \alpha_1} < \nu < \frac{1}{1 - \alpha_1} . \tag{18}$$

Then B and C have the following properties which are easily deduced from (16):

$$B_\theta - (\sqrt{K}C)_\mu = B_\mu + (\sqrt{K}C)_\theta = 0, \tag{19}$$

(20) $$\sqrt{K}C = 0, \quad B < 0 \quad \text{on } \arg(\lambda - \lambda(\Pi_1)) = 0,$$

(21) $$\sqrt{K}C = 0 \text{ in } D_\delta \text{ for sufficiently small } \delta,$$

since $0 < \frac{\nu}{\pi} \arg(\lambda - \lambda(\Pi_1)) \sim \nu(1 - \alpha_1) < 1$ by (18) and (6).

From (20) and (21), or by direct computation, it follows that $(\sqrt{K}C)_\mu > 0$ for $\arg(\lambda - \lambda(\Pi_1)) = 0$, and hence that $\sqrt{K}C$ vanishes like μ, or that C vanishes like

$$(\sqrt{K})^{-1} \int_0^\sigma \sqrt{K}\, d\sigma,$$

that is,

(22) $$C = 0 \quad \text{on } \arg(\lambda - \lambda(\Pi_1)) = 0.$$

Along any curve S_{δ_1} which is the image in the (θ, σ)-plane of a circular arc in the λ-plane of radius δ_1 and center $\lambda(\Pi_1)$, we have

$$\left| \oint_{S_{\delta_1}} B((K\omega_\theta^2\, d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2\, d\sigma) + C(K\omega_\theta^2\, d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2\, d\theta) \right|$$

$$= \left| \Re \oint_{S_{\delta_1}} \sqrt{K}(\sqrt{K}C - iB)(\omega_\theta - i\omega_\mu)^2\, dz \right|$$

$$= \left| \Re \oint_{S_{\delta_1}} \sqrt{K} e^{\pi i/2} \delta_1^{1-\nu} e^{-i\nu \arg z} (\omega_\theta - i\omega_\mu)^2\, d(\arg z) \right|$$

$$< \oint_{S_{\delta_1}} \sqrt{K}\, \delta_1^{1-\nu} |\omega_\theta - i\omega_\mu|^2 d\, |\arg z| .$$

Here

$$z = \lambda - \lambda(\Pi_1).$$

Thus the line integral tends to zero as $\delta_1 \to 0$ since, by (10),

$$K^{1/2}\, \delta_1^{1-\nu} |\omega_\theta - i\omega_\mu|^2 \to 0 \quad \text{as } \delta_1 \to 0.$$

Note from (18) that if $\alpha > 1/2$, then to satisfy (10), $\omega_\theta - i\omega_\mu$ must vanish at Π_1 if K vanishes sufficiently slowly. In particular, if K vanishes like σ, which is the case for transonic flow problems, we find that if $|\omega_\theta - i\omega_\mu|^2 = O(\delta_1^{-4/3+\nu})$, where $\nu > \alpha_1/(1 - \alpha_1)$, then (10) is satisfied.

We want to apply (16) to the domain $R = D_\delta$. First, we apply (16) to the domain $R = D_{\delta\delta_1}$ bounded by B_δ, C_0, S_{δ_1} and the θ-axis, where δ_1 is much smaller than δ.

Then we let $\delta_1 \to 0$ and obtain by (22), using the boundary condition (11) on C_0,

$$I = \iint_{D_\delta} \left\{ \frac{1}{2}\omega_\theta^2[-KB_\theta + (KC)_\sigma] + \omega_\theta\omega_\sigma[B_\sigma + KC_\theta] + \frac{1}{2}\omega_\sigma^2[B_\theta - C_\sigma] \right\} d\theta\, d\sigma$$

$$+ \int_{B_\delta} C(\Psi_\sigma d\theta - K\Psi_\theta d\sigma) + \int_{C_0} (K\omega_\theta^2 + \omega_\sigma^2)(-Bd\sigma + Cd\theta) + \int_{\sigma=0} B\Psi_\theta\, d\theta .$$

Introducing the variable μ from (3), and letting (*) denote image in the (θ, μ)-plane, we find

$$I = \iint_{D_\delta^*} \left\{ \frac{\sqrt{K}}{2}\omega_\theta^2[-B_\theta + (\sqrt{K}C)_\mu + \frac{K'}{2K^{3/2}}\sqrt{K}C] + \omega_\theta\omega_\sigma[B_\mu + (\sqrt{K}C)_\theta] \right.$$

$$\left. + \frac{1}{2\sqrt{K}}\omega_\sigma^2[B_\theta - (\sqrt{K}C)_\mu + \frac{K'}{2K^{3/2}}\sqrt{K}C] \right\} d\theta\, d\mu$$

$$+ \int_{B_\delta^*} \sqrt{K}C(\Psi_\mu d\theta - \Psi_\theta d\mu) + \int_{C_0^*} \frac{1}{\sqrt{K}}(K\omega_\theta^2 + \omega_\sigma^2)(-Bd\mu + \sqrt{K}Cd\theta)$$

$$+ \int_{\sigma=0} B\Psi_\theta\, d\theta .$$

Using (19), we find

$$I = \iint_{D_\delta^*} \frac{K'}{4K^{3/2}}(K\omega_\theta^2 + \omega_\sigma^2)d\theta\, d\mu + \int_{B_\delta^*} \sqrt{K}C(\Psi_\mu d\theta - \Psi_\theta d\mu)$$

$$+ \int_{C_0^*} \frac{1}{\sqrt{K}}(K\omega_\theta^2 + \omega_\sigma^2)\,\Re(\sqrt{K}C - iB)d(\bar{\lambda} - \bar{\lambda}(\Pi_1)) + \int_{\sigma=0} B\Psi_\theta\, d\theta ,$$

where the bar indicates the complex conjugate.

The area integral is positive, since otherwise $\omega \equiv$ constant in D_δ and, by Lemma 1, $\omega \equiv$ constant in D. The first line integral, along B_δ^*, is positive by (15). The last integral is positive by (20) and Lemma 3, for δ sufficiently small. The integral along C_0^* is nonnegative, for

$$\Re(\sqrt{K}C - iB)d(\bar{\lambda} - \bar{\lambda}(\Pi_1)) = \Re\, iz^{-\nu}d\bar{z}$$

$$= |z|^{-\nu}\,\Re\, ie^{-i\nu \arg z}d|z|e^{-i \arg z} \sim |z|^{-\nu}\sin[(\nu + 1)\arg z]\, d|z| ,$$

by (3). By (6), we have $\arg z = j - \pi = (1 - \alpha_1)\pi$, and also by (18),

$$1 < (\nu + 1)(1 - \alpha_1) < 2 - \alpha_1 .$$

Therefore the sine term is negative. Finally on C_0^*, $d|z| < 0$.

Hence Ψ does not have a local maximum at Π_1 or, similarly, at Π_2.

4. THE COUNTER-EXAMPLE

In the counter-example we violate condition (6), so that Lemma 5 does not hold. The counter-example is given here for the case $K(\sigma) = \sigma$; but by means of the formulas given in [2] it could be generalized to any $K(\sigma)$ that vanishes like σ at $\sigma = 0$.

Consider the polynomial

$$\omega = \theta\sigma\left(1 + \frac{\sigma^3}{2}\right) - \theta^3\sigma - \frac{4}{3}\theta, \tag{24}$$

which satisfies (1) and (9). (For the general case we replace this polynomial by the corresponding formal polynomial.) There is a curve satisfying $\sigma\omega_\theta\, d\sigma - \omega_\sigma\, d\theta = 0$ passing through $(\pm 1, 0)$. For if we set $v_\sigma = -K\omega_\theta$ and $v_\theta = \omega_\sigma$, we see that the curves $K\omega_\theta\, d\sigma - \omega_\sigma\, d\theta = 0$ are the curves v = constant. We find

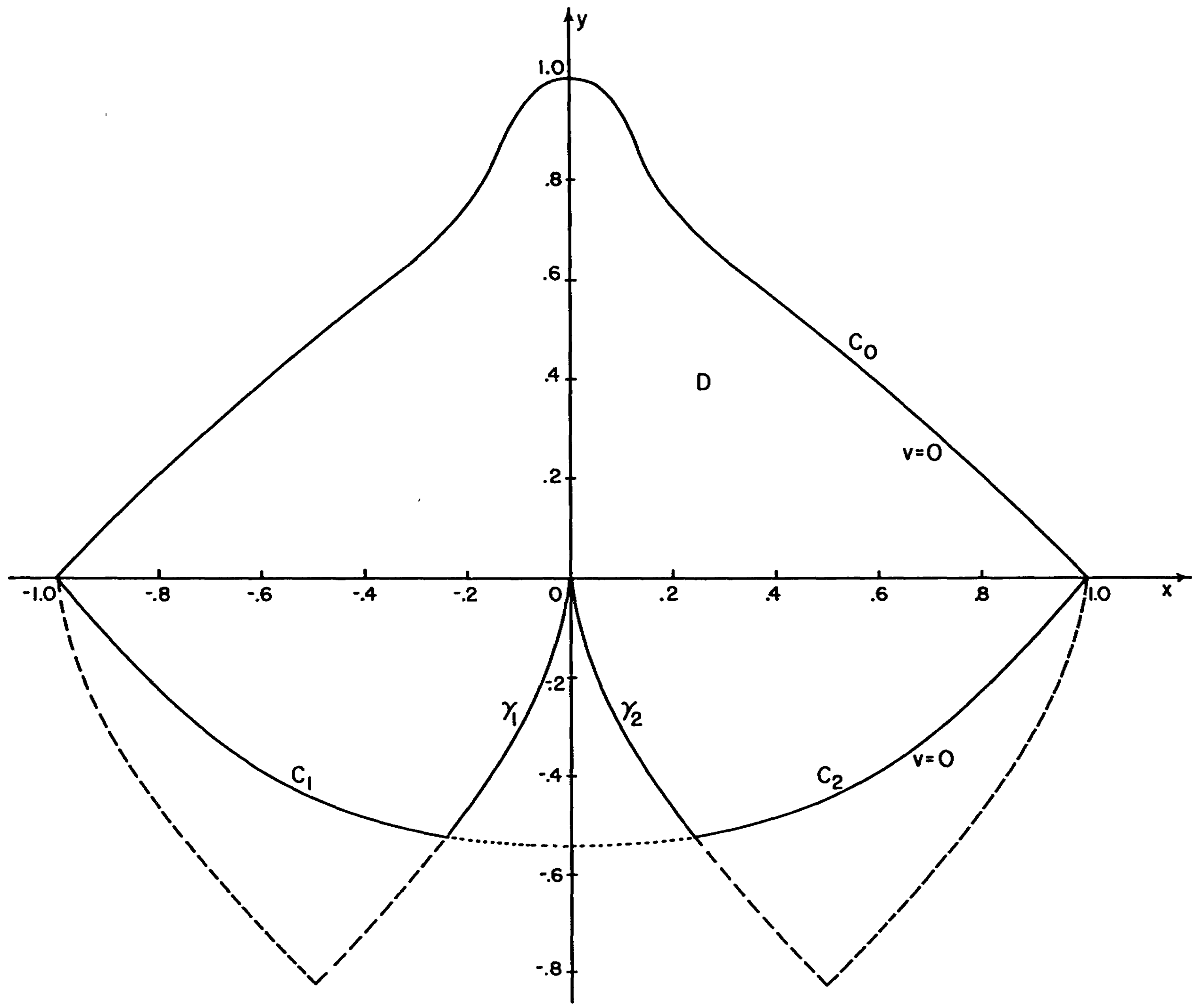

Figure 2

$$v = \frac{1}{4}(1 - \theta^2)^2 - \frac{2}{3}\sigma^2 + \sigma^3\left(\frac{1}{3} - \theta^2\right) + \frac{1}{12}\sigma^6. \tag{24}$$

We take for $C_0 + C_1 + C_2$ the portion of $v = 0$ cut out by the characteristics through the origin.

By expanding (25) about the neighborhood of the point $\Pi_1 = (-1, 0)$ and taking leading terms, it is not difficult to see that the curve $v = 0$, for $\theta < 0$, $\sigma \leq 0$, has slope greater than $-(\sqrt{-K})^{1/2}$ and less than zero. Therefore (2) is satisfied near Π_1, and by symmetry near Π_2.

A very lengthy computation would show that (2) is satisfied on the rest of the boundary $v = 0$; but a simple sketch (see Fig. 2) provides the simplest if not so rigorous way of showing that condition (2) also holds on the rest of the boundary. In addition, conditions (5) and (7) are satisfied.

Remark. It seems reasonable to conjecture that a general uniqueness theorem holds for arbitrary curves C_0, C_1 and C_2, provided (2) and (6) hold. Whether there will always exist a counter-example for arbitrary domains for which (6) does not hold is not clear.

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