ON THE PERRON-FROBENIUS THEOREM

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1. INTRODUCTION

The purpose of this note is to present another proof for the well-known theorem of Perron and Frobenius about matrices with positive elements, or rather, for the main part of it which says that *such a matrix has exactly one positive eigenvector* (that is, one whose coordinates are all positive). In the bibliography we list a few other proofs, including generalizations to function spaces. The proof given here is geometric in character, and quite elementary.

2. DEFINITIONS

Let E^n denote ordinary Euclidean n-space, with points $x = (x_1, \cdots, x_n)$, x_i real; let $A = (a_{ij})$ be an $n \times n$ matrix with real positive entries a_{ij}. We consider the linear transformation $T: E^n \to E^n$, defined by $T(x) = x' = (x_1', \cdots, x_n')$ with $x_i' = \Sigma_j a_{ij} x_j$. Let P denote the hyperplane $\{x: \Sigma x_i = 1\}$, and let S denote the (n - 1)-simplex consisting of those points of P all of whose coordinates are nonnegative, that is, the intersection of P with the positive orthant of E^n. The set S is compact and convex. The transformation T induces a transformation $\tilde{T}$ of S into itself, in an obvious way: for $x \in S$, we define $\tilde{T}(x)$ to be the point of intersection of S with the T-image of the straight line through the origin of E^n and x. It is easily verified that $\tilde{T}$ is well defined, because of the positivity of A, and that in fact $\tilde{T}(S)$ is contained in the interior S^0 of S; that is, from $x \in S$ and $x' = \tilde{T}(x)$ it follows that $x_i' > 0$ $(i = 1, \cdots, n)$. Moreover, it is clear (from considerations familiar in projective geometry) that $\tilde{T}$ is continuous and that it preserves collinearity and cross ratio.

3. THE CAYLEY METRIC

We set up a Cayley metric in the interior S^0 of S by defining, as usual, the distance d(x, y) between two distinct points x and y of S^0 to be the logarithm of the cross ratio $CR(x, y, a_1, a_2)$ of the four points x, y, a_1, a_2, where a_1, a_2 are the two points in which the line from y to x meets the boundary B of S (in the order a_1, y, x, a_2); and by defining that d(x, x) = 0. It is clear that the function d(x, y) is continuous (simultaneously in x and y), and that it is positive except for x = y. For a proof of the fact that the function d(x, y) is actually a metric, see [2, p. 158].

We now state a lemma about the cross ratio of points on a line; its proof is elementary.

LEMMA. *Let* c_1, d_1, y, x, d_2, c_2 *be six points on a (real) line in the order indicated, with* $d_1 \neq y \neq x \neq d_2$; *then* $CR(x, y, c_1, c_2) \leq CR(x, y, d_1, d_2)$, *and equality occurs only if* $c_1 = d_1$ *and* $c_2 = d_2$.

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4. PROOF OF THE THEOREM

Let x, y be two points of S^0. We claim that

$$(*) \qquad d(\tilde{T}(x), \tilde{T}(y)) < d(x, y)$$

if $x \neq y$; that is, $\tilde{T}$ is a "contraction."

Proof. As before, let a_1, a_2 denote the intersections with B of the line from y to x; let b_1, b_2 be similarly determined by the line from $\tilde{T}(y)$ to $\tilde{T}(x)$. By the projective invariance of the cross ratio, we have

$$d(x, y) = \log CR(\tilde{T}(x), \tilde{T}(y), \tilde{T}(a_1), \tilde{T}(a_2)).$$

The six points b_1, $\tilde{T}(a_1)$, $\tilde{T}(y)$, $\tilde{T}(x)$, $\tilde{T}(a_2)$, b_2 then have the order indicated on the line from $\tilde{T}(y)$ to $\tilde{T}(x)$; moreover, by the remark at the end of §2, we have $b_1 \neq \tilde{T}(a_1)$, $b_2 \neq \tilde{T}(a_2)$. Inequality (*) now follows from the lemma.

Consider now the decreasing sequence S, $\tilde{T}(S)$, $\tilde{T}(\tilde{T}(S)) = \tilde{T}^2(S)$, ⋯ of the iterated images under $\tilde{T}$ of S; let Δ be the intersection of these sets. It is clear that Δ is a nonempty compact set, and that it is contained in the interior S^0 [the last property follows from the fact that $\tilde{T}(S) \subset S^0$]. It is also clear that $\tilde{T}(\Delta) = \Delta$, i.e., that Δ is invariant under $\tilde{T}$, since $\bigcap_0^\infty \tilde{T}^i(S) = \bigcap_1^\infty \tilde{T}^i(S)$. (Incidentally, as intersection of a decreasing sequence of simplices, Δ itself is a simplex.) We claim that Δ consists of a single point. Otherwise there exist two points x_0, y_0 of Δ with maximum distance (by continuity of d and compactness of Δ):

$$d(x_0, y_0) = \max_{x,y \in \Delta} d(x, y).$$

Since $\tilde{T}(\Delta) = \Delta$, there exist $x_1, y_1 \in \Delta$ with $\tilde{T}(x_1) = x_0$, $\tilde{T}(y_1) = y_0$. But then, using the inequality (*), we have the contradiction

$$d(x_0, y_0) < d(x_1, y_1) \leq \max_{x,y \in \Delta} d(x, y) = d(x_0, y_0).$$

We have shown that $\tilde{T}$ has a fixed point, namely Δ; the relation $\bigcap_0^\infty \tilde{T}^i(S) = \Delta$ implies that for any point $x \in S$ the sequence $\tilde{T}^i(x)$ of iterated images converges to Δ ($\tilde{T}^i(x) \in \tilde{T}^i(S)$; compactness of the $\tilde{T}^i(S)$ implies that for any neighborhood V of Δ there exists a natural number i_0 with $\tilde{T}^{i_0}(S) \subset V$), so that no other point of S is fixed under $\tilde{T}$. Since fixed points of $\tilde{T}$ correspond to eigenvectors of T, and points of S^0 correspond to positive vectors, we have proved the theorem, with the sharpening that for any nonnegative vector different from 0 the sequence of iterates under T (normalized, e.g., by $\Sigma x_i = 1$) converges to the unique positive (normalized) eigenvector; it is also clear that the eigenvalue corresponding to this eigenvector is positive.

BIBLIOGRAPHY

1. P. Alexandroff and H. Hopf, *Topologie I*, Berlin, 1935, p. 480.
2. H. Busemann and P. J. Kelly, *Projective geometry and projective metrics*, New York, 1953.
3. G. Frobenius, *Über Matrizen aus positiven Elementen I, II*, S.-B. kgl. Preuss. Akad. Berlin, (1908), 471-476; (1909), 514-518.
4. R. Jentzsch, *Über Integralgleichungen mit positivem Kern*, J. Reine Angew. Math. 141 (1912), 235-244.
5. O. Perron, *Zur Theorie der Matrices*, Math. Ann. 64 (1907), 248-263.
6. M. A. Rutmann, *Sur les opérateurs totalement continus linéaires laissant invariant un certain cône*, Mat. Sbornik N.S. 8 (50), 77-96 (1940).
7. H. Wielandt, *Unzerlegbare, nicht negative Matrizen*, Math. Z. 52 (1950), 642-648.
8. J. L. Ullman, *On a theorem of Frobenius*, Michigan Math. J. 1 (1952), 189-193.

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