ON CERTAIN CONFORMAL MAPS IN SPACE

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It is well known that a differentiable homeomorphism with constant unit dilatation, defined in a two-dimensional domain, is either an analytic function of a complex variable, or else the conjugate of such a function [1; p. 1]. Studying three-dimensional quasi-conformal maps, the present author has tried to prove an analogue of this; but He has succeeded only in establishing the following result.

THEOREM. *Let* $y = y(x)$ *denote a conformal, differentiable homeomorphism, with positive Jacobian, of the open ball* $||x|| < 1$ *onto* $||y|| < 1$, *and let* $y(0) = 0$. *Then* $y(x)$ *is a rotation.*

Proof. The first quadratic form of the mapping is of the form $d\sigma^2 = \lambda^2 ds^2$ [2]. Hence the volume $V(R)$ and the area $S(R)$ of the image of $||x|| \leq R$ and $||x|| = R$, respectively, are given by

$$V(R) = \iiint\limits_{||x|| \leq R} \lambda^3 dV, \qquad S(R) = \iint\limits_{||x|| = R} \lambda^2 dS,$$

respectively. It follows at once from the isoperimetric inequality $36\pi V^2 \leq S^3$ [5; p. 530] that we have

$$\left[\frac{3}{4\pi R^3} \iiint\limits_{||x|| \leq R} \lambda^3 dV\right]^2 \leq \left[\frac{1}{4\pi R^2} \iint\limits_{||x|| = R} \lambda^2 dS\right]^3. \tag{1}$$

A judicious application of the Hölder inequality to the right-hand side of (1) yields

$$\left[\frac{3}{4\pi R^3} \iiint\limits_{||x|| \leq R} \lambda^3 dV\right]^2 \leq \left[\frac{1}{4\pi R^2} \iint\limits_{||x|| = R} \lambda^3 dS\right]^2. \tag{2}$$

The relation (2) gives us the following inequality between the volume and spherical averages:

$$\frac{3}{4\pi R^3} \iiint\limits_{||x|| \leq R} \lambda^3 dV \leq \frac{1}{4\pi R^2} \iint\limits_{||x|| = R} \lambda^3 dS. \tag{3}$$

Moreover, the analogue of (3) clearly holds for all spheres $||x - x_0|| \leq R$ lying in $||x|| < 1$; we conclude that $\lambda^3(x)$ is a subharmonic function, for $||x|| < 1$ [4; p. 7]. Therefore we have

$$\lambda^3(0) \leq \frac{3}{4\pi R^3} \iiint\limits_{||x|| \leq R} \lambda^3 dV. \tag{4}$$

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Now it is well known that the average on the right-hand side of (4) is nondecreasing with R [4; p. 8]. Hence, since we have a homeomorphism from $||x|| < 1$ onto $||y|| < 1$, we obtain $\lambda^3(0) \leq 1$ from (4), by letting $R \to 1$.

But we could also have considered the mapping of $||y|| < 1$ onto $||x|| < 1$, with differential form $ds^2 = d\sigma^2/\lambda^2$, and we would have found that $1/\lambda^3(0) \leq 1$. Hence we conclude that $\lambda(0) = 1$. Since the right-hand member of (4) is a nondecreasing function of R, and since $||x|| < 1$ is mapped onto $||y|| < 1$, we find

$$(5) \qquad 1 = \lambda^3(0) = \frac{3}{4\pi R^3} \iiint_{||x|| \leq R} \lambda^3 \, dV \leq 1 .$$

Because $\lambda^3(x)$ is subharmonic, (5) implies that $\lambda^3(x)$ is harmonic for $||x|| < 1$ [4; p. 6].

If we again apply the Hölder inequality to the isoperimetric inequality, and if we make use of the harmonic character of $\lambda^3(x)$, then we obtain

$$\lambda^2(x) = \frac{1}{4\pi R^2} \iint_{||x - x_0|| = R} \lambda^2 \, dS$$

for all spheres $||x - x_0|| = R$ lying in $||x|| < 1$. Hence $\lambda^2(x)$ is harmonic for $||x|| < 1$ [4; p.6].

Since $\lambda^2(x)$ and $\lambda^3(x)$ are both harmonic, a simple computation shows that $\lambda(x) \equiv 1$.

The preceding result is an analogue of a classical theorem of Liouville [2; p. 487], as well as an analogue of Schwarz's Lemma. It is related to a recent result due to Lavrentiev, who obtained a similar result for general image domains, but under more severe smoothness conditions on the mapping $y = y(x)$ [3].

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