MEAN-VALUES AND POLYHARMONIC POLYNOMIALS

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INTRODUCTION. It is well known that a summable function, defined in a two-dimensional domain, is harmonic if and only if it satisfies the Gauss Mean-Value Theorem. An analogous statement holds for the polyharmonic case. A function $u(x, y)$ having $2p$ continuous derivatives in a domain D is p-harmonic in D if and only if, for every $(\alpha, \beta) \in D$ and for sufficiently small positive R,

$$\frac{1}{\pi R^2} \iint_{S_R(\alpha,\beta)} u(x, y)\,dx\,dy = u(\alpha, \beta) + \sum_{j=1}^{p-1} \frac{R^{2j}}{(2^j j!)^2 (j + 1)} \triangle^j u(\alpha, \beta), \tag{0.1}$$

where $S_R(\alpha, \beta)$ is a circle with center (α, β) and radius R.

Denote by $[(\alpha, \beta), n, R, \phi]$ a regular n-polygon with center (α, β) and vertices

$$\left(\alpha + h \cos\left(\phi + \frac{2\pi}{n}k\right), \beta + h \sin\left(\phi + \frac{2\pi}{n}k\right)\right) \qquad (k = 1, \cdots, n),$$

where $R = h \cos \pi/n$. We say that a summable function $u(x, y)$ possesses the property (p, n, ϕ) in a domain D if, for every $(\alpha, \beta) \in D$, the partial derivatives $\triangle^j u(\alpha, \beta) (j = 1, \cdots, p - 1)$ exist, and if for sufficiently small R

$$\frac{1}{sR^2} \iint_{[(\alpha,\beta),n,R,\phi]} u(x, y)\,dx\,dy = u(\alpha, \beta) + \sum_{j=1}^{p-1} \frac{\gamma_{j,n} R^{2j} \triangle^j u(\alpha, \beta)}{(2^j j!)^2 (j + 1)}, \tag{0.2}$$

where s is the area of $[(0, 0), n, 1, 0]$ and the $\gamma_{j,n}$ are defined as follows:

$$\begin{aligned} \gamma_{1,n} &= \frac{2}{3} + \frac{1}{3 \cos^2 \pi/n}, \\ \gamma_{k,n} &= \frac{2k\gamma_{k-1,n}}{2k + 1} + \frac{1}{(2k + 1)\cos^{2k} \pi/n}. \end{aligned} \tag{0.3}$$

The property (p, n, ϕ) involves double integrals taken over regular polygons. Similarly, we define the properties $(p, n, \phi)'$ and $(p, n, \phi)''$ which involve, respectively, line integrals taken over the edges of regular polygons, and arithmetic means taken over the vertices of regular polygons. In the case of $(p, n, \phi)'$, condition (0.2) is replaced by

$$\frac{1}{\sigma R} \int_{\sigma_R} u(x, y)\,d\sigma = u(\alpha, \beta) + \sum_{j=1}^{p-1} \frac{\gamma_{j,n} R^{2j} \triangle^j u(\alpha, \beta)}{(2^j j!)^2}, \tag{0.4}$$

where $\sigma = 2s$. In the case $(p, n, \phi)''$, condition (0.2) is replaced by

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$$\frac{1}{n}\sum_{k=1}^{n} u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta) = \sum_{j=1}^{p-1} \frac{R^{2j} \triangle^j u(\alpha, \beta)}{(2^j j!)^2 \cos^{2j} \pi/n}. \tag{0.5}$$

The reason for the definition (0.3) will be explained in Remark a) to Theorem 2.

In this paper we discuss the problem of characterizing functions possessing one of the properties (p, n, ϕ), $(p, n, \phi)'$, $(p, n, \phi)''$. The special case $p = 1$ is already known. Beckenbach and Reade [1] have proved that a summable (continuous) function $u(x, y)$ possesses the property $(1, n, \phi)$ ($(1, n, \phi)'$) if and only if $u(x, y)$ is a harmonic polynomial of degree n and the nth derivative of $u(x, y)$ in the ϕ-direction vanishes. Walsh [4] has proved the same result for continuous functions possessing the property $(1, n, \phi)''$. The methods used in [1] and in [4] are quite different. We shall prove all the above-mentioned results by a new method which is more general. Afterwards we shall apply it to discuss the general case of (p, n, ϕ), $(p, n, \phi)'$, $(p, n, \phi)''$. The problem of characterizing polyharmonic polynomials was attacked also by Reade [3]. He introduced a mean-value property which involves integration over k regular n-gons which are homothetic, and he showed that this property characterizes k-harmonic polynomials of degree and form which depend on k and n.

1. THEOREM 1. *A necessary and sufficient condition for a summable function* $u(x, y)$ *defined in a domain* D *to possess the property* $(1, n, \phi)$ *is that* $u(x, y)$ *should be a harmonic polynomial of degree at most* n *and that the* nth *derivative of* $u(x, y)$ *in the* ϕ*-direction should vanish.*

It is sufficient to prove the theorem for the case $\phi = 0$, since the general case follows by rotation. Suppose $u(x, y)$ to possess the property $(1, n, 0)$. Then

$$u(\alpha, \beta) = \frac{1}{sR^2}\iint_{S_R} u(x, y)\,dx\,dy, \tag{1.1}$$

where $S_R = [(\alpha, \beta), n, R, 0]$. From (1.1) it is clear that $u(x, y)$ is infinitely differentiable. Applying the operator $\triangle$ to both sides of (1.1) as functions of (α, β), and using Green's Identity, we get

$$\triangle u(\alpha, \beta) = \frac{1}{sR^2}\int_{\sigma_R} \frac{\partial u}{\partial \nu}\,d\sigma, \tag{1.2}$$

where σ_R is the boundary of S_R and ν is the outwardly directed normal.

Writing (1.1) in the form

$$sR^2 u(\alpha, \beta) = \iint_{S_R} u(x, y)\,dx\,dy, \tag{1.3}$$

and differentiating twice with respect to R, we get

$$2su(\alpha, \beta) = \int_{\sigma_R} \frac{\partial u}{\partial \nu}\,d\sigma + 2\tan\frac{\pi}{n}\sum_{k=1}^{n} u(\alpha + x_k, \beta + y_k), \tag{1.4}$$

where

(1.5) $$x_k = h\cos\frac{2\pi}{n}k, \qquad y_k = h\sin\frac{2\pi}{n}k, \qquad R = h\cos\frac{\pi}{n}.$$

Noting that $s = n\tan \pi/n$ and comparing (1.2) with (1.4), we obtain

(1.6) $$\triangle u(\alpha, \beta) + \frac{2}{nR^2}\left[\sum_{k=1}^{n} u(\alpha + x_k, \beta + y_k) - nu(\alpha, \beta)\right] = 0.$$

By Taylor's Theorem,

(1.7) $$\sum_{k=1}^{n} [u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta)] = \sum_{\lambda=1}^{n} \frac{h}{\lambda!} A_\lambda u(\alpha, \beta) + O(h^{n+1}),$$

where

(1.8) $$A_\lambda u = \sum_{k=1}^{n} \left(\cos\frac{2\pi k}{n}\frac{\partial}{\partial x} + \sin\frac{2\pi k}{n}\frac{\partial}{\partial y}\right)^\lambda u.$$

Since $A_1 u = 0$ and $A_2 u = \frac{n}{2}\triangle u$, we obtain from (1.6)

$$\triangle u + \frac{2}{nR^2}\left(\frac{h^2}{2}\triangle u + O(h^3)\right) = 0,$$

so that $\triangle u = 0$. Therefore $u(x, y)$ is harmonic. We can write

(1.9) $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}.$$

Now $A_\lambda u$ is a sum of expressions of the form

$$\binom{\lambda}{\mu}\frac{\partial^\mu}{\partial x^\mu}\frac{\partial^{\lambda-\mu}}{\partial y^{\lambda-\mu}} u \sum_{k=1}^{n} \cos^\mu\frac{2\pi k}{n} \sin^{\lambda-\mu}\frac{2\pi k}{n}.$$

By symmetry it is obvious that this expression vanishes for $\lambda - \mu$ odd. For $\lambda - \mu$ even, we may (by (1.9)) substitute

$$\frac{\partial^{\lambda-\mu} u}{\partial y^{\lambda-\mu}} = i^{\lambda-\mu}\frac{\partial^{\lambda-\mu} u}{\partial x^{\lambda-\mu}}.$$

Therefore we obtain

(1.10) $$A_\lambda u = \sum_{k=1}^{n} \left(\cos\frac{2\pi k}{n}\frac{\partial}{\partial x} + i\sin\frac{2\pi k}{n}\frac{\partial}{\partial x}\right)^\lambda u = \left(\sum_{k=1}^{n} e^{\frac{2\pi\lambda i}{n}k}\right)\frac{\partial^\lambda u}{\partial x^\lambda}.$$

Since by (1.6) and (1.7) each $A_\lambda u$ must vanish, we have in particular $A_n u = 0$, or

$$\frac{\partial^n u(x, y)}{\partial x^n} = 0. \tag{1.11}$$

We conclude that $u(x, y) = \sum_{k=0}^{n-1} a_k(y)x^k$. Now by (1.9) and (1.11),

$$\frac{\partial^n u(x, y)}{\partial x^{n-2j}\partial y^{2j}} = 0,$$

and we easily see that $a_k(y)$ is a polynomial of degree at most $n - k$. We have established one part of the theorem.

Suppose now that $u(x, y)$ is a harmonic polynomial of degree at most n, and that it satisfies (1.11). We shall prove that it possesses the property $(1, n, 0)$.

It is sufficient to prove (1.1), or (1.3). Both sides of (1.3) vanish, at $R = 0$, together with their first derivatives. Therefore it is sufficient to prove that their second derivatives with respect to R are equal; that is, to prove (1.4). Since $u(x, y)$ is harmonic, and $s = n \tan \pi/n$, it remains to prove that

$$I \equiv \sum_{k=1}^{n} [u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta)] = 0.$$

But since u is a polynomial of degree at most n, (1.7), (1.8), (1.9) and (1.11) imply that

$$I = \sum_{\lambda=1}^{n} \frac{h^\lambda}{\lambda!} A_\lambda u = \frac{\partial^n u}{\partial x^n} = 0.$$

This completes the proof.

Remark. The analogous theorems concerning the properties $(1, n, \phi)'$, $(1, n, \phi)''$, can be proved by the same method. In the case of $(1, n, \phi)'$, the only change is that we differentiate the formula analogous to (1.1) only once with respect to R (and not with respect to (α, β)), and then use Green's Identity. In the case of $(1, n, \phi)''$, the formula analogous to (1.1) takes the form

$$\sum_{k=1}^{n} [u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta)] = 0.$$

It is interesting to observe that in this case our method can be generalized to 3 dimensions. The only change is in the Taylor series corresponding to $\frac{1}{n}\sum_{k=1}^{n} u(P_k) - u(P)$, where the P_k are the vertices of a regular solid with center P. Indeed, this was already done by Beckenbach and Reade [2].

2.1. We shall need the following

LEMMA. *For any numbers* a, b *and for any positive integer* n, *the following identities hold:*

$$(2.1)\qquad \sum_{k=1}^{n}\left(a\cos\frac{2\pi}{n}k + b\sin\frac{2\pi}{n}k\right)^{\ell} = \begin{cases} \frac{n}{4^m}\binom{2m}{m}(a^2+b^2)^m & (0 < \ell = 2m < n), \\ 0 & (0 < \ell = 2m-1 < n). \end{cases}$$

Consider first the case $\ell = 2m$. By symmetry,

$$\sum_{k=1}^{n} \cos^{2m-2\lambda-1}\frac{2\pi k}{n}\sin^{2\lambda+1}\frac{2\pi k}{n} = 0,$$

and using the binomial theorem, we see that it suffices to prove

$$(2.2)\qquad \binom{2m}{2\lambda}\sum_{k=1}^{n}\cos^{2m-2\lambda}\frac{2\pi k}{n}\sin^{2\lambda}\frac{2\pi k}{n} = \frac{n}{4^m}\binom{2m}{m}\binom{m}{\lambda}.$$

The proof of (2.2) is by induction on m $(1 < 2m < n)$. Assuming its truth for $m - 1$, we prove its truth for m by induction on λ. For $\lambda = 0$, (2.2) reduces to

$$(2.3)\qquad \sum_{k=1}^{n}\cos^{2m}\frac{2\pi k}{n} = \frac{n}{4^m}\binom{2m}{m}.$$

The truth of (2.3) follows from the well-known trigonometric formulas

$$\cos^{2m}x = \frac{1}{4^m}\left\{\sum_{k=0}^{m-1} 2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m}\right\},$$

$$\sum_{k=1}^{n}\cos\frac{2\pi j}{n}k = 0 \qquad (0 < j < n).$$

Assuming the truth of (2.2) for $\lambda - 1$ instead of λ, we can easily prove it for λ by substituting

$$\sin^{2\lambda}\frac{2\pi k}{n} = \sin^{2\lambda-2}\frac{2\pi k}{n}\left(1 - \cos^2\frac{2\pi k}{n}\right).$$

The case $\ell = 2m - 1$ is treated similarly. The only change is that $\sum_{k=1}^{n}\cos^{2m-1}\frac{2\pi k}{n} = 0.$

2.2. THEOREM 2. *Let* u(x, y) *be a function defined in a domain* D, *continuous with its partial derivatives of the first* 2p *orders, and possessing the property* (p, n, ϕ), *with* $n > 2p$. *Then* u(x, y) *is a* p-*harmonic polynomial of degree at most* pn, *and the* pn-*th derivative of* u(x, y) *in the* ϕ-*direction vanishes.*

As in the previous theorem, it is enough to give the proof for the case $\phi = 0$. We prove the theorem by induction. The case $p = 1$ is a part of Theorem 1. Assume that the hypothesis of Theorem 2, with p replaced by $p - 1$, imply that u is a $(p - 1)$-harmonic polynomial of degree $(p - 1)n$ and that

$$(2.4)\qquad \frac{\partial^{(p-1)n}u(x, y)}{\partial x^{(p-1)n-2j}\partial y^{2j}} = 0 \qquad (0 \leq 2j \leq (p-1)n).$$

Let $(\alpha, \beta) \in D$; then

$$(2.5)\qquad \frac{1}{sR^2}\iint_{S_R} u(x, y)\,dx\,dy = u(\alpha, \beta) + \sum_{j=1}^{p-1}\frac{\gamma_{j,n}R^{2j}\Delta^j u(\alpha, \beta)}{(2^j j!)^2(j+1)}.$$

Applying Δ to (2.5), we get

$$(2.6)\qquad \frac{1}{sR^2}\int_{\sigma_R}\frac{\partial u}{\partial \nu}d\sigma = \Delta u(\alpha, \beta) + \sum_{j=1}^{p-1}\frac{\gamma_{j,n}R^{2j}\Delta^{j+1}u(\alpha, \beta)}{(2^j j!)^2(j+1)}.$$

Multiplying (2.5) by R^2 and differentiating both sides twice with respect to R, we get

$$(2.7)\qquad \begin{aligned}&\frac{1}{sR^2}\int_{\sigma_R}\frac{\partial u}{\partial \nu}d\sigma + \frac{2}{nR^2}\sum_{k=1}^{n} u(\alpha + x_k, \beta + y_k)\\ &= \frac{2}{R^2}u(\alpha, \beta) + \frac{1}{R^2}\sum_{j=1}^{p-1}\frac{2(2j+1)\gamma_{j,n}R^{2j}\Delta^j u(\alpha, \beta)}{(2^j j!)^2}.\end{aligned}$$

Comparing (2.6) and (2.7), we obtain

$$(2.8)\qquad \begin{aligned}0 &= \Delta u(\alpha, \beta) + \sum_{j=1}^{p-1}\frac{\gamma_{j,n}R^{2j}\Delta^{j+1}u(\alpha, \beta)}{(2^j j!)^2(j+1)} - \sum_{j=1}^{p-1}\frac{2(2j+1)\gamma_{j,n}R^{2j-2}\Delta^j u(\alpha, \beta)}{(2^j j!)^2}\\ &\quad + \frac{2}{nR^2}\sum_{k=1}^{n}[u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta)].\end{aligned}$$

By Taylor's Theorem and the lemma,

$$(2.9)\qquad \sum_{k=1}^{n}[u(\alpha + x_k, \beta + y_k) - u(\alpha, \beta)] = \sum_{\lambda=1}^{p}\frac{h^{2\lambda}}{(2\lambda)!}\frac{n}{4^\lambda}\binom{2\lambda}{\lambda}\Delta^\lambda u(\alpha, \beta) + o(h^{2p}).$$

Here we have used the assumptions that $n > 2p$ and that $u(x, y)$ possesses continuous partial derivatives of the first $2p$ orders.

Substituting (2.9) in (2.8) and equating to zero the coefficients of $R^{2\lambda}$, we get the identities

$$(2.10)\qquad \begin{aligned}&\Delta u\left(1 - \frac{3}{2}\gamma_{1,n} + \frac{2}{\cos \pi/n}\right) = 0,\\ &\Delta^k u\left(\frac{\gamma_{k-1,n}}{(2^{k-1}(k-1)!)^2 k} - \frac{2(2k+1)\gamma_{k,n}}{(2^k k!)^2} + \frac{2}{(2k)!\,4^k\cos^{2k}\pi/n}\binom{2k}{k}\right) = 0 \quad (p > k),\end{aligned}$$

$$\triangle^p u\left(\frac{\gamma_{p-1,n}}{(2^{p-1}(p-1)!)^2\,p} + \frac{2}{(2p)!\,4^p \cos^{2p}\pi/n}\binom{2p}{p}\right) = 0. \tag{2.11}$$

From the definitions in (0.3) we see that the brackets in (2.10) vanish. From (2.11) we conclude that $\triangle^p u = 0$. Applying $\triangle$ to (2.5), we obtain

$$\frac{1}{sR^2}\iint_{S_R} \triangle u(x, y)\,dx\,dy = \triangle u(\alpha, \beta) + \sum_{j=1}^{p-2} \frac{\gamma_{j,n} R^{2j} \triangle^{j+1} u(\alpha, \beta)}{(2^j j!)^2 (j+1)}.$$

By the inductive assumptions, we conclude that

$$\frac{\partial^{(p-1)n} \triangle u(x, y)}{\partial x^{(p-1)n-2j} \partial y^{2j}} = 0 \qquad (0 \leq 2j \leq (p-1)n). \tag{2.12}$$

From (2.12) and the identity

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} + \triangle u$$

it follows that

$$\frac{\partial^{(p-1)n+2} u}{\partial x^{(p-1)n-2j} \partial y^{2j+2}} = -\frac{\partial^{(p-1)n+2} u}{\partial x^{(p-1)n-2j+2} \partial y^{2j}}. \tag{2.13}$$

Therefore we may substitute $\frac{\partial}{\partial y} = i\frac{\partial}{\partial x}$ in $A_\lambda u$ (for $\lambda \geq (p-1)n + 2$), and from (1.10) we have

$$A_{pn} u = n\frac{\partial^{pn} u}{\partial x^{pn}}.$$

From (2.8) we see that $A_{pn} u = 0$, so that $\frac{\partial^{pn} u}{\partial x^{pn}} = 0$. Applying (2.13), we get

$$\frac{\partial^{pn} u(x, y)}{\partial x^{pn-2j} \partial y^{2j}} = 0. \tag{2.14}$$

This implies that $u(x, y)$ is a polynomial of degree at most pn, and the proof is complete.

Remarks. a) It is clear from the proof (see formulas (2.10), (2.11)) that if $u(x, y)$ satisfies, at every $(\alpha, \beta) \in D$,

$$\frac{1}{sR^2}\iint_{S_R} u(x, y)\,dx\,dy = u(\alpha, \beta) + \sum_{j=1}^{p-1} \frac{\delta_{j,n} R^{2j} \triangle^j u(\alpha, \beta)}{(2^j j!)^2 (j+1)}, \tag{2.15}$$

and if

$$\delta_{1,n} = \gamma_{1,n}, \cdots, \delta_{k-1,n} = \gamma_{k-1,n}, \delta_{k,n} \neq \gamma_{k,n},$$

then $\triangle^k u(x, y) = 0$ and (2.15) reduces to

$$\frac{1}{sR^2}\iint_{S_R} u(x, y)\,dx\,dy = u(\alpha, \beta) + \sum_{j=1}^{k-1} \frac{\gamma_{j,n} R^{2j} \triangle^j u(\alpha, \beta)}{(2^j j!)^2 (j+1)} .$$

b) The analogous theorems concerning the properties $(p, n, \phi)'$, $(p, n, \phi)''$ can be proved in the same way.

To prove the analogue of Theorem 2 for the case of $(p, n, \phi)''$, observe first that (2.9) implies $\triangle^p u = 0$. We now apply $\triangle$ to both sides of (0.5) and use the inductive assumptions (including (2.4)). Thus we conclude that u(x, y) is a p-harmonic polynomial of degree at most pn, and the pn-th derivative of u(x, y) in the ϕ-direction vanishes.

Instead of proving directly the analogue of Theorem 2 for $(p, n, \phi)'$, we show that the properties (p, n, ϕ), $(p, n, \phi)'$ are equivalent. Indeed, multiplying (0.4) by R and differentiating with respect to R, we obtain an equivalent formula which coincides with (2.7). Since (2.7) and (2.5) are equivalent, our proof is complete.

2.3. a) The converse of Theorem 2 is not true. This is shown, for the case $p = 2$, $n = 6$, by the counter-example

$$u(x, y) = x^6 + (5a - 10)x^4y^2 + (5 - 10a)x^2y^4 + ay^6 \qquad (a \neq 1).$$

Here u(x, y) is a biharmonic polynomial of degree $n = 6$; but it does not possess the property (2, 6, 0) since, as can be verified directly, (2.8) is not satisfied.

b) We give an example for the case $p = 2$, $n = 6$, of a biharmonic polynomial of degree $n + 2 = 8$ which possesses the property (2, 6, 0):

$$u(x, y) = 3x^7y - 7x^5y^3 - 7x^3y^5 + 3xy^7 .$$

Instead of proving (2.5) for u(x, y), it is sufficient to prove (2.7), or to prove (2.6) and (2.8).

Now, since $\triangle u$ is a harmonic polynomial of degree 6 and $\partial^5 \triangle u / \partial \triangle x^6 = 0$, it follows from Theorem 1 that (2.6) holds. (2.8) can be verified directly.

c) The assumption $n > 2p$ made in Theorem 2 is really necessary. This is shown by the following counter-example:

$$p = 2, \quad n = 4, \quad \phi = \pi/4, \qquad u(x, y) = x^5y - x^3y^3 .$$

The polynomial u(x, y) possesses the property $(2, 4, \pi/4)$, as can be verified directly; but it is not biharmonic.

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