INVERSION OF TWO THEOREMS OF ARCHIMEDES

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Archimedes proved the following theorems:

THEOREM I. *The curved surface of a cylinder circumscribed about a sphere is equal to the surface of the sphere.*

THEOREM II. *The volume of a cylinder circumscribed about a sphere is* 3/2 *times that of the sphere.*

We shall invert these theorems as follows:

THEOREM 1. *If the curved surface of each right cylinder circumscribed about a convex body* K *is equal to the surface of* K, *then* K *is a sphere.*

THEOREM 2. *If the volume of each right cylinder circumscribed about a convex body* K *is* 3/2 *times that of* K, *then* K *is a sphere.*

We shall first deal with Theorem 1. Let L be the length of the closed convex curve K which is obtained by an orthogonal projection of K; let B denote the breadth of K in the direction of the projection; and let dw be the solid angle element of the directions of projection. We consider the integral

(1) $$J = \int LB\,dw,$$

extended over the whole unit sphere. It is known that L can be represented by the integral

(2) $$\int_0^{2\pi} p\,d\phi,$$

where p denotes the support function of the projection of K, and where $d\phi$ denotes the angle element of its tangent. Therefore (1) can be written in the form

(3) $$J = \int\left(\int p\,d\phi\right)B\,dw$$

Now we take advantage of the concepts of integral geometry [1]. The symbol $d\phi\,dw$ represents the density of the configuration consisting of one space direction D_1 and a direction D_2 perpendicular to D_1. Since this configuration has no invariant under motions, we have

(4) $$d\phi\,dw = C\,d\bar{\phi}\,d\bar{w}$$

where $d\bar{w}$ is the solid angle element of the normals of the plane $D_1 D_2$, and where $\bar{\phi}$ measures the rotations in this plane. C is easily shown to be 1 (for instance, by integration over the whole unit sphere). Hence, it follows that (3) can be transformed into

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(5) $$J = \int\left(\int pB\,d\bar{\phi}\right)d\bar{w} = \int\left[\int p(\bar{\phi})\left\{p\left(\bar{\phi}+\frac{\pi}{2}\right) + p\left(\bar{\phi}+\frac{3\pi}{2}\right)\right\}d\bar{\phi}\right]d\bar{w}.$$

For any convex curve, $p(\bar{\phi})$ can be expanded into a Fourier series:

(6) $$p(\bar{\phi}) = \sum_{0}^{\infty}(a_n \cos n\bar{\phi} + b_n \sin n\bar{\phi}),$$

where the a_i, b_i are integrable functions of $\bar{\theta}$, $\bar{\phi}$ ($d\bar{w} = \sin\bar{\theta}\, d\bar{\theta}\, d\bar{\phi}$), $\bar{\theta}$, $\bar{\phi}$ being polar coordinates of the projection vector

$$a_i, b_i = a_i(\bar{\theta}, \bar{\phi}), b_i(\bar{\theta}, \bar{\phi}) = a_i(\bar{w}), b_i(\bar{w}).$$

Replacing p in (5) by the series (6), we find

(7) $$J = \int\left[4\pi a_0^2(\bar{w}) + 2\pi\sum_{n=1}^{\infty}(-1)^n\left\{a_{2n}^2(w) + b_{2n}^2(\bar{w})\right\}d\bar{w}\right.$$

We now compare formula (7) with a corresponding expression for the surface S of K. Cauchy found the now well-known formula

(8) $$S = \frac{1}{\pi}\int P\,dw,$$

where P is the area of an orthogonal projection of K.

The area P can be expressed by the integral

(9) $$P = \frac{1}{2}\int_{0}^{2\pi}\left(p^2 - \left(\frac{dp}{d\bar{\phi}}\right)^2\right)d\bar{\phi}.$$

Substituting (6) in (9), we obtain

(10) $$S = \frac{1}{4\pi}\int\left[4\pi a_0^2 + 2\pi\sum_{n=1}^{\infty}(1-n^2)(a_n^2 + b_n^2)\right]d\bar{w}.$$

Comparing (7) and (10), we conclude that

(11) $$S \leq \frac{1}{4\pi}\int LB\,dw.$$

The equal sign holds only if the support function p of each orthogonal projection of K has the form

(12) $$p = a_0 + a_1\cos\phi + b_1\sin\phi.$$

Formula (12) says that every orthogonal projection of K is a circle. Therefore K is a sphere, and Theorem 1 has been proved.

It will be shown that Theorem 2 is a consequence of Theorem 1. We consider the integral extended over the whole unit sphere

$$\int V_c \, dw = \int PB \, dw . \tag{13}$$

V_c is the volume of a circumscribed cylinder, P the area of an orthogonal projection. The arithmetic mean of the volumes of the circumscribed cylinders is

$$\frac{1}{4\pi}\int PB \, dw .$$

We now form the expression

$$E = \frac{1}{4\pi}\int PB \, dw - \frac{3}{2}V , \tag{14}$$

where V is the volume of K. In order to investigate the sign of (14), we make use of a powerful method, applied with special success by G. Bol [2], namely the transition to interior parallel surfaces.

If we translate all support planes of K inwards through the same distance λ, we obtain a new convex body K_λ. It generally happens, during the shifting process, that some of the translated original support planes cease to be support planes of K_λ and become superfluous. Finally, we reach a convex body of at most two dimensions, the so-called nucleus N of K. The points of N consist of those interior points of K for which the smallest distance from the boundary of K is a maximum m.

We first calculate (14) for the nucleus. One sees immediately that in this case $E \geq 0$, since $V = 0$. The shifting process mentioned above is then performed in the opposite direction, that is, starting from the nucleus N. Let $m - \lambda$ be called μ. The nucleus corresponds to the value $\mu = 0$, the boundary of K to the value $\mu = m$. The support planes of $K_{\mu+\Delta\mu}$ consist of those of K_μ, translated outwards by $\Delta\mu$, and possibly of new support planes. In any case, $K_{\mu+\Delta\mu}$ contains the convex body $K'_{\mu+\Delta\mu}$ which may be defined in the following way. About every boundary point Q of K_μ as center, we draw a sphere with radius $\Delta\mu$. The points of K_μ, together with the points of the spheres, describe a convex body $K'_{\mu+\Delta\mu}$. From this statement we conclude that, in every direction, the quantity PB formed with respect to $K_{\mu+\Delta\mu}$ is greater than or equal to the quantity PB formed with respect to $K'_{\mu+\Delta\mu}$. Moreover, the volumes of $K_{\mu+\Delta\mu}$ and $K'_{\mu+\Delta\mu}$ differ only by $o(\Delta\mu)$, since (see [2])

$$\lim_{\Delta\mu\to 0} \frac{V(K_{\mu+\Delta\mu}) - V(K_\mu)}{\Delta\mu} = \lim_{\Delta\mu\to 0} \frac{V(K'_{\mu+\Delta\mu}) - V(K_\mu)}{\Delta\mu} = S .$$

Therefore (with easily understandable notation)

$$E(K_{\mu+\Delta\mu}) - E(K_\mu) \geq E(K'_{\mu+\Delta\mu}) - E(K_\mu) + o(\Delta\mu) . \tag{15}$$

The right-hand side can be written

$$E(K'_{\mu+\Delta\mu}) - E(K_\mu) + o(\Delta\mu) = \Delta\mu\left\{\frac{1}{4\pi}\int LB \, dw + \frac{1}{4\pi}\int P \, dw - \frac{3}{2}S\right\} + A^* + o(\Delta\mu) , \tag{16}$$

where A^* is always smaller than a number M depending on K only. (An explicit expression for M can be given very easily [2], but it is of no interest for our purposes.)

Because of (8), formula (16) can be written

$$\text{(17)} \qquad E(K'_{\mu+\Delta\mu}) - E(K_\mu) + o(\Delta\mu) = \Delta\mu\left(\frac{1}{4\pi}\int LB\,dw - S\right) + A^*\Delta\mu^2 + o(\Delta\mu).$$

We now divide the interval $[0, m]$ into equal parts by the points $0 = \mu_0, \mu_1, \mu_2, \cdots, \mu_n = m$ with $\mu_{i+1} - \mu_i = \Delta\mu$; and we add the inequalities

$$\text{(18)} \qquad E(K_{\mu_i+\Delta\mu}) - E(K_{\mu_i}) \geq E(K'_{\mu_i+\Delta\mu}) - E(K_{\mu_i}),$$

and let $\Delta\mu$ tend to zero. Using (11) and (17), we conclude that

$$E(K_m) - E(K_0) \geq 0.$$

Now $E(K_0)$, that is, the quantity E formed for the nucleus, is nonnegative. Therefore

$$E(K_m) = \frac{1}{4\pi}\int PB\,dw - \frac{3}{2}V \geq 0$$

Since LB depends continuously on μ, the equal sign holds only, as (17) shows, if for every μ

$$\frac{1}{4\pi}\int LB\,dw - S = 0.$$

In this case Theorem 1 shows that K is a sphere. Herewith, Theorem 2 has also been proved.

REFERENCES

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