CONTRIBUTIONS TO THE THEORY OF CONVEX BODIES

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1. GENERALIZATION OF THE PRINCIPAL THEOREM OF BRUNN AND MINKOWSKI

The Brunn-Minkowski theorem on closed convex bodies in n-dimensional Euclidean space can be extended by introducing a suitably defined logarithmically convex functional $\rho_K(\vec{x})$. In the present paper we give a proof of such an extension which was announced orally by the author [4], some years ago. Similarly to the way in which the Brunn-Minkowski theorem provides a means for deriving the isoperimetric inequality, our extension leads to a more general inequality. The functional $\rho_K(\vec{x})$ shall be a local function, in the interior and on the boundary of a convex body K, which depends not only on the point $\vec{x}$ ($\vec{x}$ denotes a local vector), but also on K. This dependence shall satisfy the following four conditions.

1. Continuity: If $K_1 \to K_2$ and $\vec{x}_1 \to \vec{x}_2$, then

$$\rho_{K_1}(\vec{x}_1) \to \rho_{K_2}(\vec{x}_2).$$

Here the statement $K_1 \to K_2$ means that the distance between two parallel directed support planes of K_1 and K_2 tends to zero for all directions of these planes.

2. Homogeneity: If $\lambda K + \vec{a}$ denotes a body which is obtained by applying to K a similarity transformation λK and a translation characterized by the vector $\vec{a}$, then the relation

$$\rho_{\lambda K + \vec{a}}(\lambda \vec{x} + \vec{a}) = \lambda^m \rho_K(\vec{x})$$

shall hold.

3. Logarithmic convexity: If

$$K_\theta = (1 - \theta)K_1 + \theta K_2 \qquad (0 \leq \theta \leq 1)$$

denotes the linear combination of K_1, K_2 in the Brunn-Minkowski sense, the inequality

$$\log \rho_{K_\theta}\{(1 - \theta)\vec{x}_1 + \theta \vec{x}_2\} \geq (1 - \theta) \log \rho_{K_1}(\vec{x}_1) + \theta \log \rho_{K_2}(\vec{x}_2) \tag{1}$$

shall hold, where $\vec{x}_1$, $\vec{x}_2$ are arbitrarily chosen points of K_1, K_2, respectively.

4. Nonnegativeness:

$$\rho_K(\vec{x}) \geq 0. \tag{2}$$

It is easily shown that (2) implies that $\rho_K(\vec{x})$ can vanish only at a boundary point.

Example 1. Let $a = \rho_K(\vec{x})$ denote the shortest distance of $\vec{x}$ from the boundary of K. It is evident that a satisfies the conditions 1, 2, 4. We shall prove that

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condition 3 is also fulfilled: Let $\vec{x}_1, \vec{x}_2$ be two points in K_1, K_2, respectively, and consider the point $\vec{x}_\theta = (1 - \theta)\vec{x}_1 + \theta\vec{x}_2$ in $K_\theta = (1 - \theta)K_1 + \theta K_2$. Let a_θ be the shortest distance of $\vec{x}_\theta$ from the boundary. We draw two straight line segments from $\vec{x}_1, \vec{x}_2$ to the boundaries of K_1, K_2. Let their lengths be $\tilde{a}_1, \tilde{a}_2$. They are to be parallel to a straight line segment through $\vec{x}_\theta$ of length a_θ. It is obvious that

$$a_\theta \geq (1 - \theta)\tilde{a}_1 + \theta\tilde{a}_2 \geq (1 - \theta)a_1 + \theta a_2 , \tag{3}$$

where a_1, a_2 are the shortest distances of $\bar{x}_1, \bar{x}_2$ from the boundaries of K_1, K_2. Condition (3) says that a is a convex functional under linear combinations of convex bodies. Since

$$\log a_\theta \geq \log \{(1 - \theta)a_1 + \theta a_2\} \geq (1 - \theta) \log a_1 + \theta \log a_2 ,$$

a is also a logarithmically convex functional.

Example 2. Let σ denote the distance of a point of K from a tangent plane of K with a normal parallel to a fixed direction in space. Then σ satisfies conditions 1 to 4.

THEOREM. *The* $(n + m)$th *root of the integral*

$$\int_K \rho_K dv \qquad (\rho_K \text{ homogeneous of the mth degree}),$$

extended over the volume (with element dv) *of a convex body* K, *is a convex functional under linear combinations* $K_\theta = (1 - \theta)K_1 + \theta K_2$. *In other words,*

$$\sqrt[n+m]{\int_{K_\theta} \rho_{K_\theta} dv_\theta} \geq (1 - \theta) \sqrt[n+m]{\int_{K_1} \rho_{K_1} dv_1} + \theta \sqrt[n+m]{\int_{K_2} \rho_{K_2} dv_2} . \tag{4}$$

We first prove (4) under the assumption that

$$\int_{K_1} \rho_{K_1} dv_1 = \int_{K_2} \rho_{K_2} dv_2$$

Let $x_1, \cdots, x_n$ be cartesian coordinates in R_n. Both K_1 and K_2 have two support planes (an "upper" and a "lower" one) perpendicular to the x_n-axis. Let their corresponding x_n-values be x_n^0, x_n^1, for K_1 and $\bar{x}_n^0, \bar{x}_n^1$, for K_2. The points $(\bar{x}_i)$ of K_2 are to be associated with the points (x_i) of K_1 in the following way: We determine $x_n, \bar{x}_n$ so that the planes $x_n = \text{const.}$ and $\bar{x}_n = \text{const.}$ cut equal volumes from K_1, K_2:

$$\int_{x_n^0}^{x_n} \left\{ \int \cdots \int \rho_{K_1} dx_1 \cdots dx_{n-1} \right\} dx_n = \int_{\bar{x}_n^0}^{\bar{x}_n} \left\{ \int \cdots \int \rho_{K_2} d\bar{x}_1 \cdots d\bar{x}_{n-1} \right\} d\bar{x}_n . \tag{5}$$

Thus we obtain a correspondence $\bar{x}_n = f_n(x_n)$ between $\bar{x}_n$ and x_n. Equation (5) can be written in differential form:

$$\int \cdots \int \rho_{K_1} dx_1 \cdots dx_{n-1} = \frac{d\bar{x}_n}{dx_n} \int \cdots \int \rho_{K_2} d\bar{x}_1 \cdots d\bar{x}_{n-1} . \tag{6}$$

We now consider two (n - 1)-dimensional intersections of K_1 and K_2 corresponding to the values x_n and $\overline{x}_n = f_n(x_n)$. For any pair of values x_n and $\overline{x}_n = f_n(x_n)$, the coordinate $\overline{x}_{n-1}$ can be mapped on x_{n-1} so that

$$(7)\quad \int_{x_{n-1}^0(x_n)}^{x_{n-1}} \left\{ \int \cdots \int \rho_{K_1} dx_1 \cdots dx_{n-2} \right\} dx_{n-1} = \frac{d\overline{x}_n}{dx_n} \int_{\overline{x}_{n-1}^0(x_n)}^{\overline{x}_{n-1}} \left\{ \int \cdots \int \rho_{K_2} d\overline{x}_1 \cdots d\overline{x}_{n-2} \right\} dx_{n-1}.$$

Equation (7) furnishes the relation $\overline{x}_{n-1} = f_{n-1}(x_{n-1}, x_n)$. For every pair x_{n-1}, x_n, equation (7) may be differentiated with respect to x_{n-1}:

$$(8)\quad \int \cdots \int \rho_{K_1} dx_1 \cdots dx_{n-2} = \frac{d\overline{x}_n}{dx_n} \frac{\partial \overline{x}_{n-1}}{\partial x_{n-1}} \int \cdots \int \rho_{K_2} dx_1 \cdots dx_{n-2}.$$

Continuing this procedure, we cut the intersections $x_n = \text{const.}$, $x_{n-1} = \text{const.}$ of K_1 and the corresponding intersections

$$\overline{x}_n = f_n(x_n) = \text{const.}, \qquad \overline{x}_{n-1} = f_{n-1}(x_{n-1}, x_n) = \text{const.}$$

into slices perpendicular to the x_{n-2}-axis. As before, a correspondence between the values $x_{n-2}, \overline{x}_{n-2}$ can be established:

$$(9)\quad \begin{aligned} &\int_{x_{n-2}^0(x_{n-1},x_n)}^{x_{n-2}} \left\{ \int \cdots \int \rho_{K_1} dx_1 \cdots dx_{n-3} \right\} dx_{n-2} \\ &= \frac{d\overline{x}_n}{dx_n} \frac{\partial \overline{x}_{n-1}}{\partial x_{n-1}} \int_{\overline{x}_{n-2}^0(x_{n-1},x_n)}^{\overline{x}_{n-2}} \left\{ \int \cdots \int \rho_{K_2} d\overline{x}_1 \cdots d\overline{x}_{n-3} \right\} d\overline{x}_{n-2}; \end{aligned}$$

this is equivalent to a relation

$$(10)\quad \overline{x}_{n-2} = f_{n-2}(x_{n-2}, x_{n-1}, x_n).$$

Finally we obtain a one-to-one mapping of the points of K_2, K_1 such that

$$(11)\quad \rho_{K_1} dx_1 \cdots dx_n = \rho_{K_2} d\overline{x}_1 \cdots d\overline{x}_n.$$

Equation (12) represents a generalized volume-preserving mapping of K_2 on K_1, the points of K_1 and K_2 having the "weights" ρ_{K_1}, ρ_{K_2}. The Jacobian matrix of the mapping has the form

$$(12)\quad \left(\frac{\partial \overline{x}_i}{\partial x_K} \right) = \begin{pmatrix} \frac{\partial \overline{x}_1}{\partial x_1} & \frac{\partial \overline{x}_1}{\partial x_2} & \cdots & \frac{\partial \overline{x}_1}{\partial x_n} \\ 0 & \frac{\partial \overline{x}_2}{\partial x_2} & \cdots & \frac{\partial \overline{x}_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \frac{\partial \overline{x}_n}{\partial x_n} \end{pmatrix},$$

all the elements beneath the principal diagonal being zero. The body

$$K_\theta = (1 - \theta)K_1 + \theta K_2$$

contains all points $(1-\theta)\bar{x}_1 + \theta\bar{x}_2$. A fortiori, K_θ contains the points

$$(1-\theta)\bar{x}_1 + \theta\bar{x}_2 \,, \tag{13}$$

where the $\bar{x}_2$ correspond to the x_1, by our generalized volume-preserving mapping. The point (13) has the coordinates $(1-\theta)x_i + \theta\bar{x}_i$.

The Jacobian

$$\left(\frac{\theta\{(1-\theta)x_i + \theta\bar{x}_i\}}{\partial x_K} \right)$$

has the form

$$\begin{pmatrix} (1-\theta)+\theta\dfrac{\partial\bar{x}_1}{\partial x_1} & (1-\theta)+\theta\dfrac{\partial\bar{x}_1}{\partial x_2} & \cdots & (1-\theta)+\theta\dfrac{\partial\bar{x}_1}{\partial x_n} \\ 0 & (1-\theta)+\theta\dfrac{\partial\bar{x}_2}{\partial x_2} & \cdots & (1-\theta)+\theta\dfrac{\partial\bar{x}_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (1-\theta)+\theta\dfrac{\partial\bar{x}_n}{\partial x_n} \end{pmatrix}. \tag{14}$$

Thus we see that the integrand in the integral $\int_{K_\theta} \rho_{K_\theta} dv_\theta$ in (4) satisfies the equality

$$\rho_{K_\theta} dv_\theta = \rho_{K_\theta}\left(1-\theta+\theta\frac{\partial\bar{x}_1}{\partial x_1}\right)\left(1-\theta+\theta\frac{\partial\bar{x}_2}{\partial x_2}\right)\cdots\left(1-\theta+\theta\frac{\partial\bar{x}_n}{\partial x_n}\right)dx_1\cdots dx_n. \tag{15}$$

Since

$$\log \rho_{K_1} = \log\left\{\rho_{K_2}\frac{\partial\bar{x}_1}{\partial x_1}\frac{\partial\bar{x}_2}{\partial x_2}\cdots\frac{\partial\bar{x}_n}{\partial x_n}\right\},$$

we conclude from (1) and

$$\log\left(1-\theta+\theta\frac{\partial\bar{x}_i}{\partial x_i}\right) \geq \theta\log\frac{\partial\bar{x}_i}{\partial x_i} \tag{16}$$

that

$$\int_{K_\theta} \rho_{K_\theta} dv_\theta \geq \int_{K_1} \rho_{K_1} dv_1 \,. \tag{17}$$

(As the development shows, $\dfrac{\partial\bar{x}_i}{\partial x_i}$ is never negative, and it can not be zero except on the boundary.)

We see further that the more general relation

$$\int_{D_\theta} \rho_{K_\theta} dv_\theta \geq \int_D \rho_{K_1} dv_1$$

holds, where the integral on the right-hand side is extended over an arbitrary domain D of K_1. The integral on the left-hand side is extended over a domain in K_θ which arises from D by applying to D the generalized volume-preserving mapping. From this statement we derive immediately the result

$$\int_{K_\theta} \sigma_{K_\theta} \rho_{K_\theta} dv_\theta \geq (1 - \theta) \int_{K_1} \sigma_{K_1} \rho_{K_1} dv_1 + \theta \int_{K_2} \sigma_{K_2} \rho_{K_2} dv_2, \tag{18}$$

where σ_K denotes a convex functional obeying the same laws as ρ_K, in addition to the following:

$$\sigma_{K_\theta}\{(1 - \theta)\vec{x}_1 + \theta\vec{x}_2\} \geq (1 - \theta)\sigma_{K_1}(\vec{x}_1) + \theta\sigma_{K_2}(\vec{x}_2). \tag{19}$$

Here the local vectors $\vec{x}_1$, $\vec{x}_2$ are arbitrary points of K_1, K_2.

The decision on the validity of the equal sign can be made without difficulty. Equation (17) shows that we must have

$$\frac{\partial \bar{x}_i}{\partial x_i} \equiv 1,$$

identically for every direction of the coordinate axes. Therefore

$$\bar{x}_i = x_i + a_i, \qquad a_i = \text{const.}$$

In other words, equality in (17) and (18) takes place only if K_1, K_2 can be transformed into each other by a translation. This concludes the proof of the theorem for the special case where

$$\int_{K_1} \rho_{K_1} dv_1 = \int_{K_2} \rho_{K_2} dv_2.$$

We now consider the case where

$$\int_{K_1} \rho_{K_1} dv_1 \not\equiv \int_{K_2} \rho_{K_2} dv_2.$$

By subjecting K_2 to a similarity transformation $\overline{K}_2 = \lambda K_2$ and choosing λ suitably, we arrive at the relation

$$\int \rho_{K_1} dv_1 = \int \rho_{\overline{K}_2} d\bar{v}_2 = \lambda^{m+1} \int_{K_2} \rho_{K_2} dv_2.$$

Substituting $\overline{K}_\theta = (1 - \theta)K_1 + \theta\overline{K}_2$, we obtain from (17)

$$(20)\qquad \int_{\bar{K}_\theta} \rho_{\bar{K}_\theta} dv_\theta \geq \int \rho_{K_1} dv_1 = (1-\theta)\int_{K_1} \rho_{K_1} dv_1 + \theta\lambda^{m+n}\int_{K_2} \rho_{K_1} dv_2 ,$$

where ρ_K is supposed to be homogeneous of the mth degree in R_n. Since

$$(21)\qquad \bar{K}_\theta = (1-\theta)K_1 + \theta\bar{K}_2 = (1-\theta)K_1 + \theta\lambda K_2 ,$$

the following substitution suggests itself:

$$(22)\qquad \begin{aligned} 1 - \theta &= \sigma(1-\mu), \\ \theta\lambda &= \sigma\mu , \end{aligned}$$

where μ and θ run from 0 to 1. By combining (20), (21) and (22), we find that

$$(23)\qquad \sigma^{m+n}\int_{K_\mu} \rho_{K_\mu} dv_\mu \geq (1-\theta)\int_{K_1} \rho_{K_1} dv_1 + \theta\lambda^{m+n}\int_{K_2} \rho_{K_2} dv_2 .$$

We extract the (m+n)th root on both sides of (23) and take advantage of the elementary relation

$$\{(1-\theta)a + \theta b\}^{1/n} \geq (1-\theta)a^{1/n} + \theta b^{1/n};$$

this leads to the inequality

$$\sigma\left[\int_{K_\mu} \rho_{K_\mu} dv_\mu\right]^{1/(m+n)} \geq (1-\theta)\left[\int_{K_1} \rho_{K_1} dv_1\right]^{1/(m+n)} + \theta\lambda\left[\int_{K_2} \rho_{K_2} dv_2\right]^{1/(m+n)}$$

or, with reference to (22),

$$\left[\int_{K_\mu} \rho_{K_\mu} dv_\mu\right]^{1/(m+n)} \geq (1-\mu)\left[\int \rho_{K_1} dv_1\right]^{1/(m+n)} + \mu\left[\int_{K_2} \rho_{K_2} dv_2\right]^{1/(m+n)}.$$

The equal sign characterizes the case where K_2, K_1 can be transformed into each other by a similarity transformation and a translation. We shall say, with Minkowski, that K_1 and K_2 are homothetic.

2. APPLICATIONS WITH $\rho \equiv 1$

Let us first apply (18) to polyhedra in R . Let K_1 be a polyhedron with q lateral surfaces, and $\bar{K}_2$ a polyhedron circumscribed about the unit sphere, with q lateral surfaces parallel to those of K_1. We project K_1 and K_2 orthogonally on the same R_{n-1}, the volumes of the convex projections being P_1, P_2. By means of a similarity transformation applied to K_2 ($K_2 \rightarrow RK_2$), we can arrange that the projections P_1, P_2 of K_1 and K_2 have equal volumes. The linear combination

$$K_\theta = (1-\theta)K_1 + \theta K_2$$

is also a polyhedron with q lateral surfaces parallel to those of K_1, K_2. For the sake of simplicity, we assume the center of the sphere inscribed in K_2 to be the origin 0 of our cartesian coordinate system. Moreover, we suppose 0 to be in the interior of K_1. Let p_i $(i = 1, \cdots, q)$ be the distances of the q lateral surfaces of K_1 from 0. Let R be the distance of those of K_2 from 0. The distances of the lateral surfaces of K_θ from 0 are

$$(1 - \theta)p_i + \theta R .$$

K_θ can be generated in the following way: First we subject K_1 to a similarity transformation $\overline{K}_1 + (1 - \theta)K_1$. Then we translate the lateral surfaces of $\overline{K}_1$ outward by the amount θR. Let s_i be the $(n - 1)$-dimensional volumes of the lateral surfaces of K_1, V_1 the volume, S_1 the surface of K_1. The volume V_θ of K_θ certainly satisfies the inequality

$$V_\theta \geq (1 - \theta)^n V + \theta(1 - \theta)^{n-1} R \sum_{i-1}^{q} s_i = (1 - \theta)^n V + \theta(1 - \theta)^{n-1} RS. \tag{24}$$

An upper bound on V_θ is found by means of the following method:

We consider a pyramid determined by a lateral surface A_i of $\overline{K}_1$ ($(1 - \theta)p_i$ is its distance from 0) and 0. The lateral surfaces passing through 0 are cut by a plane which is parallel to A_i and lies at a distance $(1 - \theta)p_i + \theta R$ from 0.

This intersection determines, together with 0, a new pyramid with the volume v_i. It is obvious that

$$V_\theta \leq \sum_{i=1}^{q} v_i$$

or, more explicitly,

$$V_\theta \leq \sum \frac{(1 - \theta)^{n-1} s_i}{n} \{(1 - \theta)p_i + R\theta\}^n \{(1 - \theta)p_i\}^{-(n-1)};$$

that is,

$$V_\theta \leq (1 - \theta)^n V + \theta(1 - \theta)^{n-1} RS + \sum_{j=2}^{\infty} \alpha_j \theta^j, \tag{25}$$

where the series is convergent for $\theta < 1$. The explicit values of the α_j are of no interest for our purposes. In any case, (24) and (25) show that $dV_\theta/d\theta|_{\theta=0}$ exists and that

$$\left.\frac{dV_\theta}{d\theta}\right|_{\theta=0} = -nV + RS. \tag{26}$$

We now specialize (18) as follows: We extend the integrals over the orthogonal projections of K_1, K_2, K_θ and substitute $\rho_K \equiv 1$. The quantities σ_{K_1}, σ_{K_2}, σ_{K_θ} may be the lengths of secants perpendicular to the projection plane. They are evidently convex functionals. We have

$$V_\theta = \int \sigma_{K_\theta} dv_\theta, \qquad V_1 = \int \sigma_{K_1} dv_1, \qquad V_2 = \int \sigma_{K_2} dv_2 .$$

At $\theta = 0$, the differential quotient with respect to θ of the left-hand side of (18) must be greater or equal to that of the right-hand side, because of the inequality and the fact that both sides of (18) are equal for $\theta = 0$. Therefore we obtain, by means of (26),

$$\begin{aligned} -nV_1 + RS_1 &\geq -V_1 + V_2, \\ -(n-1)V_1 + RS_1 &\geq V_2 . \end{aligned} \tag{27}$$

If we designate the volume of $\overline{K}_2$ (polyhedron similar to K_2, circumscribed about the unit sphere, as mentioned above) by $\widetilde{V}$, then

$$V_2 = R^n \widetilde{V},$$

and equation (27) takes the form

$$-(n-1)V_1 + RS_1 \geq R^n \widetilde{V} . \tag{28}$$

If we assume K_1, K_2 to have equal surface areas $S_1 = S_2$, we can always find an R_{n-1} such that the projections of K_1, K_2 on R_{n-1} have equal volumes and (19) can be applied. The reason is that the surface area S of a convex body in R_n can be found, according to Cauchy, by integrating its $(n-1)$-dimensional orthogonal projections over all directions:

$$S = \frac{2}{\omega_{n-1}} \int P d\omega,$$

where ω_{n-1} denotes the volume of the unit sphere in R_{n-1}, where P is the volume of an orthogonal projection, and where $d\omega$ is the solid angle element of their directions. Since

$$S_1 = S_2 = nR^{n-1}\widetilde{V},$$

R can be expressed in terms of S_1 and $\widetilde{V}$:

$$R = (S_1/n\widetilde{V})^{1/(n-1)} . \tag{29}$$

Replacing R in (28) by the expression (29), we obtain the result

$$S_1^n \geq n^n \widetilde{V} V_1^{n-1} . \tag{30}$$

Inequality (30) represents an isoperimetric inequality between the surface area and the volume of a polyhedron. It was first proved by G. Bol and the author [2] by means of other methods. It is remarkable that the value of R given by (29) is the minimum of the polynomial

$$R^n \widetilde{V} - RS_1 + (n-1)V_1 \qquad (0 \leq R) . \tag{31}$$

The inequality (30) can be improved. (In the two-dimensional case an inequality similar to (28) has been derived by Bonnesen.) For this purpose we first return to the geometrical meaning of R. We consider again our bodies K_1 and $\overline{K}_2$

(circumscribed about the unit sphere with lateral surfaces parallel to those of K_1). Let P_1, $\widetilde{P}$ be their orthogonal projections on an R_{n-1}. Then (29) holds for every R satisfying the condition

$$R^{n-1} = P_1/\widetilde{P} \tag{32}$$

Let R_{max} denote the maximum of R reached for a certain direction of projection, and R_{min} the minimum. Let R, as defined by (29), be called R_0. Then R_0 makes the polynomial (31) a minimum, as said before. Inequality (28) shows that

$$R^n_{max}\widetilde{V} = R_{max}S + (n-1)V \leq 0 . \tag{33}$$

(We now omit the subscript 1 in S_1, V_1.)

We now write

$$R_{max} = R_0 + \mu .$$

Taking into account the fact that the value of the expression in (33) is not smaller than

$$R_0^n\widetilde{V} - R_0S + (n-1)V ,$$

we obtain

$$R_0^n\widetilde{V} - R_0S + (n-1)V \leq -\widetilde{V}\sum_{i=2}^{n}\binom{n}{i}\mu^iR_0^{n-i} . \tag{34}$$

In the same manner we find, writing $R_{min} = R_0 - \sigma$, that

$$R_0^n\widetilde{V} - R_0S + (n-1)V \leq -\widetilde{V}\sum_{i=2}^{n}\binom{n}{i}\sigma^i(-1)^iR_0^{n-i} . \tag{35}$$

The right-hand sides of (34) and (35) are negative. Inequalities (34) and (35) are therefore sharper inequalities than (31). Since

$$\Sigma\binom{n}{i}(-1)^i\sigma^iR_0^{n-i} \geq \frac{n}{2}\sigma^2R_0^{n-2} ,$$

we can replace (34) and (35) by the less sharp but more elegant inequalities

$$R_0^n\widetilde{V} - R_0S + (n-1)V \leq -\binom{n}{2}\mu^2R_0^{n-2}\widetilde{V} , \tag{36}$$

$$R_0^n\widetilde{V} - R_0S + (n-1)V \leq -\frac{n}{2}\sigma^2R_0^{n-2}\widetilde{V} . \tag{37}$$

By adding (36) and (37), we obtain

$$\begin{aligned} &R_0^n\widetilde{V} - R_0S + (n-1)V \\ &\leq -\frac{n}{4}\widetilde{V}R_0^{n-2}\left[\left(\frac{P}{\widetilde{P}}\right)_{max}^{\frac{1}{n-1}} - \left(\frac{P}{\widetilde{P}}\right)_{min}^{\frac{1}{n-1}}\right]^2 - \frac{n(n-2)}{2}\widetilde{V}R_0^{n-2}\left[\left(\frac{P}{\widetilde{P}}\right)_{max}^{\frac{1}{n-1}} - \left(\frac{P}{\widetilde{P}}\right)_0^{\frac{1}{n-1}}\right]^2 . \end{aligned} \tag{38}$$

The symbols require no further explanation.

We introduce (29) into (38) and obtain, finally

$$n^{\frac{-n}{n-1}}S^{\frac{n}{n-1}} - \widetilde{V}^{\frac{1}{n-1}}V$$

$$\geq \frac{1}{4(n-1)}n^{\frac{1}{n-1}}\widetilde{V}^{\frac{2}{n-1}}S^{\frac{n-2}{n-1}}\cdot\left[\left(\frac{P}{\overline{\overline{P}}}\right)_{max}^{\frac{1}{n-1}} - \left(\frac{P}{\overline{\overline{P}}}\right)_{min}^{\frac{1}{n-1}}\right]^2 \tag{39}$$

$$+ \frac{n-2}{2(n-1)}\widetilde{V}^{\frac{2}{n-1}}n^{\frac{1}{n-1}}S^{\frac{n-2}{n-1}}\left[\left(\frac{P}{\overline{\overline{P}}}\right)_{max}^{\frac{1}{n-1}} - \left(\frac{P}{\overline{\overline{P}}}\right)_0^{\frac{1}{n-1}}\right]^2.$$

Now let us increase the number of lateral surfaces beyond any fixed limit in such a way that K_1 tends to a regularly curved convex body and $\overline{K}_2$ tends to the unit sphere. We shall confine our considerations to the two- and three-dimensional cases. Since volume, surface area, and curve length depend continuously on convex bodies, we are allowed to apply (39) to bodies without vertices and edges: In this case we have to substitute in R_2:

$\widetilde{P} \equiv 2$, P_{max} = maximum breadth of the curve, P_{min} = minimum breadth of curve.

Denoting by L and F the length of the curve K_1 and the area of the enclosed region, we obtain again Bonnesen's result

$$L^2 - 4\pi F \geq \frac{\pi^2}{2}(P_{max} - P_{min})^2, \tag{40}$$

a famous improvement on the well-known plane isoperimetric inequality.

We now specialize (40) to R_3. (As far as I know this is the first analogue to Bonnesen's inequality (40).)

In R_3 we have to substitute

$\widetilde{P} \equiv \pi$,

P_{max} = orthogonal projection of maximum area of K_1,

P_{min} = orthogonal projection of minimum area of K_1,

$P_0 = S/4$.

The result is:

$$\frac{1}{3}S^{3/2} - (4\pi)^{1/2}V \geq \frac{\sqrt{S}}{2}\left(\sqrt{P_{max}} - \sqrt{P_{min}}\right)^2 + \frac{S^{1/2}}{4}\left(\sqrt{P_{max}} - \frac{1}{2}\sqrt{S}\right)^2. \tag{41}$$

We recognize (41) as an improvement on the isoperimetric inequality in R_3:

$$S^3 - 36\pi V^2 \geq 0.$$

Another application of (18) follows. We take a convex body K_1 and take K_2 as a sphere with the same volume V as K_1, and such that K_1 and K_2 are tangent to the same R_{n-1}. We denote by s_i (i = 1, 2) the distance of a point of K_i from the R_{n-1}. The integral $\int s_i dv_i$ has the meaning

$$I_i = \int s_i \, dv_i = \sigma_i V,$$

where σ_i represents the distance of the center of gravity of K_i (the interior filled with mass of constant density) from R_{n-1}. Since K_2 is a sphere, we have the relation $\sigma_2 = \sqrt[n]{V/\omega_n}$, where ω_n is the volume of the unit sphere. We consider the integral

$$I_\theta = \int s_\theta \, dv_\theta,$$

extended over the linear combination $K_\theta = (1 - \theta)K_1 + \theta K_2$. We see immediately that

$$\left.\frac{dI_\theta}{d\theta}\right|_{\theta=0} = -nI_1 + \sigma_2 V + \sigma_2 \int s_1 d\sigma_1,$$

where $d\sigma_1$ denotes the surface element of K_1. In the same manner as before, we conclude from (18), after substituting $\rho_K \equiv 1$, $\sigma_K = s_K$, that

$$-nI_1 + \sigma_2 V + \sigma_2 \int s_1 d\sigma_1 \geq -I_1 + \sigma_2 V. \tag{42}$$

Since

$$\int s_1 d\sigma_1 = \tau_1 \cdot S,$$

where τ_1 is the distance of the center of gravity of K_1 from R_{n-1} (the surface covered with mass of constant density), we derive from (42) in an elementary way (again omitting the now superfluous subscript 1), the inequality

$$\frac{S^n}{n^n \omega_n V^{n-1}} \geq (\sigma/\tau)^n. \tag{43}$$

This inequality is also an improvement on the classical isoperimetric inequality since, for every body with $\sigma \neq \tau$, there exist tangential R_{n-1} for which $\sigma > \tau$.

3. APPLICATIONS WITH $\rho \neq 1$

We consider a convex central surface S with center C. We shall derive an isoperimetric inequality between the principal moments of inertia of the surface (constant surface density 1):

$$I_1^{(s)} \geq I_2^{(s)} \geq I_3^{(s)}$$

and those of the volume (constant volume density 1):

$$I_1^{(r)} \geq I_2^{(r)} \geq I_3^{(r)} .$$

Let us draw a plane P through C. The distance from P of an interior point of S is called s. We consider the integral extended over the volume of S:

$$\text{(44)} \qquad I \doteq \int s^2 \, dv .$$

The interior points of S can be mapped in a one-to-one way on the interior points of a sphere K with an equal value I, and so that the plane P is transformed into itself. Let s_K denote the distances of the corresponding interior points of K from P, and s_θ those of the points of $\{(1 - \theta)S + \theta K\}'$ from P. The symbol $\{(1 - \theta)S + \theta K\}'$ denotes the body consisting of the points $(1 - \theta)\vec{x}_S + \theta \vec{x}_K$, where $\vec{x}_S$, $\vec{x}_K$ are points of S and K associated with each other by the mapping. Since s_θ^2 is a logarithmically convex function of θ, and $\{(1 - \theta)S + \theta K\}'$ is contained in $(1 - \theta)S + \theta K$ (taken in the Brunn-Minkowski sense), our first theorem is applicable. Hence

$$\text{(45)} \qquad I_\theta \geq I_\theta' \geq I ,$$

where I_θ, I_θ' denote the integrals extended over $(1 - \theta)S + \theta K$ and $\{(1 - \theta)S + \theta K\}'$, respectively. From (45) it follows, in a manner analogous to the preceding examples, that

$$\text{(46)} \qquad \left.\frac{dI_\theta}{d\theta}\right|_{\theta=0} = -5I + R\int s^2 \, d\sigma \geq 0 ,$$

where R is the radius of K and where the integral

$$\text{(47)} \qquad \int s^2 \, d\sigma$$

is extended over the surface of S. Since

$$\text{(48)} \qquad I = 4\pi R^5/15 ,$$

(46) implies the inequality

$$\text{(49)} \qquad \left(\int s^2 \, d\sigma\right)^5 \geq \frac{2500}{3}\pi \left(\int s^2 \, dv\right)^4 .$$

The equal sign characterizes the spheres.

In order to derive from (49) an isoperimetric inequality between the $I_j^{(s)}$ and the $I_j^{(r)}$, we remark that there exists a plane P_r, passing through C, and with the following property: The integral $\int s^2 \, dv$ has the same value with respect to every plane P perpendicular to any direction in P_r. Well-known considerations about the circular intersections of an ellipsoid show that, for such a P, the formula

$$\int s^2 \, dv = \frac{1}{2}\left(I_1^{(r)} - I_2^{(r)} + I_3^{(r)}\right)$$

holds. There exists another plane P_s such that

$$\int s^2 d\sigma = \frac{1}{2}\left(I_1^{(s)} - I_2^{(s)} + I_3^{(s)}\right),$$

for any plane perpendicular to any direction in P_s. If we now form the integrals $\int s^2 dv$ and $\int s^2 d\sigma$ with respect to a plane the normal of which coincides with the direction of the intersection of P_r and P_s, we obtain

$$\left(I_1^{(s)} - I_2^{(s)} + I_3^{(s)}\right)^5 \geq \frac{5000\pi}{3}\left(I_1^{(r)} - I_2^{(r)} + I_3^{(r)}\right)^4. \tag{50}$$

The inequality (49) can be generalized. For instance, relations similar to (49) exist for any momenta $\int s^\nu dv$, $\int s^\nu d\sigma$ $(\nu \geq 0)$ provided that the convex surface has a center.

Another application of (18) improves inequalities, first derived by G. Bol [1] concerning the integrals $\int a^n dv$ $(n \geq 0)$, where a denotes the shortest distance of an interior point from the boundary of a convex body K_1 (which is not assumed to have a center). Denoting by s the distance of an interior point from a support plane, we consider the integrals

$$I_n = \int s a^n dv. \tag{51}$$

We have already shown that a, and therefore a^n, are logarithmically convex functionals, and that s is a convex functional. The relation (18) is therefore applicable to (51) if we substitute $\sigma_K = s$, $\rho_K = a^n$. In our case, K_2 is a sphere (of R) possessing the same $\int a^n dv$ as K_1. From (18) we conclude again that

$$\left.\frac{dI_n}{d\theta}\right|_{\theta=0} \geq 0, \tag{52}$$

or that

$$-(n + 4)I_n + R\int a^n dv + nR\int s a^{n-1} dv \geq - I_n + \left(\int s a^n dv\right)_{K_2}.$$

Because of the relations

$$\left(\int s a^n dv\right)_{K_2} = R\left(\int a^n dv\right)_{K_2} = R\left(\int a^n dv\right)_{K_1},$$

the inequality (52) can be written

$$-(n + 3)I_n + nR\int s a^{n-1} dv \geq 0.$$

We introduce the notation

$$\int a^n dv = V_n,$$

and denote by σ_n the distance of the center of gravity of the body (covered with the mass density a^n) from the support plane. Since

$$V_n = \int a^n \, dv = \frac{8\pi R^{n-3}}{(n+1)(n+2)(n+3)},$$

we obtain

$$\frac{n^{(n+3)}(n+1)(n+2)}{8\pi(n+3)^{n+2}} \cdot \frac{V_{n-1}^{n+3}}{V_n^{n+2}} \geq \left(\frac{\sigma_n}{\sigma_{n-1}}\right)^{n+3} . \tag{53}$$

(53) improves Bol's inequalities, just as (43) improves the classical isoperimetric inequality.

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