# The Dirichlet Problem for the Complex Monge–Ampère Operator: Stability in $L^2$

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## 0. Introduction

Consider a strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ and the Dirichlet problem for the generalized complex Monge–Ampère operator:

$$\begin{cases} \varphi \in \mathrm{PSH} \cap L^\infty(\Omega) \\ (dd^c\varphi)^n = g\,dV \\ \lim_{z\to\xi} \varphi(z) = h(\xi) \quad \text{for all } \xi \in \partial\Omega. \end{cases}$$

We prove that if $0 \le g \in L^2(\Omega)$ and $h \in C(\partial\Omega)$, then there is a unique solution $\varphi = U(h,g)$. This is not true if we require only that $g \in L^1(\Omega)$; see [7]. We also prove that the solution $U(h,g)$ is continuous not only at the boundary but in all of $\bar{\Omega}$. In fact we prove the estimate

$$\sup_{\Omega} |U(h_1,g_1) - U(h_2,g_2)| \le \sup_{\partial\Omega} |h_1 - h_2| + C\left(\int_\Omega |g_1 - g_2|^2\right)^{1/2n},$$

which is our main result and generalizes an inequality of Gaveau [8], who used the $L^\infty$-norm instead of $L^2$. In the proof we use a comparison between convex and plurisubharmonic functions and their real and complex Monge–Ampère measures, based on an idea of Cheng and Yau discussed in [1]. We use the following notation: Let $\mathrm{PSH}(\Omega)$ and $\mathrm{CVX}(\Omega)$ denote the cones of plurisubharmonic and convex functions on $\Omega$, respectively; let $dV$ denote $2n$-dimensional Lebesgue measure; let $|\cdot|_\Omega$ and $|\cdot|_{\partial\Omega}$ denote the sup-norm on $\Omega$ and $\partial\Omega$, and let $\|\cdot\|_{p,\Omega}$ denote the $L^p$-norm on $\Omega$; for a matrix $A$, let $A'$ denote its transpose; the identity matrix is denoted by $I$.

## 1. The Generalized Real and Complex Monge–Ampère Operators

Let $\Omega$ be a strictly convex domain in $\mathbf{C}^n$ $(n \ge 2)$, and let $J: \mathbf{C}^n \to \mathbf{R}^n \times \mathbf{R}^n$: $(x+iy) \mapsto (x,y)$ be the canonical isomorphism. For $\varphi \in C^2(\Omega; \mathbf{R})$, define the real $n \times n$ matrices

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$$A=\left[\frac{\partial^2(\varphi\circ J^{-1})}{\partial x_j\partial x_k}\right],\qquad B=\left[\frac{\partial^2(\varphi\circ J^{-1})}{\partial x_j\partial y_k}\right],\qquad C=\left[\frac{\partial^2(\varphi\circ J^{-1})}{\partial y_j\partial y_k}\right].$$

The real and complex Hessians of $\varphi$, denoted $H_{\mathbf{R}}(\varphi)$ and $H_{\mathbf{C}}(\varphi)$, are defined as

$$H_{\mathbf{R}}(\varphi)=\begin{bmatrix} A & B^t \\ B & C\end{bmatrix},$$

a real $2n\times 2n$ matrix, and

$$H_{\mathbf{C}}(\varphi)=(A+C)+i(B^t-B)=4\left[\frac{\partial^2\varphi}{\partial z_j\partial \bar{z}_k}\right],$$

a complex $n\times n$ matrix.

The real Monge–Ampère operator on $\varphi$ is defined as

$$\begin{aligned} M_{\mathbf{R}}(\varphi)&=\det H_{\mathbf{R}}(\varphi)\,dV \\ &=(-2i)^n d\left(\frac{\partial\varphi}{\partial z_1}\right)\wedge d\left(\frac{\partial\varphi}{\partial \bar{z}_1}\right)\wedge\cdots\wedge d\left(\frac{\partial\varphi}{\partial z_n}\right)\wedge d\left(\frac{\partial\varphi}{\partial \bar{z}_n}\right),\end{aligned}$$

where $d=\partial+\bar{\partial}$ and $dV=(i/2)^n dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$, the Lebesgue measure on $\mathbf{R}^{2n}$. The complex Monge–Ampère operator on $\varphi$ is defined as

$$M_{\mathbf{C}}(\varphi)=\det H_{\mathbf{C}}(\varphi)\,dV=\frac{1}{n!}(dd^c\varphi)^n$$

where $d^c=i(\bar{\partial}-\partial)$.

Note that if $\varphi\in\mathrm{PSH}(\Omega)$ then $M_{\mathbf{C}}(\varphi)\geq 0$, and if $\varphi\in\mathrm{CVX}(\Omega)$ then also $M_{\mathbf{R}}(\varphi)\geq 0$. The operators $M_{\mathbf{C}}$ and $M_{\mathbf{R}}$ can be extended to $\mathrm{PSH}\cap C(\Omega)$ and $\mathrm{CVX}(\Omega)$, as positive measures, and the following continuity property holds.

THEOREM 1. *If $\varphi_j$, $\varphi_j\in\mathrm{CVX}(\Omega)$ or $\mathrm{PSH}\cap C(\Omega)$ $(j\in\mathbf{N})$, and $\varphi_j\to\varphi$ uniformly on compact subsets when $j\to\infty$, and $\psi\in C_0(\Omega)$, then*

$$\lim_{j\to\infty}\int_\Omega \psi M(\varphi_j)=\int_\Omega\psi M(\varphi)$$

*for $M=M_{\mathbf{R}}$ and $M_{\mathbf{C}}$, respectively.*

*Proof.* See [11, Prop. 3.1] and [2, Prop. 2.3].

LEMMA 1. *The operators $M_{\mathbf{R}}$ and $M_{\mathbf{C}}$ are superadditive; that is, if $\varphi_1,\varphi_2\in$ $\mathrm{CVX}(\Omega)$ or $\mathrm{PSH}\cap C(\Omega)$ then*

$$M(\varphi_1)+M(\varphi_2)\leq M(\varphi_1+\varphi_2)$$

*for $M=M_{\mathbf{R}}$ or $M_{\mathbf{C}}$, respectively.*

*Proof.* See [11, Prop. 3.3] and [2, Prop. 2.8].

The operator $M_{\mathbf{C}}$ is further extended to $\mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(\Omega)$ as a nonnegative measure, and is then weakly continuous on monotone sequences. See [2, Prop. 2.9] and also [6] and [10] for details.

## 2. The Dirichlet Problem

Let $\Omega$ be bounded and strictly convex, assume that $h \in C(\partial\Omega)$ and that $g \geq 0$ is a measurable function on $\Omega$, and consider the Dirichlet problems

$$(1)\qquad \begin{cases} \varphi \in \mathrm{CVX}(\Omega) \\ M_{\mathbf{R}}(\varphi) = g\,dV \\ \lim\limits_{z \to \xi} \varphi(z) = h(\xi) \quad \text{for all } \xi \in \partial\Omega \end{cases}$$

and

$$(2)\qquad \begin{cases} \varphi \in \mathrm{PSH} \cap L^\infty(\Omega) \\ M_{\mathbf{C}}(\varphi) = g\,dV \\ \lim\limits_{z \to \xi} \varphi(z) = h(\xi) \quad \text{for all } \xi \in \partial\Omega. \end{cases}$$

The solutions, if they exist, are denoted $U_{\mathbf{R}}(h, g)$ and $U_{\mathbf{C}}(h, g)$.

THEOREM 2.

(i) *If* $0 \leq g \in L^1(\Omega)$ *then* (1) *has a unique solution* $U_{\mathbf{R}}(h, g)$.
(ii) *If* $0 \leq g \in C(\bar{\Omega})$ *then* (2) *has a unique solution* $U_{\mathbf{C}}(h, g)$, *and this solution is in* $C(\bar{\Omega})$.

*Proof.* See [11, Thm. 4.1] and [2, Thm. D].

The uniqueness in Theorem 2 follows from the following comparison principle.

THEOREM 3. *Assume that* $h_1, h_2 \in C(\partial\Omega)$ *and that* $g_1, g_2 \in L^1(\Omega)$. *If* $h_1 \leq h_2$ *and* $g_1 \geq g_2$, *then* $U_{\mathbf{R}}(h_1, g_1) \leq U_{\mathbf{R}}(h_2, g_2)$. *If in addition* $g_1, g_2 \in C(\bar{\Omega})$, *then also* $U_{\mathbf{C}}(h_1, g_1) \leq U_{\mathbf{C}}(h_2, g_2)$.

*Proof.* See [11, Lemma 3.6] and [2, Thm. A].

COROLLARY 1.

$$U(h_1, g_1) + U(h_2, g_2) \leq U(h_1 + h_2, g_1 + g_2)$$

*and*

$$|U(h_1, g_1) - U(h_2, g_2)| \leq -U(-|h_1 - h_2|, |g_1 - g_2|)$$

*for* $U = U_{\mathbf{R}}$ *and* $U = U_{\mathbf{C}}$.

*Proof.* Use the comparison principle and the superadditivity.

Concerning higher regularity, the following is known.

THEOREM 4. *If* $\partial\Omega \in C^\infty$, $h \in C^\infty(\partial\Omega)$, *and* $g \in C^\infty(\bar{\Omega})$ $(g > 0)$, *then for* $U = U_{\mathbf{R}}$ *and* $U = U_{\mathbf{C}}$, $U(h, g) \in C^\infty(\bar{\Omega})$.

*Proof.* See [4, Thm. 1.1] and [3, Thm. 1.1].

REMARK. We use only the regularity result for $U_{\mathbf{R}}$, and we would do with a classical one (e.g., [9, Thm. 17.24]).

THEOREM 5. *If* $h \in C(\partial\Omega)$ *and* $g \in L^1(\Omega)$ $(g \geq 0)$, *then*

$$\inf_{\partial\Omega} h - C\|g\|_{1,\Omega}^{1/2n} \leq U_{\mathbf{R}}(h,g) \leq \sup_{\partial\Omega} h$$

*for some constant C depending only on* $\Omega$, *and*

$$|U_{\mathbf{R}}(h_1,g_1) - U_{\mathbf{R}}(h_1,g_2)|_{\Omega} \leq |h_1 - h_2|_{\partial\Omega} + C\|g_1 - g_2\|_{1,\Omega}^{1/2n}$$

*for all* $h_1, h_2 \in C(\partial\Omega)$ *and nonnegative* $g_1, g_2 \in L^1(\Omega)$.

*Proof.* The first inequality follows from [11, Lemma 3.5]; see also [9, Lemma 9.2]. The second inequality follows from the first and Corollary 1.

## 3. Solvability and Stability of the Complex Dirichlet Problem in $L^2$ for Bounded Strictly Convex Domains

For the reader's convenience, we include the following elementary facts (see also [10]).

LEMMA 2. *Let* $T: \mathbf{C}^n \to \mathbf{C}^n: z \mapsto z[P+iQ]$ *(P, Q real* $n \times n$ *matrices) be a* $\mathbf{C}$*-linear mapping, and let* $J: \mathbf{C}^n \to \mathbf{R}^n \times \mathbf{R}^n: (x+iy) \mapsto (x,y)$ *be the canonical isomorphism. Then* $JTJ^{-1}: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ *is given by*

$$(x,y) \mapsto (x,y)\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix}$$

*and* $|\det T|^2 = \det(JTJ^{-1})$.

*Proof.* The simple computations

$$\begin{aligned} JTJ^{-1}(x,y) &= JT(x+iy) = J[(x+iy)(P+iQ)] \\ &= J[xP - yQ + i(yP + xQ)] = (xP - yQ, yP + xQ) \end{aligned}$$

and

$$\begin{aligned} |\det T|^2 &= \det(P+iQ)\det(P-iQ) = \det\begin{bmatrix} P+iQ & 0 \\ 0 & P-iQ \end{bmatrix} \\ &= \det\begin{bmatrix} P+iQ & Q \\ 0 & P-iQ \end{bmatrix} = \det\begin{bmatrix} P+iQ & Q \\ iP-Q & P \end{bmatrix} = \det\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \end{aligned}$$

prove the lemma. □

THEOREM 6. *If* $0 \leq g \in C(\bar{\Omega})$ *and* $h \in C(\partial\Omega)$, *then*

$$U_{\mathbf{R}}(h, g^2) \leq U_{\mathbf{C}}(h, g).$$

*Proof.* Put $\varphi = U_{\mathbf{R}}(h, g^2)$, a convex function. By the comparison principle, it is sufficient to prove that

(3) $$\int_\Omega \psi M_{\mathbf{C}}(\varphi) \geq \int_\Omega \psi g\, dV \quad \text{for all } 0 \leq \psi \in C_0(\Omega).$$

Fix $\psi$. We can assume that $\partial\Omega \in C^\infty$. Otherwise, replace $\Omega$ by a strictly convex subdomain of $\Omega$ with $C^\infty$ boundary containing $\operatorname{supp}\psi$, and replace $h$ by the boundary values of $\varphi$ on this domain. Take a sequence $h_j \in C^\infty(\partial\Omega)$ converging uniformly to $h$, and a sequence $0 < g_j \in C^\infty(\bar{\Omega})$ converging uniformly to $g$. Put $\varphi_j = U_{\mathbf{R}}(h_j, g_j)$. Then $\varphi_j \to \varphi$ uniformly by Theorem 5, and $\int \psi M_{\mathbf{C}}(\varphi_j) \to \int \psi M_{\mathbf{C}}(\varphi)$ by Theorem 1. Furthermore, $\int \psi g_j\, dV \to \int \psi g\, dV$. Thus it is sufficient to prove (3) for $\partial\Omega \in C^\infty$, $0 < g \in C^\infty(\bar{\Omega})$, and $h \in C^\infty(\partial\Omega)$. But then $\varphi \in C^\infty(\bar{\Omega})$ by Theorem 4, so it suffices to prove that $(\det H_{\mathbf{C}}(\varphi))^2 \geq g^2 = \det H_{\mathbf{R}}(\varphi)$ or, by Lemma 2, that

(4) $$\det(JH_{\mathbf{C}}(\varphi)J^{-1}) \geq \det H_{\mathbf{R}}(\varphi).$$

To prove this, put

$$S = JH_{\mathbf{C}}(\varphi)J^{-1} = \begin{bmatrix} A+C & B^t - B \\ B - B^t & C+A \end{bmatrix}$$

by Lemma 2, and

$$S_1 = H_{\mathbf{R}}(\varphi) = \begin{bmatrix} A & B^t \\ B & C \end{bmatrix},$$

which is nonnegative since $\varphi$ is convex, and

$$S_2 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B^t \\ B & C \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} C & -B \\ -B^t & A \end{bmatrix}.$$

Then $S_1$ and $S_2$ have the same eigenvalues. In particular, $S_2$ is also nonnegative, and $\det S_1 = \det S_2$. Furthermore, $S = S_1 + S_2$, and the superadditivity of the determinant on nonnegative matrices (see the proof of Proposition 3.3 in [11]) gives $\det S \geq \det S_1 + \det S_2 \geq \det S_1$, which proves (4) and the theorem. □

THEOREM 7. *If $h \in C(\partial\Omega)$ and $0 \leq g \in L^2(\Omega)$, then $U_{\mathbf{C}}(h, g)$ exists and is in* $C(\bar{\Omega})$. Furthermore,

$$\inf_{\partial\Omega} h - C\|g\|_{2,\Omega}^{1/n} \leq U_{\mathbf{C}}(h, g) \leq \sup_{\partial\Omega} h$$

*and*

$$|U_{\mathbf{C}}(h_1, g_1) - U_{\mathbf{C}}(h_2, g_2)|_\Omega \leq |h_1 - h_2|_{\partial\Omega} + C\|g_1 - g_2\|_{2,\Omega}^{1/n}$$

*for all $h_1, h_2 \in C(\partial\Omega)$ and nonnegative $g_1, g_2 \in L^2(\Omega)$.*

*Proof.* It is sufficient to prove the inequalities for $g \in C(\bar{\Omega})$. The existence of $U_{\mathbf{C}}(h, g)$ for $g \in L^2(\Omega)$ then follows by approximation, using the second inequality and Theorem 1. Now Theorem 5 and Theorem 6 give

$$\inf_{\partial\Omega} h - C\|g^2\|_{1,\Omega}^{1/2n} \leq U_{\mathbf{R}}(h, g^2) \leq U_{\mathbf{C}}(h, g) \leq \sup_{\partial\Omega} h,$$

which proves the first inequality. The second inequality follows from the first and Corollary 1. This proves the theorem. □

REMARK. Theorem 7 answers the question of stability given in [1, p. 20], where also the first inequality is stated and is attributed to Cheng and Yau.

## 4. Solution of the Dirichlet Problem in $L^2$ for Strictly Pseudoconvex Domains

THEOREM 8. *Let $\Omega$ be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$, and let $h \in C(\partial\Omega)$ and $0 \le g \in L^2(\Omega)$. Then the following Dirichlet problem has a unique solution:*

$$\begin{cases} \varphi \in \mathrm{PSH} \cap C(\bar{\Omega}) \\ (dd^c\varphi)^n = g\,dV \\ \lim_{z \to \xi} \varphi(z) = h(\xi) \quad \textit{for all } \xi \in \partial\Omega. \end{cases}$$

*Proof.* The estimates in Theorem 7 hold also in this case. For take a bounded strictly convex domain $\tilde{\Omega}$ containing $\Omega$ and extend $|g_1 - g_2|$ by zero to an $L^2$-function on $\tilde{\Omega}$. Then $-U_{\mathrm{C}}(0, |g_1 - g_2|)$ is dominated by the corresponding function relative to $\tilde{\Omega}$ and Theorem 8 now follows from Theorem 7. □

REMARK. Let $\Omega$ be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$, $h \in C(\partial\Omega)$, and $H: \mathbf{R} \times \Omega \to [0, \infty]$ such that the function $z \mapsto \sup\{H(t, z): t \le \sup h\}$ is in $L^2(\Omega)$ and $H(\cdot, z)$ is continuous on $]-\infty, \sup h]$ for all fixed $z \in \Omega$. Then the following Dirichlet problem is solvable:

$$\begin{cases} \varphi \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega}) \\ (dd^c\varphi)^n = H(\varphi, z)\,dV \\ \lim_{z \to \xi} \varphi(z) = h(\xi) \quad \text{for all } \xi \in \partial\Omega. \end{cases}$$

Via Theorem 8, this is proved exactly as the theorem in [5].

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