A Mean-Value Theorem for Character Sums

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The classical mean-value theorem for Dirichlet polynomials asserts that

$$S(\lambda)=\frac{1}{\phi(q)}\sum_{\chi(q)}\left|\sum_{n\le N}\lambda_n\chi(n)\right|^2=(1+O(q^{-1}N))\|\lambda\|, \tag{1}$$

where

$$\|\lambda\|=\sum_{\substack{n\le N\\(n,q)=1}}|\lambda_n|^2$$

and the implied constant is absolute; cf. [M, Thm. 6.2], where also the result is refined. This result is best possible when $N<q$, in which case the result is true without the error term. In this paper we consider the case $N>q$, and improve on the result for special sequences. We consider convolutions $\lambda=\alpha*\beta*\gamma$ and $N=KLM$, where $\alpha=(\alpha_k)_{k\le K}$, $\beta=(\beta_l)_{l\le L}$, and $\gamma=(\gamma_m)_{m\le M}$, with α_k,β_l arbitrary and $\gamma_m=1$.

Let

$$S^*(\lambda)=\frac{1}{\phi(q)}\sum_{\substack{\chi(q)\\ \chi\ne\chi_0}}\left|\sum_{n\le N}\lambda_n\chi(n)\right|^2.$$

THEOREM 1. *We have*

$$S^*\ll\|\alpha\|\|\beta\|\|\gamma\|[1+q^{-3/4}(K+L)^{1/4}(KL)^{5/4}+q^{-1}(KL)^{7/4}]q^{\epsilon}.$$

REMARKS. Actually, by using (1) and restricting $M\ll q^{1/2}$ (without loss of generality), the term $q^{-1}(KL)^{7/4}$ can be dropped. In this paper, in order to keep the exposition clear, we present only the proof of a special case which does, however, contain all of the basic ideas.

This work was originally motivated by applications to character sums and Dirichlet L-functions. The Pólya–Vinogradov theorem [P; V] gave the first significant estimates in this area:

$$\sum_{m\le M}\chi(m)\ll q^{1/2}\log q$$

so that

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(2) $$\sum_{m \le M} \chi(m) \ll M^{1-\delta}$$

for $M > q^{1/2+\epsilon}$ and some $\delta = \delta(\epsilon) > 0$, and

(3) $$L(s, \chi) \ll q^{1/4} \log q$$

for $\operatorname{Re} s \ge \frac{1}{2}$, with an implied constant depending on s.

The problem of obtaining (2) for shorter intervals is important and difficult. It was only forty years later that Burgess succeeded in obtaining (2) for $M > q^{1/4+\epsilon}$, at first for prime moduli [B1] and then more generally [B2]. He also gave specific values for δ which, among other things, improve (3) to

(4) $$L(s, \chi) \ll q^{3/16+\epsilon}.$$

Burgess uses deep estimates for complete character sums due to Weil. Burgess' technique could be combined with a weaker result of Davenport [D] that preceded Weil's. This would give, for instance, (2) for $M > q^{4/9+\epsilon}$. Davenport's estimate, like Weil's, is based on the arithmetic of function fields and is not a simple one to obtain.

There has also been more recent work of interest in this area due to Gallagher [G], Hildebrand [H], Elliott [E], and Graham and Ringrose [GR].

Theorem 1 yields the following corollaries.

COROLLARY 1. *Let χ be a nonprincipal character* mod p. *We then have, for $M \ge p^{5/11+\epsilon}$,*

$$\sum_{m \le M} \chi(m) \ll M^{1-\delta},$$

where δ and the implied constant may depend on ϵ.

COROLLARY 2. *With χ as above, we have*

$$L(s, \chi) \ll p^{5/22+\epsilon} \quad \textit{for} \ \operatorname{Re} s \ge \tfrac{1}{2},$$

with an implied constant depending on ϵ and s.

In fact, both Theorem 1 and the corollaries can be quantitatively sharpened by using more sophisticated tools, for example, estimates for Kloosterman sums. We purposely refrain from using these in order to show that the Pólya–Vinogradov barrier can be beaten without "advanced technology"; in fact, we use only an estimate for the sum of a geometric progression. Since we are not straining for the best result, we make further compromises for technical simplification by proving only a special case of Theorem 1 that still suffices for Corollaries 1 and 2. Thus we make the following assumptions:

(5) q is prime;

(6) $\gamma(m) = f(m)$, where f is a smooth real function supported on $(\frac{1}{2}M, M)$ with derivatives satisfying

(7) $$f^{(j)} \ll M^{-j}, \quad j = 0, 1, 2, \ldots,$$

where the implied constant depends on j.

Proof of Theorem 1. We note that we can also assume $K, L, M < q$; otherwise, the result follows from the classical mean-value theorem for Dirichlet polynomials.

To the sum S^* we add a corresponding contribution from the principal character, namely,

$$S' = \frac{1}{q-1}\left|\sum_{k\le K}\alpha_k\right|^2\left|\sum_{l\le L}\beta_l\right|^2\left|\sum_{m\le M}f(m)\right|^2,$$

so that

$$S = S^* + S' = \sum_{k_1 l_1 m_1 \equiv k_2 l_2 m_2 \,(\mathrm{mod}\, q)}\cdots\sum \alpha_{k_1}\bar\alpha_{k_2}\beta_{l_1}\bar\beta_{l_2}f(m_1)f(m_2).$$

We split

$$S = \sum_{|r|<R} S_r,$$

where S_r denotes the contribution from terms with $k_1 l_1 m_1 - k_2 l_2 m_2 = qr$ and $R = KLMq^{-1}$. For $r = 0$ (the diagonal) we have the trivial estimate

$$S_0 \ll \|\alpha\|\|\beta\|\|\gamma\|q^{\epsilon}.$$

For $r \neq 0$, we put $\delta = (k_1 l_1, k_2 l_2)$, $n_1 = k_1 l_1 \delta^{-1}$, $n_2 = k_2 l_2 \delta^{-1}$, and $s = r\delta^{-1}$, so that $(n_1, n_2) = 1$ and $n_1 m_1 - n_2 m_2 = qs$. Equivalently,

$$m_1 \equiv qs\bar n_1 \,(\mathrm{mod}\, n_2).$$

For given δ, n_1, n_2, s we sum over m_1 using Poisson's formula and obtain

$$\sum_{m_1, m_2} f(m_1)f(m_2) = \sum_{m \equiv qs\bar n_1 \,(\mathrm{mod}\, n_2)} f(m) f\left(\frac{mn_1 - qs}{n_2}\right)$$
$$= \frac{1}{n_1 n_2}\sum_h e\left(-hqs\frac{\bar n_1}{n_2}\right)\int f\left(\frac{x}{n_1}\right) f\left(\frac{x-qs}{n_2}\right) e\left(\frac{hx}{n_2 n_2}\right) dx.$$

The terms with $h = 0$ give the main contribution:

$$T_s = \frac{1}{n_1 n_2}\int f\left(\frac{x}{n_1}\right) f\left(\frac{x-qs}{n_2}\right) dx.$$

Sum this over $s \neq 0$ to obtain

$$\sum_{s\neq 0} T_s = \frac{1}{n_1 n_2}\int f\left(\frac{x}{n_1}\right)\sum_{s\neq 0} f\left(\frac{x-qs}{n_2}\right) dx$$
$$= \frac{1}{n_1 n_2}\int f\left(\frac{x}{n_1}\right)\left(\int f\left(\frac{x-qy}{n_2}\right) dy + O(1)\right) dx$$
$$= \frac{1}{q}\left(\int f(x)\right)^2 + O\left(\frac{M}{n_2}\right).$$

By symmetry we may take the error term to be $O(M/n_1)$ and therefore $O(M/(n_1+n_2))$. These terms give a contribution to S of

$$\frac{1}{q}\sum_{\substack{k_1, l_1\\ k_2, l_2}} \alpha_{k_1}\bar\alpha_{k_2}\beta_{l_1}\bar\beta_{l_2}\left(\int f(x)\,dx\right)^2 + O\left(M\sum_{\substack{k_1, l_1\\ k_2, l_2}} \frac{|\alpha_{k_1}\alpha_{k_2}\beta_{l_1}\beta_{l_2}|(k_1 l_1, k_2 l_2)}{k_1 l_1 + k_2 l_2}\right) =$$

$$=\frac{1}{q}\sum_{\substack{k_1,l_1\\k_2,l_2}}\alpha_{k_1}\bar{\alpha}_{k_2}\beta_{l_1}\bar{\beta}_{l_2}\Big(\sum_m f(m)+O(1)\Big)^2+O(M\|\alpha\|\|\beta\|q^{\epsilon})$$
$$=S'+O(\|\alpha\|\|\beta\|\|\gamma\|(KLq^{-1}+q^{\epsilon})).$$

Thus the terms with $h=0$ cancel the main term, apart from admissible error terms.

We next consider the terms with $h\neq 0$. First we truncate. Integrate by parts j times, yielding

$$\int f\Big(\frac{x}{n_1}\Big)f\Big(\frac{x-qs}{n_2}\Big)e\Big(\frac{hx}{n_1n_2}\Big)dx \ll \Big(\frac{n_1n_2}{h}\Big)^j\int\Big|\frac{d^j(ff)}{dx^j}\Big|dx$$
$$\ll\Big(\frac{n_1n_2}{h}\Big)^j\Big(\frac{1}{n_1}+\frac{1}{n_2}\Big)^jM^{-j}M$$
$$\ll\Big(\frac{n_1+n_2}{hM}\Big)^jM\ll\Big(\frac{KL}{\delta hM}\Big)^jM,$$

by Leibniz' rule and (7). Hence if $|h|\geq H\doteq(KL/M\delta)q^{\epsilon}$ then the integral $\ll(\delta hq)^{-2}$, provided $j>j_0(\epsilon)$, so these terms contribute a negligible amount to S.

Now let $0<|h|<H$. These remaining terms give a contribution to S of $V=\sum_{0<\delta<R}V_{\delta}$, where

$$V_{\delta}=\sum_{\substack{0<|h|<H\\0<|s|<R/\delta}}\sum_{(k_1l_1,k_2l_2)=\delta}\alpha_{k_1}\bar{\alpha}_{k_2}\beta_{l_1}\bar{\beta}_{l_2}e\Big(-hqs\frac{\overline{k_1l_1/\delta}}{k_2l_2/\delta}\Big)$$
$$\times\int f\Big(\frac{x\delta}{k_1l_1}\Big)f\Big(\frac{x\delta-q\delta s}{k_2l_2}\Big)e\Big(\frac{hx\delta^2}{k_1l_1k_2l_2}\Big)\frac{\delta^2}{k_1l_1k_2l_2}\,dx.$$

We separate l_1 (in f) from the other variables. We first make the change of variable $x\to x(l_1k_2l_2)/\delta$, so the integral becomes

$$\int f\Big(\frac{xk_2l_2}{k_1}\Big)f\Big(xl_1-\frac{qs}{\delta k_2l_2}\Big)e\Big(hx\frac{\delta}{k_1}\Big)\frac{\delta}{k_1}\,dx.$$

To separate l_1 from the other parameters write

$$f(x)=\int\hat{f}(y)e(xy)\,dy,$$

where $\hat{f}$ is the Fourier transform, so that

$$\int|\hat{f}(y)|\,dy\ll\int_{-\infty}^{\infty}\min\Big(M,\frac{1}{y^2M}\Big)dy\ll 1.$$

Hence

$$f\Big(xl_1-\frac{qs}{\delta k_2l_2}\Big)=\int\hat{f}(y)e\Big(xyl_1-\frac{yqs}{\delta k_2l_2}\Big)dy,$$

giving

$$V_\delta \ll \delta \int_y \int_x \sum_{s,h} \sum_{\substack{k_2,l_2 \\ \delta | k_2 l_2}} \sum_{k_1} |\alpha_{k_1}\alpha_{k_2}\beta_{l_2}| f\left(\frac{k_2 l_2 x}{k_1}\right) \left| \sum_{l_1} \right| |\hat{f}(y)| \frac{dx}{k_1} dy$$

where

$$\sum_{l_1} = \sum_{\substack{l_1 \\ (l_1 k_1, l_2 k_2) = \delta}} \beta_{l_1} e\left(-hqs \frac{\overline{k_1 l_1/\delta}}{k_2 l_2/\delta} + xyl_1\right).$$

Hence

$$V_\delta \ll \frac{\delta M}{KL} \sum_{s,h} \sum_{k_2,l_2} \sum_{k_1} |\alpha_{k_1}\alpha_{k_2}\beta_{l_2}| \left| \sum_{l_1} \tilde{\beta}_{l_1} e\left(-hqs \frac{\overline{k_1 l_1/\delta}}{k_2 l_2/\delta}\right) \right|$$

for some $\tilde{\beta}_l$ with $|\tilde{\beta}_l| = |\beta_l|$. Fixing attention on (k_1, δ), we have for some $\delta^* | \delta$

$$V_\delta \ll \frac{\delta\tau(\delta)M}{KL} \sum_{s,h} \sum_{k_2,l_2} \sum_{\substack{k_1 \\ (k_1,\delta)=\delta^*}} |\alpha_{k_1}\alpha_{k_2}\beta_{l_2}| \left| \sum_{l_1} \right|.$$

Furthermore, by Cauchy's inequality, we get

$$V_\delta \ll \frac{\delta\tau(\delta)M}{KL} \left(\frac{RH}{\delta}\right)^{1/2} \|\alpha\| \|\beta\|^{1/2} \left(\sum_{k_2,l_2} \sum_{k_1,s,h} \left| \sum_{l_1} \tilde{\beta}_{l_1} e\left(-hqs \frac{\overline{k_1 l_1/\delta}}{k_2 l_2/\delta}\right) \right|^2 \right)^{1/2}.$$

Put $d = k_2 l_2/\delta$, so $d \le D$ with $D = KL/\delta$. Letting $\nu_d(x)$ be the number of solutions to $-hqs(\overline{k_1/\delta^*}) \equiv x \pmod d$, we can write

$$\sum\sum \left|\sum\right|^2 \ll q^\epsilon \sum_{d \le D} \sum_{x \pmod d} \nu_d(x) \left| \sum_l \beta_l e\left(\frac{x\bar{l}}{d}\right) \right|^2,$$

where $\beta_l = \tilde{\beta}_{l\delta/\delta^*}$.

We again apply Cauchy's inequality, obtaining

$$\sum\sum \left|\sum\right|^2 \ll q^\epsilon (XY)^{1/2},$$

where

$$X = \sum_d d \sum_{x \pmod d} \nu_d^2(x) = \sum_d d \#\{sh(k/\delta^*) \equiv s'h'(k'/\delta^*) \bmod d\}$$
$$\ll D\left(D + \frac{RHK}{\delta}\right) \frac{RHK}{\delta} q^\epsilon,$$

and, as we shall prove:

PROPOSITION. *We have*

$$Y \doteq \sum_{d \le D} d^{-1} \sum_{x \pmod d} \left| \sum_{l < L} \beta_l e\left(x \frac{\bar{l}}{d}\right) \right|^4 \ll \|\beta\|^2 (D + L^2) L^\epsilon.$$

Substituting in these estimates for X and Y, we have

$$V_\delta \ll \frac{\delta\tau(\delta)M}{KL} \left(\frac{RH}{\delta}\right)^{1/2} \|\alpha\| \|\beta\| \left[\frac{KL}{\delta^3} RHK(KL + RHK)(KL + L^2)\right]^{1/4} q^\epsilon.$$

Substituting for R and H and summing over δ,

$$V = \sum_\delta V_\delta \ll \|\alpha\| \|\beta\| \|\gamma\| (KL)^{5/4} q^{-1+\epsilon} (q + K^2 L)^{1/4} (K + L)^{1/4}.$$

By the symmetry in the problem we may replace K^2L by $\min(K^2L, KL^2)$, and this is bounded by $(KL)^2/(K+L)$. This gives Theorem 1, subject to the proposition.

Proof of Proposition. The left-hand side is

$$Y \le \sum_{d\le D} \sum_{\bar{l}_1+\bar{l}_2\equiv \bar{l}_3+\bar{l}_4 \,(\mathrm{mod}\, d)} |\beta_{l_1}\beta_{l_2}\beta_{l_3}\beta_{l_4}|$$
$$\le \sum_{d\le D} \sum_{l_2l_3l_4+l_1l_3l_4\equiv l_1l_2l_4+l_1l_2l_3 \,(\mathrm{mod}\, d)} = \sum_d \sum_= + \sum_d \sum_{\ne}$$
$$\ll D \sum_{1/l_1+1/l_2=1/l_3+1/l_4} |\beta_{l_1}\cdots\beta_{l_4}| + \sum_{l_1,\dots,l_4} |\beta_{l_1}\cdots\beta_{l_4}|\tau(l_2l_3l_4+\cdots).$$

The second sum $\ll L^2\|\beta\|^2L^\epsilon$; the first,

$$\ll \sum_{l_1,l_2} |\beta_{l_1}|^2|\beta_{l_2}|^2 \sum_{1/l_3+1/l_4=1/l_1+1/l_2} 1.$$

Now consider the equation

$$(l_3+l_4)l_1l_2 = l_3l_4(l_1+l_2).$$

Write $(l_3,l_4)=\Delta$, $l_3=\Delta d_3$, and $l_4=\Delta d_4$, where $(d_3,d_4)=1$. Then $d_3d_4\Delta \mid l_1l_2$ so, given l_1, l_2, there are at most L^ϵ possibilities. Thus $\sum_d \sum_= \ll D\|\beta\|^2L^\epsilon$, proving the result. □

Proof of Corollaries. We take $q=p$ and $\alpha_k=\bar{\chi}(k)$, $\beta_l=\bar{\chi}(l)$ $(k\le K,\ l\le L)$.

The contribution of $\chi\ (\mathrm{mod}\, p)$ to the sum S^* is bounded below by

$$\frac{(KL)^2}{p}\left|\sum_{m\le M}\chi(m)\right|^2.$$

The contribution to S^* from each of the other nonprincipal characters ψ $(\mathrm{mod}\, p)$ is greater than or equal to 0.

We have, by choosing $K=L=p^{3/11}$ and applying Theorem 1,

$$\left|\sum_{m\le M}\chi(m)\right| \ll M^{1/2}p^{5/22+\epsilon}.$$

This gives Corollary 1 at once and also, by a standard technique [B2], gives Corollary 2. □

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