Minimal Hypersurfaces Foliated by Spheres

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1. Introduction

Let M^n be an n-dimensional submanifold of $\mathbf{R}^{n+1}$. If there is a 1-parameter family of hyperplanes of $\mathbf{R}^{n+1}$ whose intersections with M^n are round spheres, then we refer to M^n as being foliated by spheres. If M^n is an (open) subset of $N^n \subset \mathbf{R}^{n+1}$, and the intersections with a family of planes are pieces of round spheres, then we say that M^n is foliated by pieces of spheres.

This article consists of two main results and a corollary. First, if $M^n \subset \mathbf{R}^{n+1}$ $(n \geq 3)$ is a minimal submanifold and M^n is foliated by spheres in parallel hyperplanes, then M^n is rotationally symmetric about an axis containing the centers of all the spheres. Second, if $M^n \subseteq \mathbf{R}^{n+1}$ $(n \geq 3)$ is a minimal submanifold and M^n is foliated by pieces of spheres, then the hyperplanes containing these spheres must be parallel. M^n is not assumed complete. However, from our two main results, if such an M^n is complete then it is a higher-dimensional catenoid. The author would like to thank Richard Schoen for helpful discussions.

In an 1867 article by Riemann and Hattendorff [12], it was shown that a minimal surface in $\mathbf{R}^3$ that is foliated by circles in parallel planes must be either a piece of a catenoid or the example now called the "Riemann staircase." In 1869, Enneper [4] showed that if a minimal surface is foliated by circles or by circular arcs, then the planes containing the circles or circular arcs must be parallel. A lengthier discussion of these results is available in the new English edition of Nitsche's book [9].

In 1956, Shiffman considered the related problem with boundaries [14]. If a minimal surface $M^2 \subseteq \mathbf{R}^3$ is bounded by convex curves in parallel planes, and if M^2 is topologically an annulus, then the intersections of M^2 with all other parallel planes are also convex curves. Similarly, if the boundaries are circles in parallel planes then the intermediate cross-sections must be circles.

There is a conjecture of William Meeks that the topological assumption is unnecessary in Shiffman's results (cf. [6, p. 87]).

In 1983, Schoen [13] developed a version of Alexandrov reflection [1] that applies to minimal submanifolds with boundary. In our first result we will

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be using Schoen's methods to establish symmetries of the submanifolds in question.

2. Notation and Methods

A submanifold is called *minimal* if its mean curvature vanishes identically. In this study, our submanifolds will be expressed as level sets of smooth functions. For a graph over $\mathbf{R}^n$ the mean curvature is

$$nH = \frac{1}{\sqrt{1+|\nabla f|^2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}\right) f_{ij}.$$

The mean curvature function was originally defined for parametrized surfaces in $\mathbf{R}^3$, therefore including graphs of functions. It is actually defined for any locally two-sided hypersurface of a Riemannian manifold, and, up to a sign change that follows a change in choice of unit normal, is an invariant quantity.

If M^n is a submanifold of $\mathbf{R}^{n+1}$ that is given as the level set $F=0$ for some function F, then the mean curvature of M is given by

$$nH = \frac{1}{|\nabla F|} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \left(\delta_{ij} - \frac{F_i F_j}{|\nabla F|^2}\right) F_{ij}.$$

If one substitutes $F = f(x_1, \ldots, x_n) - x_{n+1}$, then a short calculation will show that the mean curvature for the level surface $F=0$ is the same as that given above for the graph $x_{n+1} = f(x_1, \ldots, x_n)$. Meanwhile, some rearrangement puts the above expression in the form of a quadratic form acting on the gradient vector of F, that is,

$$nH = \frac{1}{|\nabla F|^3} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (|\nabla F|^2 \delta_{ij} - F_i F_j) F_{ij},$$

$$nH = \frac{1}{|\nabla F|^3} \left(\Delta F |\Delta F|^2 - \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} F_{ij} F_i F_j\right),$$

and

$$nH = \frac{1}{|\nabla F|^3} (\nabla F^t \cdot B \cdot \nabla F),$$

where B is the symmetric matrix

$$B = \Delta F \cdot I - \text{Hess}\, F \quad \text{or} \quad B_{ij} = \Delta F \delta_{ij} - F_{ij}.$$

In another setting, suppose M^n is a level set $f=0$ of some function f that is expressed in a coordinate system $\{y^i\}$ on $\mathbf{R}^{n+1}$. To check possible minimality of M, we first construct the gradient vector field of f, defined thus:

$$\nabla f = \sum g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}.$$

The unit normal $\vec{N}$ to all the level sets of f is now the gradient vector of f divided by its own length, deduced from

$$|\nabla f|^2 = \sum g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}.$$

Meanwhile, for any vector field $\vec{Y} = \sum a_i(\partial/\partial y^i)$, the divergence of y is given by

$$\nabla \cdot \vec{Y} = \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial y^i}(a_i \sqrt{g}).$$

Altogether, M is minimal if and only if the divergence of the unit normal field $\vec{N}$ is identically 0 on M.

For our first result, we will also need a result of Schoen [13] that is based on the maximum principle for elliptic partial differential equations. Essentially, we will need to show that since the boundary B of our minimal submanifold M^n in $\mathbf{R}^{n+1}$ enjoys some reflection symmetries, so does the submanifold M^n itself.

We will simply repeat the definitions appropriate to Schoen's Theorem 2. Let $p: \mathbf{R}^{n+1} \to \pi_0$ be an orthogonal projection onto the hyperplane π_0. A subset N will be called a *graph* over π_0 if the projection of N into π_0 is one-to-one. If a graph N is the closure of a C^2 submanifold, N will be said to have *locally bounded slope* if the vector that is normal to π_0 is not tangent to N, except possibly at points in $N \cap \pi_0$. Finally, N is a *digraph* over π_0 if the intersection of N with either of the closed half-spaces that have π_0 as a boundary is a graph with locally bounded slope.

Suppose that B is a digraph over a plane π_0. Suppose further that B is contained in the cylinder $p^{-1}(\partial\Omega)$, where $\Omega \subset \pi_0$ is a bounded C^2 domain and $\partial\Omega$ has nonpositive mean curvature as a subset of π_0. Schoen's Theorem 2 states that if B is invariant under reflection through π_0, if M is a minimal submanifold with $\partial M = B$, and if the interior of M is contained in the interior of $p^{-1}(\Omega)$, then M is also invariant under reflection through π_0. In this paper, $p^{-1}(\partial\Omega)$ will usually include pieces of other hyperplanes, with one component of B in each. In fact, ∂M will usually have exactly two components, and these will be compact subsets of hyperplanes, not always parallel.

In what follows, the word "plane" will be understood to mean "n-dimensional affine subspace of $\mathbf{R}^{n+1}$," and the word "sphere" will mean a round S^{n-1} in such a plane. For example, if $n=2$, "sphere" would refer to a circle in $\mathbf{R}^3$.

3. First Result

THEOREM 1. *If M^n is a minimal submanifold in $\mathbf{R}^{n+1}$ with $n \geq 3$, and if M is foliated by round spheres lying in parallel planes, then M is a hypersurface of revolution.*

Proof. Pick two particular hyperplanes that intersect M in S^{n-1}'s. Then move M by rigid motions to arrange that these planes, perhaps called π_0 and π_1, are parallel to the plane $x_{n+1}=0$. Additionally, arrange that the centers of both the spheres $M \cap \pi_0$ and $M \cap \pi_1$ lie in the $x_1 x_{n+1}$ plane.

We need to show that the center of every parallel spherical cross-section lies in the $x_1 x_{n+1}$ plane. We may consider $B=(\pi_0 \cap M)\cup(\pi_1 \cap M)$ to be the boundary of the subset of M lying between the two planes π_0 and π_1. B is invariant under each of the reflections $x_2 \to -x_2, \dots, x_n \to -x_n$. By Schoen's Theorem 2, M itself inherits these reflection symmetries, at least between the planes π_0 and π_1. Since the two planes could have been chosen as far apart as desired, we conclude that all of M does indeed inherit the symmetries mentioned. In particular, the center of every spherical cross-section of M lies in the same 2-plane, that where $x_2=0, \dots, x_n=0$.

It is now clear that M is the level set of a smooth function f. Let us rename the x_{n+1} coordinate direction by $t=x_{n+1}$. We need two functions of t, $r(t)$ and $c(t)$, to denote (resp.) the radius and the x_1 coordinate of the center of the sphere in hyperplane $x_{n+1}=t$. M is the set of points where

$$0=f=-r(t)^2+(x_1-c(t))^2+\sum_{i=2}^{n} x_i^2.$$

To calculate the mean curvature of M, it is only necessary to calculate the gradient and the matrix of second partials of f and combine them properly:

$$f=-r(t)^2+(x_1-c(t))^2+\sum_{i=2}^{n} x_i^2;$$

$$\tfrac{1}{2}\nabla f=(x_1-c, x_2, \dots, x_n, -(x_1-c)c'-rr');$$

$$|\tfrac{1}{2}\nabla f|^2=r^2+(rr'+(x_1-c)c')^2.$$

Defining $a=(c-x_1)c''+c'^2-r'^2-rr')$, we find that

$$\frac{1}{2}\text{Hessian}(f)=\begin{pmatrix} 1 & 0 & \cdots & 0 & -c' \\ 0 & 1 & 0 & \vdots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -c' & 0 & \cdots & 0 & a \end{pmatrix},$$

$$\tfrac{1}{2}\Delta f=n-(x_1-c)c''-rr''+c'^2-r'^2.$$

By defining

$$b_{11}=n-1-(x_1-c)c''+c'^2-rr''-r'^2,$$

we have

$$\frac{1}{2}B=\begin{pmatrix} b_{11} & 0 & \cdots & 0 & c' \\ 0 & b_{11} & 0 & \vdots & 0 \\ \vdots & 0 & b_{11} & 0 & \vdots \\ 0 & \cdots & 0 & b_{11} & 0 \\ c' & 0 & \cdots & 0 & n \end{pmatrix}.$$

We may now evaluate the quadratic form B on the vector ∇f, by forming the matrix product $\frac{1}{8}\nabla f \cdot B \cdot \nabla f^t$. Restricting to the submanifold M, that is, requiring that

$$(x_1-c(t))^2+\sum_{i=2}^{n} x_i^2=r(t)^2,$$

we conclude that the condition that M be minimal is that

$$0=r^2 b_{11}+n[-(x_1-c)'-rr']^2+2c'(x_1-c)[-(x_1-c)c'-rr'].$$

The functions r, c, r', c', r'', c'' are functions of $t=x_{n+1}$ only, or (more to the point) have constant values on any fixed plane $x_{n+1}=t$. We may therefore examine the values of these functions at different locations in M and find equations that hold simultaneously. In particular, if we evaluate the equation that describes minimality along the three subsets of M wherein (i) $x_1=c(t)+r(t)$, (ii) $x_1=c-r$, and (iii) $x_1=c$, we get the following three (slightly rearranged) equations:

$$r(r''+c'')=(n-1)(1+(r'+c')^2);$$
$$r(r''-c'')=(n-1)(1+(r'-c')^2);$$
$$rr''=(n-1)(1+r'^2)+c'^2.$$

Taking half the sum of the first two equations, we have

$$rr''=(n-1)(1+r'^2+c'^2).$$

Subtracting the third equation, we finally obtain

$$(n-2)c'^2=0.$$

Since $n \geq 3$, we conclude that $c(t)$ is constant and hence M is a hypersurface of revolution. This completes the proof of Theorem 1. □

REMARK 1. Notice that our method gives no conclusion when $n=2$, and in fact, for the Riemann staircase, c' is not zero.

REMARK 2. In keeping with our next results, we might instead complete this proof by an argument based on considering the minimality equation as a polynomial in x_1-c. The coordinate x_1, restricted to M, takes all values from $c(t)-r(t)$ to $c(t)+r(t)$, so x_1-c ranges from $-r$ to r. Rearranging the equation for minimality gives us

$$r^2(rr''+(x_1-c)c'')=r^2(n-1+c'^2-r'^2)$$
$$+(rr'+(x_1-c)c')(nrr'+(n-2)(x_1-c)c').$$

Regarding this as an equation between polynomials in x_1-c, we find that the highest-degree term present is the quadratic term $(x_1-c)c'\cdot(n-2)(x_1-c)c'$, or $(n-2)c'^2\cdot(x_1-c)^2$. The equality of the polynomials gives us that the quadratic coefficient is zero, or $(n-2)c'^2=0$.

4. Second Result

THEOREM 2. *If M^n is a minimal submanifold in $\mathbf{R}^{n+1}$ with $n \geq 3$, and if M is foliated by pieces of round spheres lying in a 1-parameter family of planes, then the planes in the family are actually parallel.*

Given a family of hyperplanes in $\mathbf{R}^{n+1}$, we wish to construct a coordinate system on an open subset of $\mathbf{R}^{n+1}$ that respects foliation. The first construct is the unit vector field that is normal to the hyperplanes, named N_0. Next, pick a particular integral curve γ of the field N_0 parametrized by arc length. Then a hyperplane is labelled π_t precisely when the point $\gamma(t)$ lies in the hyperplane. Equivalently, the function t which is arc length along γ is extended to be constant on each plane in the family. It is important to notice that the gradient vector field ∇t is not assumed to have constant length.

Next, we construct a moving frame that respects the foliation. As in the Frenet frame for a space curve, we construct the vector N_1 by requiring that

$$\dot{N}_0 = \kappa_0 N_1,$$

where the dot in $\dot{N}_0$ means $\partial N_0/\partial t$. N_2 is constructed by requiring that

$$\dot{N}_1 = -\kappa_0 N_0 + \kappa_1 N_2.$$

For $1 \leq i \leq n-1$,

$$\dot{N}_i = -\kappa_{i-1} N_{i-1} + \kappa_i N_{i+1},$$

while

$$\dot{N}_n = -\kappa_{n-1} N_{n-1}.$$

To save space, we will cease writing out the separate cases that apply for the subscripts 1 and n. From now on, all summations will be written with the understanding that

$$\kappa_{-1} = 0, \quad \kappa_n = 0, \quad v_0 = 0, \quad v_{n+1} = 0, \quad N_{n+1} = 0.$$

We now extend the Frenet frame by requiring that each of the fields N_i remain constant on each hyperplane π_t. Accordingly, we will be able to perform all necessary calculations by using the vectors N_i and the various curvature functions κ_i just as these are defined on γ. While a different choice of γ would make for different values of the κ's, it is nevertheless true that the planes π_t must be parallel if κ_0 is equal to 0.

The hypothesis of our main theorem is that each hyperplane intersects the submanifold in a geometrically round sphere, or at least in an open subset of such a sphere. Each hyperplane thus contains the center of its sphere, and we will call this center $c(t)$ in π_t. We introduce some functions $a_i(t)$ and $\alpha_i(t)$ by equating

$$c(t) = \gamma(t) + \sum_{i=1}^{n} a_i(t) N_i \quad \text{and} \quad \dot{c} = \sum_{k=0}^{n} \alpha_i N_i.$$

We define a map $\vec{X}$ from $\mathbf{R}^{n+1}$ to $\mathbf{R}^{n+1}$ by

$$\vec{X}(t, \vec{v}) = c(t) + r(t) \sum_{i=1}^{n} v_i N_i,$$

where $r(t)$ is the radius of the appropriate sphere. To check where this will provide a local coordinate system, we will check that the determinant of the inner-product matrix g_{ij} is nonzero. We will write g_{0i} for the inner product of $\partial\vec{X}/\partial t$ and $\partial\vec{X}/\partial v_i$, and g_{00} for the squared length of $\partial\vec{X}/\partial t$. Then

$$\dot{X}=\frac{\partial \vec{X}}{\partial t}=\dot{c}+\dot{r}\sum_{j=1}^{n} v_j N_j+r\sum_{j=1}^{n} v_j \dot{N}_j$$

$$=(\alpha_0-rv_1\kappa_0)N_0+\sum_{j=1}^{n}(\alpha_j+\dot{r}v+rv_{j-1}\kappa_{j-1}-rv_{j+1}\kappa_j)N_j;$$

$$\frac{\partial \vec{X}}{\partial v_i}=rN_i.$$

For $i,j\geq 1$, $g_{ij}=\delta_{ij}r^2$. Next,

$$g_{0i}=\left(\dot{c}+\dot{r}\sum_{j=1}^{n} v_j N_j+r\sum_{j=1}^{n} v_j \dot{N}_j\right)\cdot(rN_i)$$

$$=r\alpha_i+r\dot{r}v_i+r^2\sum_{j=1}^{n} v_j\dot{N}_j N_i$$

$$=r\alpha_i+r\dot{r}v_i+r^2(v_{i-1}\kappa_{i-1}-v_{i+1}\kappa_i).$$

We will repeatedly use the simplifications resulting from the equation

$$\sum_{i=1}^{n} g_{0i}v_i=r\sum_{i=1}^{n}\alpha_i v_i+r\dot{r}\sum_{i=1}^{n} v_i^2.$$

Introducing the common notations

$$\alpha\cdot v=\sum_{i=1}^{n}\alpha_i v_i \quad\text{and}\quad |v|^2=v\cdot v=\sum_{i=1}^{n} v_i^2,$$

we may write

$$\sum_{i=1}^{n} g_{0i}v_i=r[(\alpha\cdot v)+\dot{r}|v|^2].$$

Also,

$$g_{00}=\left|\dot{c}+\dot{r}\sum_{j=1}^{n} v_j N_j+r\sum_{j=1}^{n} v_j\dot{N}_j\right|^2$$

$$=\left|\sum_{j=1}^{n}\alpha_i N_i+\sum_{j=1}^{n}\dot{r}v_j N_j+r\sum_{j=1}^{n} v_j(-\kappa_{j-1}N_{j-1}+\kappa_j N_{j+1})\right|^2$$

$$=(\alpha_0-rv_i\kappa_0)^2+\sum_{j=1}^{n}(\alpha_j+\dot{r}v_j-rv_{j+1}\kappa_j+rv_{j-1}\kappa_{j-1})^2$$

$$=(\alpha_0-rv_1\kappa_0)^2+\frac{1}{r^2}\sum_{j=1}^{n} g_{0j}^2.$$

Altogether, the inner product matrix is the following, given the quantities g_{00} and g_{0i} as calculated:

$$G=\begin{pmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0n}\\ g_{01} & r^2 & 0 & \cdots & 0\\ g_{02} & 0 & r^2 & & \vdots\\ \vdots & \vdots & & r^2 & 0\\ g_{0n} & 0 & \cdots & 0 & r^2\end{pmatrix}.$$

Expanding by minors, we find that the determinant g of this matrix is given by

$$g=\det(G)=r^{2n-2}\left(r^2 g_{00}-\sum_{i=1}^{n} g_{0i}^2\right).$$

However,

$$r^2 g_{00}-\sum_{j=1}^{n} g_{0i}^2=r^2|\dot{X}|^2-r^2\sum_{j=1}^{n}(\dot{X}\cdot N_j)^2 r^2(\dot{X}\cdot N_0)^2$$
$$=r^2(\alpha_0-rv_1\kappa_0)^2.$$

Therefore,

$$g=\det(G)=r^{2n}(\alpha_0-rv_1\kappa_0)^2.$$

At this point, we may pause to consider the effect of the case $g=0$, when $\vec{X}$ ceases to be an immersion. Since v_1 is free to take all values from -1 to 1, the condition $g\equiv 0$ implies that both $\alpha_0\equiv 0$ and $r\kappa_0\equiv 0$, or with nonzero radius that $\kappa_0\equiv 0$. In turn, this would imply that the map $\vec{X}$ that we defined is simply a peculiar parametrization for a *single* hyperplane in $\mathbf{R}^{n+1}$. Such a hyperplane is a perfectly legal minimal submanifold, since it is totally geodesic. Meanwhile, we will begin to assume that for a small t interval and a small v_1 interval, g can be required to be nonzero. This allows us to use $(t, v_1, \ldots, v_n)$ as a coordinate system on an open subset of $\mathbf{R}^{n+1}$. We will eventually show that the minimality of M^n implies that $\kappa_0=0$, meaning that the planes π_t are parallel.

Our submanifold M^n is described as the level set of a function, namely,

$$0\equiv f(t, v_1,\ldots, v_n)=v_1^2+\cdots+v_n^2-1.$$

In a general Riemannian manifold, the condition that a level set be a *minimal* submanifold is that the divergence of the unit normal field $\nabla f/|\nabla f|$ be 0 when restricted to the submanifold in question. The *gradient* and *divergence* are defined as follows: for some function f,

$$\nabla f=\sum g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^j},$$

and

$$|\nabla f|^2=\sum g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}.$$

For a vector field $\vec{Y}=\sum \alpha^i(\partial/\partial y^i)$,

$$\nabla\cdot\vec{Y}=\frac{1}{\sqrt{g}}\sum\frac{\partial}{\partial y^i}(\alpha^i\sqrt{g}).$$

With $f(t, v_1,\ldots, v_n)=v_1^2+\cdots+v_n^2-1$,

$$\frac{\partial f}{\partial t}=0,\qquad \frac{\partial f}{\partial v_j}=2v_j,$$

$$\frac{1}{2}\nabla f=\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}v_i\frac{\partial}{\partial v_j}+\sum_{i=1}^{n} g^{i0}v_i\frac{\partial}{\partial t}.$$

However, since the indices in g_{ij} range from 0 to n, we notice that

$$\sum_{j=0}^{n} g_{ij} g^{jk} = \delta_i^k$$

and

$$\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k - g_{i0} g^{0k}.$$

Altogether, for the particular function f above,

$$\frac{1}{4}|\nabla f|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} v_i v_j.$$

Thus M^n is minimal if and only if it satisfies the following equation:

$$0 = \frac{\partial}{\partial t}\left(\sqrt{g}\frac{\sum_{k=1}^{n} g^{0k} v_k}{\sqrt{\sum_{k=1}^{n}\sum_{l=1}^{n} g^{kl} v_k v_l}}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial v_i}\left(\sqrt{g}\frac{\sum_{k=1}^{n} g^{ik} v_k}{\sqrt{\sum_{k=1}^{n}\sum_{l=1}^{n} g^{kl} v_k v_l}}\right).$$

It will not be necessary to explicitly calculate the entries g^{ij} of the matrix G^{-1}, because these are always summed against other quantities. Instead, we introduce the auxiliary functions w^i, defined by

$$w^i = \sum_{k=1}^{n} g^{ik} v_k.$$

These functions obey the rules

$$\sum_{l=0}^{n} g_{0l} w^l = 0 \quad \text{and} \quad \sum_{l=0}^{n} g_{il} w^l = v_i.$$

If $i \geq 1$ then $g_{ij} = r^2 \delta_{ij}$, so the second sum above gives

$$g_{i0} w^0 + r^2 w^i = v_i,$$

or $w^i = r^{-2}(v_i - g_{0i} w^0)$. Substituting these values into $\sum_{l=0}^{n} g_{0l} w^l = 0$, we find that

$$g_{00} w^0 + \sum_{i=1}^{n} g_{0i} r^{-2}(v_i - g_{0i} w^0) = 0.$$

Continuing,

$$\left(g_{00} - r^{-2} \sum_{i=1}^{n} g_{0i}^2\right) w^0 = -r^{-2} \sum_{i=1}^{n} g_{0i} v_i,$$

and from our earlier calculation of the determinant g of G, we know that

$$g_{00} - r^{-2} \sum_{i=1}^{n} g_{0i}^2 = (\alpha_0 - r v_1 \kappa_0)^2.$$

While calculating the metric coefficients g_{0i}, we also noted that

$$\sum_{i=1}^{n} g_{0i} v_i = r[(\alpha \cdot v) + \dot{r}|v|^2].$$

Taken together, these show that

$$w^0 = \frac{-((\alpha\cdot v)+\dot r|v|^2)}{r(\alpha_0 - rv_1\kappa_0)^2}.$$

Remembering that $w^i = r^{-2}(v_i - g_{0i}w^0)$, we also find that

$$w^i = r^{-2}\left(v_i + g_{0i}(\alpha_0 - rv_i\kappa_0)^{-2}r^{-2}\sum_{j=1}^n g_{0j}v_j\right)$$

and

$$\sum_{i=1}^n w^i v_i = r^{-2}\left(|v|^2 + (\alpha_0 - rv_1\kappa_0)^{-2}r^{-2}\left(\sum_{j=1}^n g_{0j}v_j\right)^2\right),$$

while again using $\sum_{i=1}^n g_{0i}v_i = r[(\alpha\cdot v)+\dot r|v|^2]$ gives us finally that

$$\sum_{i=1}^n w^i v_i = r^{-2}[|v|^2 + (\alpha_0 - rv_1\kappa_0)^{-2}((\alpha\cdot v)+\dot r|v|^2)^2],$$

or

$$\sum_{i=1}^n w^i v_i = r^{-2}[|v|^2 + r^2(\alpha_0 - rv_1\kappa_0)^2(w^0)^2].$$

Recalling the definitions of the functions w^i, we find that

$$\sum_{i=1}^n\sum_{j=1}^n g^{ij}v_iv_j = \sum_{k=1}^n w^k v_k.$$

We may now rewrite the equation that would demonstrate minimality for M as

$$0 = \frac{\partial}{\partial t}\left(\sqrt{g}\frac{w^0}{\sqrt{\sum_{k=1}^n w^k v_k}}\right) + \sum_{i=1}^n \frac{\partial}{\partial v_i}\left(\sqrt{g}\frac{w^i}{\sqrt{\sum_{k=1}^n w^k v_k}}\right).$$

We had previously found that $\sqrt{g} = r^n(\alpha_0 - rv_1\kappa_0)$ and $w^i = r^{-2}(v_i - g_{0i}w^0)$, so minimality for M depends on the truth of the equation

$$\begin{aligned}0 = {}& \frac{\partial}{\partial t}\left(r^{n+1}(\alpha_0 - \kappa_0 v_1 r)\frac{w^0}{\sqrt{(v\cdot v)+r^2(\alpha_0-\kappa_0v_1r)^2(w^0)^2}}\right)\\ &+ \sum_{i=1}^n \frac{\partial}{\partial v_i}\left(r^{n-1}(\alpha_0-\kappa_0v_1r)\frac{v_i - g_{0i}w^0}{\sqrt{(v\cdot v)+r^2(\alpha_0-\kappa_0v_1r)^2(w^0)^2}}\right).\end{aligned} \tag{4.1}$$

Because $r(\alpha_0 - r\kappa_0v_1)^2w^0 = -(\alpha\cdot v + \dot r(v\cdot v))$ and $r^2(\alpha_0 - r\kappa_0v_1)^4(w^0)^2 = (\alpha\cdot v + \dot r(v\cdot v))^2$, we can multiply through and obtain

$$\begin{aligned}0 = {}& \frac{\partial}{\partial t}\left(r^n\frac{r(\alpha_0 - r\kappa_0v_1)^2w^0}{\sqrt{(\alpha_0-r\kappa_0v_1)^2(v\cdot v)+r^2(\alpha_0-\kappa_0v_1r)^4(w^0)^2}}\right)\\ &+ \sum_{i=1}^n \frac{\partial}{\partial v_i}\left(r^{n-2}\frac{r(\alpha_0-r\kappa_0v_1)^2(v_i - g_{0i}w^0)}{\sqrt{(\alpha_0-r\kappa_0v_1)^2(v\cdot v)+r^2(\alpha_0-\kappa_0v_1r)^4(w^0)^2}}\right)\end{aligned} \tag{4.2}$$

or

$$\begin{aligned}0 = {}& \frac{\partial}{\partial t}\left(r^n\frac{-(\alpha\cdot v+\dot r(v\cdot v))}{\sqrt{(\alpha_0-r\kappa_0v_1)^2(v\cdot v)+(\alpha\cdot v+\dot r(v\cdot v))^2}}\right)\\ &+ \sum_{i=1}^n \frac{\partial}{\partial v_i}\left(r^{n-2}\frac{r(\alpha_0-r\kappa_0v_1)^2v_i + (\alpha\cdot v+\dot r(v\cdot v))g_{0i}}{\sqrt{(\alpha_0-r\kappa_0v_1)^2(v\cdot v)+(\alpha\cdot v+\dot r(v\cdot v))^2}}\right).\end{aligned} \tag{4.3}$$

Given the above equation which must be satisfied, we will define the abbreviations

$$D=(\alpha_0-r\kappa_0 v_1)^2(v\cdot v)+(\alpha\cdot v+\dot{r}(v\cdot v))^2,$$

$$T_0=-r^n(\alpha\cdot v+\dot{r}(v\cdot v)),$$

and

$$T_i=r^{n-1}(\alpha_0-r\kappa_0 v_1)^2 v_i+r^{n-2}(\alpha\cdot v+\dot{r}(v\cdot v))g_{0i}.$$

Repeating $\sum_{i=1}^{n} v_i g_{0i}=r(\alpha\cdot v+\dot{r}(v\cdot v))$ with the above gives us

$$\sum_{i=1}^{n} v_i T_i=r^{n-1}[\,(\alpha_0-r\kappa_0 v_1)^2(v\cdot v)+(\alpha\cdot v+\dot{r}((v\cdot v))^2]$$

or

$$\sum_{i=1}^{n} v_i T_i=r^{n-1}D.$$

We now get the shortest looking version of the minimality condition for M^n, that the following equation should be true on M:

$$0=\frac{\partial}{\partial t}\left(\frac{T_0}{\sqrt{D}}\right)+\sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{T_i}{\sqrt{D}}\right). \tag{4.4}$$

If we multiply the above equation by $2D^{3/2}$, we get

$$0=\left(2\frac{\partial T_0}{\partial t}D-T_0\frac{\partial D}{\partial t}\right)+\sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D-T_i\frac{\partial D}{\partial v_i}\right). \tag{4.5}$$

We then define $\mathbf{P}$ by

$$\mathbf{P}=\left(2\frac{\partial T_0}{\partial t}D-T_0\frac{\partial D}{\partial t}\right)+\sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D-T_i\frac{\partial D}{\partial v_i}\right).$$

To complete our theorem, we regard the preceding equation as a polynomial equation $\mathbf{P}=0$ in the variables v_i, with coefficients that are functions of the independent variable t. We will postpone the details of the calculation of the leading coefficients of $\mathbf{P}$ until the appendix, and simply present the coefficients themselves now. Restricting $\mathbf{P}$ to M^n will be effected by reducing the polynomial modulo $(v\cdot v)-1\equiv 0$, after performing all the differentiations required. The polynomial is eventually seen to have degree exactly four, with the highest-degree terms being given by

$$2nr^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2+(\alpha\cdot v)^2).$$

What is the effect of requiring that equation (4.5) hold on the set M where $(v\cdot v)-1\equiv 0$? If $n=2$, (4.5) can be rewritten with $v_1=\cos\theta$, $v_2=\sin\theta$, and a trigonometric polynomial results from the right-hand side of (4.5). The trigonometric polynomial can be uniquely written as a finite Fourier series, and the resulting Fourier coefficients set to 0.

If $n\geq 3$, we cannot readily rewrite the resulting polynomial equation as a Fourier series equation. However, it is still possible to work out some linear basis for the equivalence classes of polynomials modulo $(v\cdot v)-1\equiv 0$. In particular, such a basis can be arranged entirely of homogeneous polynomials, allowing us to separately require that the highest-degree part of our polynomial, $2nr^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2+(\alpha\cdot v)^2)$, reduce to the zero polynomial modulo $(v\cdot v)-1\equiv 0$. Furthermore, the polynomials that vanish on the unit sphere $\sum_{i=1}^n v_i^2$ make up a prime ideal in the ring of polynomials in the variables v_i. Since our homogeneous polynomial $2nr^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2+(\alpha\cdot v)^2)$ is already factored, we find that setting it to the zero polynomial gives either $\kappa_0\equiv 0$ or

$$r^2\kappa_0^2 v_1^2+(\alpha\cdot v)^2\equiv 0. \tag{4.6}$$

If $n=2$, this last condition will not be enough to provide the desired conclusion $\kappa_0=0$. Since $(v\cdot v)-1\equiv 0$, a basis for the quadratic polynomials is $v_1 v_2$ and $v_1^2-v_2^2$. We therefore may only conclude that

$$r^2\kappa_0^2+\alpha_1^2+\alpha_2^2=0 \quad \text{and} \quad 2\alpha_1\alpha_2=0.$$

This does show that at least one of the α's is 0. If we knew that specifically $\alpha_2=0$, it would then follow that $r^2\kappa_0^2+\alpha_1^2=0$, or both $\kappa_0=0$ and $\alpha_1=0$. However, this just shows that we can only assume $\alpha_1=0$, with $\alpha_2=\pm r\kappa_0$, and the status of κ_0 still uncertain.

If $n\geq 3$, these degree-4 homogeneous parts of our polynomial do provide enough information to conclude that $\kappa_0=0$. For example, for $n=3$, we get exactly five equations from the quadratic polynomial (4.6):

$$\alpha_1\alpha_2=0, \quad \alpha_1\alpha_3=0, \quad \alpha_2\alpha_3=0,$$
$$\alpha_2^2-\alpha_3^2=0, \quad \text{and} \quad r^2\kappa_0^2+\alpha_1^2-\alpha_3^2=0.$$

The mixed terms $\alpha_i\alpha_j=0$ tell us that no more than one of the α_i's is *not* 0. The the fourth equation improves that to give that both α_3 and α_2 are 0. The fifth equation finally shows that, indeed, $\kappa_0=0$.

The situation for all $n\geq 3$ is quite similar to the above description of the case $n=3$. In setting

$$r^2\kappa_0^2 v_1^2+(\alpha\cdot v)^2\equiv 0$$

modulo $(v\cdot v)-1\equiv 0$, we have only to consider a single further relation holding among the variables v_i, and this can be fixed as the substitution

$$v_n^2=1-v_1^2-\cdots-v_{n-1}^2.$$

This results in the following equations:

(i) $\alpha_i\alpha_j=0$ for all $1\leq i\neq j\leq n$,
(ii) $\alpha_j^2-\alpha_n^2=0$ for all $2\leq j\leq n-1$, and
(iii) $r^2\kappa_0^2+\alpha_1^2-\alpha_n^2=0$.

The equations listed as (i) and (ii) show that $\alpha_2=0,\ldots,\alpha_n=0$. Equation (iii) now reads

$$r^2\kappa_0^2+\alpha_1^2=0.$$

Since the radius r is strictly positive, and the squared terms are in any case nonnegative, this forces α_1 and finally κ_0 to be 0. This completes the proof of Theorem 3. □

REMARK. At this point, we may redo our first result using only local assumptions. Since $\kappa_0 = 0$, the fourth-degree terms in $\mathbf{P}$ vanish. Further calculation shows that the third-degree terms all cancel, and the quadratic terms are exactly those in $(2n-4)\alpha_0^2 r^{n-1}(\alpha\cdot v)^2$. Since M is assumed nonplanar, α_0 is not zero. Since n is not 2 (else we would allow the Riemann staircase), $2n-4$ is not zero. After applying the condition $v\cdot v = 1$, an argument similar to the end of our second main result shows that all the other α's are zero, and the pieces of spheres in M are coaxial. Using uniqueness of continuation by the maximum principle, we obtain the following corollary.

COROLLARY. *If M^n is a minimal submanifold in $\mathbf{R}^{n+1}$ with $n \geq 3$, M is complete and nonplanar, and an open subset of M is foliated by pieces of spheres, then M is a higher-dimensional catenoid.*

5. Appendix

We still need to confirm the assertion that the polynomial given by

$$\mathbf{P} = \left(2\frac{\partial T_0}{\partial t}D - T_0\frac{\partial D}{\partial t}\right) + \sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D - T_i\frac{\partial D}{\partial v_i}\right)$$

has the specified degree-4 coefficients, and that these are the highest-degree coefficients remaining when $\mathbf{P}$ is restricted to M. The various symbols have already been defined as follows:

$$D = (\alpha_0 - r\kappa_0 v_1)^2(v\cdot v) + (\alpha\cdot v + \dot r(v\cdot v))^2,$$

$$T_0 = -r^n(\alpha\cdot v + \dot r(v\cdot v)),$$

$$T_i = r^{n-1}(\alpha_0 - r\kappa_0 v_1)^2 v_i + r^{n-2}(\alpha\cdot v + \dot r(v\cdot v))g_{0i},$$

$$g_{0i} = r\alpha_i + r\dot r v_i + r^2(v_{i-1}\kappa_{i-1} - v_{i+1}\kappa_i),$$

with the understanding that

$$\kappa_{-1}=0, \quad \kappa_n = 0, \quad v_0 = 0, \quad v_{n+1}=0, \quad N_{n+1}=0.$$

The omnipresent simplifications are

$$\sum_{i=1}^{n} v_i g_{0i} = r(\alpha\cdot v + \dot r(v\cdot v))$$

and

$$\sum_{i=1}^{n} v_i T_i = r^{n-1}[(\alpha_0 - r\kappa_0 v_1)^2(v\cdot v) + (\alpha\cdot v + \dot r(v\cdot v))^2],$$

or $\sum_{i=1}^{n} v_i T_i = r^{n-1}D$. Furthermore, when restricted to M^n,

$$(v\cdot v) - 1 \equiv 0.$$

Perhaps the first step is to show that the t derivative terms have degree no higher than 3, and can thus be ignored. Specifically,

$$D|_M \equiv (\alpha_0 - r\kappa_0 v_1)^2 + ((\alpha\cdot v) + r)^2,$$

which is evidently of degree 2 in the v_i's. Further,

$$T_0|_M \equiv -r^n((\alpha\cdot v)+r),$$

which is linear in the v_i's. Differentiation by t does not affect the degree of these polynomials, so $\partial D/\partial t$ and $\partial T_0/\partial t$ have degrees 2 and 1, respectively. We conclude that

$$\left(\frac{\partial T_0}{\partial t}D - T_0\frac{\partial D}{\partial t}\right)$$

is only of degree 3 when restricted to M.

The next task is to examine

$$\sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D\right) \quad \text{or} \quad 2D\sum_{i=1}^{n}\left(\frac{\partial T_i}{\partial v_i}\right).$$

We already know that $D|_M$ has degree 2, so we need only search among the terms $\partial T_i/\partial v_i$ for degree-2 coefficients:

$$\frac{\partial T_i}{\partial v_i} = r^{n-1}(\alpha_0 - r\kappa_0 v_1)^2 + 2r^{n-1}(\alpha_0 - r\kappa_0 v_1)(-r\kappa_0)\delta_{1i}v_i$$
$$+ r^{n-2}(\alpha_i + 2\dot{r}v_i)g_{0i} + r^{n-2}(\alpha\cdot v + \dot{r}(v\cdot v))\frac{\partial g_{0i}}{\partial v_i}.$$

After reducing modulo $(v\cdot v)-1\equiv 0$, the remaining quadratic terms are

$$r^{n+1}\kappa_0^2 v_1^2(1+2\delta_{1i}) + 2r^{n-2}\dot{r}v_i g_{0i}.$$

Combining $\sum_{i=1}^n v_i g_{0i} = r(\alpha\cdot v + \dot{r}(v\cdot v))$ with $(v\cdot v)-1\equiv 0$ shows that the sum

$$\sum_{i=1}^{n}\frac{\partial T_i}{\partial v_i}$$

has quadratic terms equal to

$$r^{n+1}\kappa_0^2 v_1^2 \sum_{i=1}^{n}(1+2\delta_{1i}) = (n+2)r^{n+1}\kappa_0^2 v_1^2,$$

and so $\sum_{i=1}^n(2(\partial T_i/\partial v_i)D)$ has quartic coefficients identical with those of $(2n+4)r^{n+1}\kappa_0^2 v_1^2 D$.

It remains only to show that the quartic terms in $(\sum_{i=1}^n(\partial D/\partial v_i)T_i)|_M$ do actually amount to those in $4r^{n+1}\kappa_0^2 v_1^2 D$, when both are restricted to the sphere $v\cdot v = 1$. From the definition $D = (\alpha_0 - r\kappa_0 v_1)^2(v\cdot v) + (\alpha\cdot v + \dot{r}(v\cdot v))^2$, we find partial derivatives

$$\frac{\partial D}{\partial v_i} = 2(\alpha_0 - r\kappa_0 v_1)(-r\kappa_0)\delta_{1i}(v\cdot v) + 2(\alpha_0 - r\kappa_0 v_1)^2 v_i$$
$$+ 2(\alpha\cdot v + \dot{r}(v\cdot v))(\alpha_i + 2\dot{r}v_i).$$

Reducing this mod $(v\cdot v)-1\equiv 0$ gives us

$$\left.\frac{\partial D}{\partial v_i}\right|_M = 2(\alpha_0 - r\kappa_0 v_1)(-r\kappa_0)\delta_{1i} + 2(\alpha_0 - r\kappa_0 v_1)^2 v_i$$
$$+ 2(\alpha\cdot v + \dot{r})(\alpha_i + 2\dot{r}v_i).$$

Including the above values gives

$$\left(\sum_{i=1}^{n}\frac{\partial D}{\partial v_i}T_i\right)\Bigg|_M = 2(\alpha_0 - r\kappa_0 v_1)(-r\kappa_0)T_1 + 2(\alpha_0 - r\kappa_0 v_1)^2\sum_{i=1}^{n} v_i T_i$$
$$+2(\alpha\cdot v+\dot r)\sum_{i=1}^{n}\alpha_i T_i + 4(\alpha\cdot v+\dot r)\dot r\sum_{i=1}^{n} v_i T_i.$$

Examining this in light of

$$\left(\sum_{i=1}^{n} v_i T_i\right)\Bigg|_M \equiv r^{n-1}(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2) + \text{lower-degree terms},$$

we find that the possible degree-4 terms in $(\sum_{i=1}^{n}(\partial D/\partial v_i)T_i)\,|_M$ are found in

$$2r^2\kappa_0^2 v_1 T_1 + 2(\alpha_0 - r\kappa_0 v_1)^2 r^{n-1}(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2) + 2(\alpha\cdot v)\sum_{i=1}^{n}\alpha_i T_i.$$

Remembering $T_i = r^{n-1}(\alpha_0 - r\kappa_0 v_1)^2 v_i + r^{n-2}(\alpha\cdot v + \dot r(v\cdot v))g_{0i}$, so that

$$T_i\,|_M \equiv r^{n+1}\kappa_0^2 v_1^2 v_i + \text{lower-degree terms},$$

we find the cubic terms in

$$\left(\sum_{i=1}^{n}\alpha_i T_i\right)\Bigg|_M \equiv r^{n+1}\kappa_0^2 v_1^2(\alpha\cdot v) + \text{lower-degree terms}.$$

Emphasizing that

$$T_1\,|_M \equiv r^{n+1}\kappa_0^2 v_1^3 + \text{lower-degree terms},$$

the quartic terms in $(\sum_{i=1}^{n}(\partial D/\partial v_i)T_i)\,|_M$ are found among

$$2r^2\kappa_0^2 v_1 r^{n+1}\kappa_0^2 v_1^3 + 2r^2\kappa_0^2 v_1^2 r^{n-1}(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2) + 2(\alpha\cdot v) r^{n+1}\kappa_0^2 v_1^2\cdot(\alpha\cdot v).$$

This simplifies to

$$2r^{n+3}\kappa_0^4 v_1^4 + 2r^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2) + 2r^{n+1}\kappa_0^2 v_1^2(\alpha\cdot v)^2$$

or

$$\left(\sum_{i=1}^{n}\frac{\partial D}{\partial v_i}T_i\right)\Bigg|_M \equiv 4r^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2) + \text{lower-degree terms}.$$

To sum up, we have shown that the degree-4 terms of the polynomial

$$\mathbf{P} = \left(2\frac{\partial T_0}{\partial t}D - T_0\frac{\partial D}{\partial t}\right) + \sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D - T_i\frac{\partial D}{\partial v_i}\right)$$

when $\mathbf{P}$ is restricted modulo $(v\cdot v) - 1 \equiv 0$, are precisely given by

$$2nr^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2). \qquad \square$$

References

1. A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large, I–V,* Amer. Math. Society Transl. Ser. 2, 21, pp. 341–411, Amer. Math. Soc., Providence, R.I., 1962. First appeared in Vestnik Leningrand Univ. (in Russian) in 1956, 1957, and 1958.

2. J. L. M. Barbosa and A. G. Colares, *Minimal surfaces in R^3*, Lecture Notes in Math., 1195, Springer, Berlin, 1986.
3. M. P. do Carmo, *Differential geometry of curves and surfaces.* Prentice-Hall, Englewood Cliffs, N.J., 1976.
4. A. Enneper, *Die cyklischen Flächen,* Z. Math. Phys. 14 (1869), 393–421.
5. H. B. Lawson, Jr., *Lectures on minimal submanifolds,* Publish or Perish, Berkeley, 1980.
6. W. H. Meeks, III, *Lectures on Plateau's problem,* I.M.P.A., Rio de Janeiro, 1978.
7. ———, *The topological uniqueness of minimal surfaces in three dimensional Euclidean space,* Topology 20 (1981), 389–410.
8. ———, *A survey of the geometric results in the classical theory of minimal surfaces,* Bol. Soc. Brasil. Mat. 12 (1981), 29–86.
9. J. C. C. Nitsche, *Lectures on minimal surfaces,* Vol. 1, Cambridge Univ. Press, Cambridge, 1989.
10. ———, *Minimal surfaces and partial differential equations.* Studies in Partial Differential Equations (W. Littman, ed.), pp. 69–142, Mathematical Association of America, Washington, D.C., 1982.
11. R. Osserman, *A survey of minimal surfaces,* Dover, New York, 1986.
12. B. Riemann. *Über die Fläche vom Kleinsten Inhalt bei gegebener Begrenzung.* Gesammelte Mathematische Werke, pp. 283–315, Leipzig, 1876.
13. R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces,* J. Differential Geom. 18 (1983), 791–809.
14. M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes,* Ann. of Math. (2) 63 (1956), 77–90.
15. D. J. Struik, ed., *A source book in mathematics, 1200–1800,* Harvard Univ. Press, Cambridge, Mass., 1969.

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