Compact Symplectic Manifolds with Free Circle Actions, and Massey Products

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1. Introduction

Only recently have examples of compact symplectic manifolds with no positive definite Kähler metrics been found. The first examples were given by Kodaira [Kod] and independently by Thurston [Th]. In fact, as is becoming apparent, in any even dimension greater than 2 the compact non-Kähler symplectic manifolds are fairly widespread. There are (see, e.g., [BG], [CFG], [CFL]) many nilmanifolds that are symplectic, but none can carry a positive definite Kähler metric. However, some symplectic manifolds with no positive definite Kähler metric possess indefinite Kähler metrics.

In order to find more classes of symplectic manifolds, especially ones with no positive definite Kähler metric, we generalize the constructions of [CFG] and [FGG]. In the latter paper symplectic structures on compact 4-dimensional nilmanifolds were described; the manifolds were viewed as circle bundles over 3-manifolds B which themselves are circle bundles over tori. In order to generalize this construction to surfaces of higher genus we were led to a somewhat different description, which we present in Section 2, where B is viewed as a mapping torus of a symplectic diffeomorphism of a 2-torus. It is this description that generalizes. We then observed that the construction given in Section 2 always produced symplectic manifolds with free S^1-actions, and we wondered if a general symplectic manifold with a free circle action arises in this way. In addressing this question we realized that the construction of Section 2 could be generalized further, and that this broader construction gave rise to the general symplectic manifold with a free circle action. Again, many of these more general symplectic manifolds do not have positive definite Kähler metrics. This broader construction is given in Section 5. In Section 6 we show that all symplectic manifolds with free S^1-actions arise in this way.

The construction of Section 2 begins with a compact symplectic manifold U together with a symplectic diffeomorphism $\varphi: U \to U$. From this we construct a bundle U_φ over S^1 with fiber U and monodromy φ. Next, if b is an element of $H^1(U, \mathbf{Z})$ invariant under φ, we use b to construct a principal

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circle bundle E over U_φ and a symplectic form on E. In the more general construction given in Section 5 we begin with a manifold U, a diffeomorphism $\varphi: U \to U$, and a certain type of 1-parameter family of symplectic structures $t \mapsto \Omega_t$ periodic under φ. Once again there is a symplectic form on the total space built out of the symplectic forms on U.

As indicated above, one aspect of the construction that interests us is the question of which of the symplectic manifolds that we construct carry no positive definite Kähler metrics. We address this question explicitly in Theorem 6, where we show that as long as the class b is nonzero the construction produces a symplectic manifold with no positive definite Kähler metric.

We shall assume throughout that all manifolds and maps are of class C^∞.

2. Construction of Symplectic Manifolds from Lower-Dimensional Symplectic Manifolds and Symplectic Diffeomorphisms

Let U be a compact oriented differentiable manifold and $\varphi: U \to U$ a diffeomorphism which preserves the orientation. The mapping torus U_φ of φ is just $U \times [0,1]$ with the ends identified by φ. It is naturally a C^∞ manifold, because it is the quotient of $U \times \mathbf{R}$ by the infinite cyclic group generated by $(u,t) \mapsto (\varphi(u), t-1)$. First we note the following lemma.

LEMMA 1. *A C^∞-form ω on $U \times [0,1]$ glues up to give a C^∞-form on U_φ, provided that for all $n > 0$ we have*

$$\varphi^*\left(\left.\frac{\partial^n \omega}{\partial t^n}\right|_{U\times\{0\}}\right) = \left.\frac{\partial^n \omega}{\partial t^n}\right|_{U\times\{1\}}.$$

Next we observe the following.

LEMMA 2. *For any $b \in H^1(U, \mathbf{R})$ with $\varphi^* b = b$ there is a closed 1-form β on U_φ such that the restriction of β to $U \times \{0\}$ is a representative for b.*

Proof. Let $\tilde{b}$ be a closed 1-form on U representing $b \in H^1(U, \mathbf{R})$. As $\varphi^* b = b$, we have $\varphi^*\tilde{b} - \tilde{b} = df$ for some C^∞ function $f: U \to \mathbf{R}$. Let $\psi: [0,1] \to [0,1]$ be a C^∞ function identically 1 near 0 and identically 0 near 1. We write

$$\tilde{\beta}(x,t) = \psi(t)\tilde{b}(x) + (1-\psi(t))\varphi^*\tilde{b}(x) - \psi'(t) f(x)\, dt.$$

Then $\tilde{\beta}$ is a 1-form on $U \times [0,1]$, which by Lemma 1 glues up to make a 1-form β on U_φ. Clearly, $d\tilde{\beta} = 0$ and $[\tilde{\beta}|_{U\times\{0\}}] = b \in H^1(U, \mathbf{R})$. □

The natural map $p: U_\varphi \to S^1$ defined by $p(u,t) = e^{2\pi i t}$ is the projection of a C^∞ locally trivial fiber bundle. Let ω be the volume form of S^1, normalized so that it represents a generator of $H^1(S^1, \mathbf{Z})$. Put $\alpha = p^*(\omega)$; then the cohomology class $[\alpha]$ of α is an integral class in $H^1(U_\varphi, \mathbf{Z})$.

LEMMA 3. *Let $b \in H^1(U, \mathbf{Z})$ with $\varphi^* b = b$. Then the class $[\beta]$ of the form β given in Lemma 2 is integral, and its image in $H^1(U_\varphi, \mathbf{R})$ is unique up to multiples of $[\alpha]$. Finally, the class $[\alpha] \cup [\beta] \in H^2(U_\varphi, \mathbf{R})$ is the image of b under the natural map a in the Wang sequence*

$$\cdots \longrightarrow H^1(U, \mathbf{Z}) \xrightarrow{\varphi^* - 1} H^1(U, \mathbf{Z}) \xrightarrow{a} H^2(U_\varphi, \mathbf{Z}) \longrightarrow H^2(U, \mathbf{Z}) \longrightarrow \cdots.$$

Proof. This is immediate from the definition of the Wang sequence. □

As a consequence of Lemma 3, given an invariant class $b \in H^1(U, \mathbf{Z})$, the class $[\alpha] \cup [\beta] \in H^2(U_\varphi, \mathbf{R})$ is independent of the choice of β, as in Lemma 2.

With these preliminaries out of the way, we are ready to proceed to the first case of the main construction.

THEOREM 4. *Let U be a compact manifold and assume Ω_U is a symplectic form on U. Suppose there is a symplectic diffeomorphism $\varphi: U \to U$ such that the induced cohomology map $\varphi^*: H^1(U, \mathbf{Z}) \to H^1(U, \mathbf{Z})$ has an element $b \in H^1(U, \mathbf{Z})$ with $\varphi^* b = b$. Let β be any form of the type constructed in Lemma 2. Let $\pi: E \to U_\varphi$ be the principal circle bundle whose Euler class is $a(b) \in H^2(U_\varphi, \mathbf{Z})$. Then Ω_U gives rise to a symplectic form Ω_E on E.*

Proof. Since $\varphi^*(\Omega_U) = \Omega_U$, the pullback of Ω_U to $U \times [0,1]$ via the natural projection is a closed 2-form which descends to a closed 2-form Ω on U_φ. Moreover, the restriction of Ω to each fiber is Ω_U, so Ω never vanishes on U_φ.

It is well known that corresponding to $a(b) \in H^2(U_\varphi, \mathbf{Z})$ there is a circle bundle $\pi: E \to U_\varphi$. By Lemma 3 there is a connection form η for this bundle whose curvature form is $\alpha \wedge \beta$ (i.e., $\pi^*(\alpha \wedge \beta) = d\eta$; see, e.g., [Kob]). Define

$$\Omega_E = \pi^*(\Omega) + \pi^*(\alpha) \wedge \eta.$$

Then Ω_E is closed, and it is not hard to see that Ω_E has maximal rank; thus Ω_E is a symplectic form on E. □

We also have the following corollary.

COROLLARY 5. *If the symplectic form Ω_U is integral, then the symplectic form Ω_E constructed in Theorem 4 is also integral.*

3. When is E a Kähler Manifold?

Now we take up the question of whether manifolds like the ones constructed in Theorem 4 have positive definite Kähler metrics. If φ is isotopic to the identity, $b = 0$, and U has a positive definite Kähler metric, then E is diffeomorphic to the product of U with a torus. Clearly, E has a positive definite Kähler metric. Our main result along these lines is the following.

THEOREM 6. *Let U be a compact oriented manifold, let $\varphi: U \to U$ be an orientation-preserving diffeomorphism, and let $b \in H^1(U, \mathbf{Z})$ be invariant* *under φ. If $b \neq 0$ in $H^1(U, \mathbf{R})$, then the total space E of the circle bundle over U_φ with Euler class $a(b)$ carries no positive definite Kähler metric.*

There are two cases, depending on whether or not b is in the image of $(\varphi^* - 1)$. We do the simplest case first.

PROPOSITION 7. *Use the notation as in Theorem 6. If b is not in the image of $(\varphi^* - 1): H^1(U, \mathbf{R}) \to H^1(U, \mathbf{R})$, then E carries no positive definite Kähler metric.*

Proof. It follows from the assumption that b is not in the image of $\varphi^* - 1$, the Wang exact sequence

$$\cdots \longrightarrow H^1(U, \mathbf{R}) \xrightarrow{\varphi^* - 1} H^1(U, \mathbf{R}) \xrightarrow{a} H^2(U_\varphi, \mathbf{R}) \to \cdots,$$

and Lemma 3 that the 2-form $\alpha \wedge \beta$ represents a nonzero class in $H^2(U_\varphi, \mathbf{R})$. Therefore the principal circle bundle $\pi: E \to U_\varphi$ determined by it is nontrivial.

Since $\pi^*(\alpha \wedge \beta) = d\eta$, the Massey product

$$\mu = \langle [\pi^*(\alpha)], [\pi^*(\alpha)], [\pi^*(\beta)] \rangle = [\pi^*(\alpha) \wedge \eta]$$

is defined in E. This closed 2-form evaluates nontrivially on the 2-dimensional torus T in E sitting over a section of $p: U_\varphi \to S^1$. Since any element in the image of $\pi^*: H^2(U_\varphi, \mathbf{R}) \to H^2(E, \mathbf{R})$ vanishes on T, it follows that μ is not in $\pi^*(H^2(U_\varphi, \mathbf{R})) \subset H^2(E, \mathbf{R})$. It follows from the Gysin exact sequence for $\pi: E \to U_\varphi$ that the map $\pi^*: H^1(U_\varphi, \mathbf{R}) \to H^1(E, \mathbf{R})$ is surjective, so that the Massey product μ is nontrivial modulo the ideal

$$H^1(E, \mathbf{R}) \cdot ([\pi^*(\alpha)], [\pi^*(\beta)]).$$

Hence, E has no positive definite Kähler metric by the main theorem of [DGMS]. □

Now we turn to the case when $b \neq 0$ but $b \in \operatorname{Im}(\varphi^* - 1)$. Showing that E is not formal as defined in [DGMS], and thus has no Kähler metric, essentially amounts to proving that E has nonzero higher Massey products.

Let $V \subset H^1(U, \mathbf{R})$ be the generalized eigenspace of φ^* associated to the eigenvalue 1; thus V is the maximal φ^*-invariant subspace on which $(\varphi^* - 1)$ is nilpotent. Clearly, $b \in V$. There is $n \geq 2$ such that

$$b \in \operatorname{Im}(\varphi^* - 1)^{n-1} \quad \text{but} \quad b \notin \operatorname{Im}(\varphi^* - 1)^n.$$

Choose $b_n \in V$ with $(\varphi^* - 1)^{n-1} b_n = b$, and set

$$b_i = (\varphi^* - 1)^{n-i} b_n$$

for $1 \leq i \leq n-1$. Then $b = b_1 = (\varphi^* - 1) b_2, \ldots, b_{n-1} = (\varphi^* - 1) b_n$.

Next we construct an abstract differential graded algebra

$$A^* = \Lambda^*(\bar{a}, \bar{b}_1, \ldots, \bar{b}_n),$$

where $\operatorname{degree}(\bar{a}) = \operatorname{degree}(\bar{b}_i) = 1$. We define the differential on A^* by setting

$$d\bar{a} = d\bar{b}_1 = 0 \quad \text{and} \quad d\bar{b}_i = \bar{b}_{i-1} \wedge \bar{a} \quad \text{for } 2 \leq i \leq n,$$

and then extending to all of A^* by the Leibnitz rule and linearity. Clearly, A^* is built up from the trivial differential graded algebra by a sequence of Hirsch extensions of differential graded algebras. (See [DGMS, p. 249] for a definition of Hirsch extension, which is called elementary extension there.) Furthermore, the cohomology classes of $\bar{b}_1$ and $\bar{a}$ form a basis for $H^1(A^*)$, and $\bar{b}_n \wedge \bar{a}$ is a closed 2-form in A^* representing a cohomology class which is not divisible by the cohomology class of $\bar{a}$.

Let $\Omega^*(B)$ denote the de Rham complex of a manifold B.

LEMMA 8. *Suppose U, φ, and b are as in Theorem 6. Suppose further that $b \neq 0$ in $H^1(U, \mathbf{R})$ but that $b \in \operatorname{Im}(\varphi^* - 1)$. Then there is a map*

$$\Psi: A^* \to \Omega^*(U_\varphi)$$

of differential graded algebras sending $\bar{a}$ to α and $\bar{b}_1$ to the form β constructed in the proof of Lemma 2. Furthermore, $\Psi(\bar{b}_n \wedge \bar{a})$ is a closed form representing a class in $H^2(U_\varphi, \mathbf{R})$ which is not in the image of

$$\bigcup [\alpha]: H^1(U_\varphi, \mathbf{R}) \to H^2(U_\varphi, \mathbf{R}).$$

Let us accept this lemma for the moment and use it to take care of the case $b \neq 0$, $b \in \operatorname{Im}(\varphi^* - 1)$.

PROPOSITION 9. *Let U, φ, and $b \in H^1(U, \mathbf{Z})$ be as in Theorem 6. Suppose that $b \neq 0$ in $H^1(U, \mathbf{R})$ but that $b \in \operatorname{Im}(\varphi^* - 1)$. Then E has no positive definite Kähler metric.*

Proof. We begin by showing that neither U_φ nor E is formal. Were U_φ formal then by definition there would be a diagram of differential graded algebras

$$\Omega^*(U_\varphi) \xleftarrow{i} \mathfrak{M} \xrightarrow{j} \{H^*(U_\varphi, \mathbf{R}), d = 0\},$$

where i and j induce isomorphisms on cohomology, and where

$$H^*(\Omega^*(U_\varphi)) \xrightarrow{(i^*)^{-1}} H^*(\mathfrak{M}) \xrightarrow{j^*} H^*(U_\varphi, \mathbf{R})$$

is the de Rham isomorphism. (For example, one could take $\mathfrak{M}$ to be the minimal model for $\Omega^*(U_\varphi)$.)

By Lemma 8 we have $\Psi: A^* \to \Omega^*(U_\varphi)$. Since A^* is a sequence of Hirsch extensions and since i^* is a homotopy equivalence of differential graded algebras, there is a map $\bar{\Psi}: A^* \to \mathfrak{M}$ such that $i \circ \bar{\Psi}$ is homotopic to Ψ. In particular, $i^* \circ \bar{\Psi}^* = \Psi^*$ in cohomology. We consider $j \circ \bar{\Psi}: A^* \to \{H^*(U_\varphi, \mathbf{R}), d = 0\}$. Clearly, $j \circ \bar{\Psi}(\bar{a}) = [\alpha]$ and $j \circ \bar{\Psi}(\bar{b}_i)$ is some class $a_i \in H^1(U_\varphi, \mathbf{R})$ for $1 \leq i \leq n$. Thus, $j \circ \bar{\Psi}(\bar{b}_n \wedge \bar{a}) = a_n \cup [\alpha] \in H^2(U_\varphi, \mathbf{R})$. Since

$$j^* \circ \bar{\Psi}^* = \Psi^*: H^*(A^*) \to H^*(U_\varphi, \mathbf{R}),$$

this implies that $\Psi(\bar{b}_n \wedge \bar{a})$ is a closed form in the ideal $[\alpha] \cdot H^*(U_\varphi, \mathbf{R})$. This contradicts the last assertion in Lemma 8, and establishes that U_φ is not formal.

Now let us deduce that E is not formal either. Recall that E is a principal S^1-bundle over U_φ with connection form η satisfying $d\eta = \pi^*(\alpha\wedge\beta)$, where $\pi: E\to U_\varphi$ is the natural projection map. Since $b\in \operatorname{Im}(\varphi^*-1)$, the form $\alpha\wedge\beta$ is exact in U_φ. Thus, there is a connection 1-form η for $E\to U_\varphi$ which is closed. Then the natural map

$$\Lambda^*(\eta)\otimes\Omega^*(U_\varphi)\to\Omega^*(E)$$

induces an isomorphism in cohomology. Hence the minimal model for $\Omega^*(E)$ is isomorphic to $\Lambda^*(\eta)\otimes\mathfrak{M}(U_\varphi)$, where $\mathfrak{M}(U_\varphi)$ is the minimal model for $\Omega^*(U_\varphi)$. In particular, $H^*(E)\equiv\Lambda^*(\eta)\otimes H^*(U_\varphi)$. Were E formal, then there would be a map of differential algebras

$$\rho:\Lambda^*(\eta)\otimes\mathfrak{M}(U_\varphi)\to\{H^*(E), d=0\}$$

inducing the identity in cohomology. The composition

$$\mathfrak{M}(U_\varphi)\hookrightarrow\Lambda^*(\eta)\otimes\mathfrak{M}(U_\varphi)\xrightarrow{\rho}\{H^*(E),d=0\}\to\{H^*(U_\varphi),d=0\}$$

would also induce the identity on cohomology, contradicting the fact that U_φ is not formal.

Since E is not formal, by [DGMS] it does not have a positive definite Kähler metric. □

NOTE 10. By a similar argument one can show that if U_φ satisfies the conclusion of Lemma 8, then the total space of *any* circle bundle over U_φ is not formal and hence does not have a positive definite Kähler metric.

Proof of Lemma 8. For $1\le i\le n$ we inductively define 1-forms $\tilde\beta_i(x,t)$ on $U\times[0,1]$ such that:

$$(1)\quad\begin{cases}\text{(a) each } \tilde\beta_i(x,t),\ 1\le i\le n, \text{ descends to a 1-form } \beta_i \text{ on } U_\varphi;\\ \text{(b) } d\beta_1=0 \text{ and } d\beta_i=\beta_{i-1}\wedge\alpha \text{ for } 2\le i\le n;\\ \text{(c) for all } t\in[0,1],\ \beta_i|_{U\times\{t\}} \text{ is closed and}\\ \quad [\tilde\beta_i|_{U\times\{t\}}]\equiv b_i \bmod(b_1,\dots,b_{i-1}).\end{cases}$$

We set $\tilde\beta_1=\tilde\beta$ as in Lemma 3. Clearly, (a)–(c) hold for $\tilde\beta_1$. Suppose that inductively we have defined $\beta_1,\dots,\beta_{i-1}$ as required for $2\le i\le n$. Consider

$$\tilde\gamma(x,t) = -\int_0^t\tilde\beta_{i-1}(x,s)\,ds.$$

By (c), for $\tilde\beta_{i-1}$ we see that $\tilde\gamma|_{U\times\{t\}}$ is closed and that

$$[\tilde\gamma|_{U\times\{t\}}]\equiv t b_{i-1}\bmod(b_1,\dots,b_{i-2}).$$

In particular, if

$$(2)\qquad \gamma=-\int_0^1\tilde\beta_{i-1}(x,s)\,ds$$

then γ is a closed 1-form on U with $[\gamma]\equiv b_{i-1}\bmod(b_1,\dots,b_{i-2})$. Thus, $[\gamma]=(\varphi^*-1)[\xi]$ for some closed form ξ with $[\xi]\equiv b_i\bmod(b_1,\dots,b_{i-1})$. We write

$$\gamma = (\varphi^* - 1)\xi + df \tag{3}$$

and we define

$$\tilde{\beta}_i(x,t) = -\int_0^t \tilde{\beta}_{i-1}(x,s)\,ds + \xi + d(\psi(t)\cdot(f\circ\varphi^{-1}(x))),$$

where ψ is in Lemma 2.

Clearly,

$$\varphi^*\tilde{\beta}_i(x,0) = \varphi^*\xi + df \quad \text{and} \quad \tilde{\beta}_i(x,1) = \gamma + \xi.$$

Hence, by Lemma 3 and equations (2) and (3), we see that condition (a) holds for $\tilde{\beta}_i$. Also, $d\tilde{\beta}_i(x,t) = \tilde{\beta}_{i-1}(x,t)\wedge dt$. This proves (b).

Furthermore, $[\tilde{\beta}_i|_{U\times\{t\}}] \equiv [\xi] \bmod(b_1,\ldots,b_{i-1})$. As $[\xi] = b_i \bmod(b_1,\ldots, b_{i-1})$, we obtain (c).

The inductive construction of the β_i is finished, thus completing the proof of Lemma 8, and hence of Proposition 9 and Theorem 6. □

COROLLARY 11. *Let U be a compact manifold, and let Ω be a symplectic form on U. Suppose $\varphi: U\to U$ is a diffeomorphism with $\varphi^*\Omega = \Omega$. Suppose also that $b\in H^1(U,\mathbf{Z})$ is invariant under φ and that the image of b in $H^1(U,\mathbf{R})$ is nontrivial. Let $E\to U_\varphi$ be the circle bundle with Euler class $a(b)$. Then E has symplectic structure, but has no positive definite Kähler metrics.*

Proof. This is immediate from Theorems 4 and 6. □

4. Examples

Here are two classes of examples of Corollary 11.

(1) Let U be a Riemann surface of genus $g\geq 1$, and let $b\in H^1(U,\mathbf{Z})$ be nontrivial. Consider the circle bundle $E\to U\times S^1$ with Euler class $b\otimes[\alpha]$, where $[\alpha]$ is the generator of $H^1(S^1,\mathbf{Z})$. Then E has a symplectic form but no positive definite Kähler metric. (That E has no positive definite Kähler metrics follows already from the fact that $b_1(E) = 2g+1$.)

(2) Let U be a Riemann surface of genus $g\geq 1$, and let $\tau: U\to U$ be a Dehn twist about a topologically nontrivial closed curve γ. There is a volume form on U invariant by τ. Let $b\in H^1(U,\mathbf{Z})$ be any nontrivial class with zero homological intersection with γ. Let $E\to U_\tau$ be the S^1-bundle with Euler class $a(b)$. Then E has a symplectic structure but no positive definite Kähler metric. (Notice that $b_1(E) = 2g$ if $[\gamma]\neq 0$ in $H_1(U,\mathbf{Z})$, and $b_1(E) = 2g+1$ if $[\gamma] = 0$.)

If $U = T^2$, then both examples yield a 4-manifold diffeomorphic to $N^3\times S^1$, where N is the Heisenberg nilmanifold of dimension 3. These are exactly the examples constructed in [FGG].

5. A More General Construction

First of all we shall give some simple examples of 1-parameter families of symplectic forms.

LEMMA 12. *Let X be a Kähler manifold and let $\varphi: X \to X$ be a complex analytic diffeomorphism. Denote by F the Kähler form of X and by g the corresponding (positive definite) Kähler metric. Then*

$$t \to (1-t)F + t\varphi^*(F)$$

is a C^∞ 1-parameter family of symplectic forms for $0 \le t \le 1$.

Proof. Since the set of positive definite metrics is convex, $g_t = (1-t)g + t\varphi^*(g)$ is a positive definite metric. It is easy to verify that g_t is Kählerian, and that its Kähler form is $(1-t)F + t\varphi^*(F)$. □

NOTE 13. If U is a Riemann surface with volume form Ω and if $\varphi: U \to U$ is an orientation-preserving diffeomorphism, then an argument along the same lines shows that there is a 1-parameter family Ω_t of volume forms on U connecting Ω to $\varphi^*\Omega$.

In this section it will be convenient to describe the mapping torus a little differently than in Section 2. Let X be an oriented manifold together with an orientation-preserving diffeomorphism $\varphi: X \to X$. Fix $r > 0$ and denote by

$$X_\varphi = \frac{X \times \mathbf{R}}{(x,t) \sim (\varphi(x), t-r)}$$

the mapping torus of φ. Then the canonical 1-form dt on $\mathbf{R}$ can be considered as a 1-form on $X \times \mathbf{R}$, and since it is invariant by the diffeomorphism $(x,t) \to (\varphi(x), t-r)$, it descends to a 1-form α on the quotient X_φ.

The purpose of this section is to prove the following.

THEOREM 14. *Let X be a $(2n-2)$-dimensional compact oriented manifold together with an orientation-preserving diffeomorphism $\varphi: X \to X$. Suppose that $\xi \in H^2(X, \mathbf{Z})$ is an integral class invariant under φ. Let $\mathbf{c}$ be any class in $H^2(X_\varphi, \mathbf{Z})$ such that*

$$\mathbf{c}\,|_{X \times \{0\}} = \xi.$$

Let $M_{\mathbf{c}} \to X_\varphi$ be the S^1-bundle with Euler class $\mathbf{c}$. The following condition is sufficient for $M_{\mathbf{c}}$ to have a symplectic structure invariant under the S^1-action:

> *There is a 1-parameter family Ω_t of symplectic structures on X with $\varphi^*\Omega_{t-r} = \Omega_t$ and with $[\Omega_t] = [\Omega_0] + t\xi$.*

Let us examine the special case $\xi = 0$. Any $\mathbf{c} \in H^2(X_\varphi, \mathbf{Z})$ whose restriction to $X \times \{0\}$ is trivial is of the form $a(\mathbf{b})$ for some $\mathbf{b} \in H^1(X, \mathbf{Z})$. In this case the 1-parameter family of symplectic forms Ω_t has constant cohomology class, but are not required to be constant forms. Thus, Theorem 14 in the special case when $\xi = 0$ generalizes the construction in Section 2.

To prove Theorem 14 we need several lemmas. The first two are easy exercises.

LEMMA 15. *Any class $f \in H^2(X_\varphi, \mathbf{R})$ which restricts to zero class on $X \times \{0\}$ has a 2-form representative in the ideal of α.*

LEMMA 16. *The family of 2-forms Ω_t on X gives rise to a 2-form $\tilde{\Omega}$ on $X\times\mathbf{R}$ which is invariant under the diffeomorphism $(x,t)\mapsto(\varphi(x),t-r)$. Hence $\tilde{\Omega}$ defines a 2-form Ω on X_φ.*

LEMMA 17. *Under the hypotheses of Theorem 14 there is a 2-form γ on X_φ such that*

(i) $d\gamma=0$;
(ii) $[\gamma]=c\in H^2(X_\varphi,\mathbf{Z})$;
(iii) $d\Omega=\alpha\wedge\gamma$.

Suppose for the moment that Lemma 17 is true.

Proof of Theorem 14. Let γ be the 2-form given by Lemma 17. Conditions (i) and (ii) imply that there is a connection 1-form $\eta\in\Lambda^1(M_c)$ with $d\eta=\pi^*\gamma$. We define a 2-form Ω_c on M_c by

$$\Omega_c=\pi^*(\Omega)+\pi^*(\alpha)\wedge\eta.$$

Then, by part (iii) of Lemma 17, we have

$$\begin{aligned} d\Omega_c&=\pi^*(d\Omega)-\pi^*(\alpha)\wedge d\eta\\ &=\pi^*(d\Omega-\alpha\wedge\gamma)\\ &=0. \end{aligned}$$

To show that Ω_c is nondegenerate, we note that

$$\Omega_c^n=n\pi^*(\Omega^{n-1})\wedge\pi^*(\alpha)\wedge\eta=n\pi^*(\Omega^{n-1}\wedge\alpha)\wedge\eta.$$

Of course, Ω^{n-1} never vanishes on tangent space to the fibers $X\times\{t\}\hookrightarrow X_\varphi$, since Ω restricts to a symplectic form on each $X\times\{t\}\hookrightarrow X_\varphi$. Hence $\Omega^{n-1}\wedge\alpha$ is nowhere zero on X_φ. Since η is also nonzero on tangent vectors to fibers of $\pi:M_c\to X_\varphi$, it follows that $\pi^*(\Omega^{n-1}\wedge\alpha)\wedge\eta$ is nowhere zero on M_c. Consequently, Ω_c is a symplectic form on M_c. □

Proof of Lemma 17. To construct a 2-form γ on X_φ satisfying (i), (ii), and (iii), we begin by computing $d\tilde{\Omega}$ on $X\times\mathbf{R}$ and thus also $d\Omega$ on X_φ. Clearly, $\tilde{\Omega}|_{X\times\{t\}}$ is closed, and $\tilde{\Omega}$ does not involve dt. Hence

$$d\tilde{\Omega}=\frac{\partial\tilde{\Omega}}{\partial t}\wedge dt.$$

Moreover, $\partial\tilde{\Omega}/\partial t$ descends to a (possibly nonclosed) 2-form Ω_1 on X_φ with

$$d\Omega=\Omega_1\wedge\alpha.$$

Since $\tilde{\Omega}|_{X\times\{t\}}$ is closed, the form $(\partial\tilde{\Omega}/\partial t)|_{X\times\{t\}}$ is closed. In fact, since we have $[\tilde{\Omega}|_{X\times\{t\}}]=[\tilde{\Omega}|_{X\times\{0\}}]+t\xi$, it follows that

$$\left[\frac{\partial\tilde{\Omega}}{\partial t}\bigg|_{X\times\{t\}}\right]=\xi.$$

Clearly,

$$d\left(\frac{\partial\tilde{\Omega}}{\partial t}\right)=\frac{\partial^2\tilde{\Omega}}{\partial t^2}\wedge dt.$$

The form $\partial^2\tilde{\Omega}/\partial t^2$ descends to a 2-form Ω_2 on X_φ with $d\Omega_1=\Omega_2\wedge\alpha$. Since $[(\partial\tilde{\Omega}/\partial t)|_{X\times\{t\}}]$ is constant, the class $[(\partial^2\tilde{\Omega}/\partial t^2)|_{X\times\{t\}}]$ vanishes for all t. Now we find a 1-form $\tilde{\gamma}_2$ that descends to a 1-form γ_2 on X_φ such that, for all t,

$$d(\tilde{\gamma}_2|_{X\times\{t\}})=\left.\frac{\partial^2\tilde{\Omega}}{\partial t^2}\right|_{X\times\{t\}},$$

so that $d\gamma_2-\Omega_2$ is in the ideal generated by α. Let γ_1 be the 2-form on X_φ given by $\gamma_1=\Omega_1-\gamma_2\wedge\alpha$; then

$$d\gamma_1=d\Omega_1-d\gamma_2\wedge\alpha=(\Omega_2-d\gamma_2)\wedge\alpha=0.$$

Also, considering $X\times\{0\}$ as a submanifold of X_φ, we have

$$\gamma_1|_{X\times\{0\}}=\Omega_1|_{X\times\{0\}},$$

so that the cohomology class $[\gamma_1]|_{X\times\{0\}}$ equals ξ. By hypothesis $c|_{X\times\{0\}}=\xi$, and hence the cohomology class $c-[\gamma_1]\in H^2(X_\varphi,\mathbf{R})$ is such that it is the zero class when restricted to $X\times\{0\}$. Now Lemma 15 implies that there is a closed 2-form $\nu\wedge\alpha$ on X_φ which represents the cohomology class $c-[\gamma_1]$ (i.e., $[\nu\wedge\alpha]=c-[\gamma_1]$). We define the 2-form γ on X_φ by

$$\gamma=\gamma_1+\nu\wedge\alpha.$$

Then

$$d\gamma=d\gamma_1+d(\nu\wedge\alpha)=0.$$

Also, the cohomology class of γ in $H^2(X_\varphi,\mathbf{R})$ is

$$[\gamma]=[\gamma_1]+[\nu\wedge\alpha]=c.$$

Hence γ satisfies conditions (i) and (ii) of Lemma 17. Since

$$\gamma\wedge\alpha=\gamma_1\wedge\alpha=\Omega_1\wedge\alpha=d\Omega,$$

γ also satisfies condition (iii). This completes the proof of Lemma 17. □

Using Theorem 14, we can construct more examples along the lines of those in Section 4. Let U be a compact Riemann surface with genus g at least 1. Then a Kähler form Ω of U is a volume form. If $\varphi\colon U\to U$ is any orientation-preserving diffeomorphism, then there is a 1-parameter family of symplectic structures connecting Ω to $\varphi^*\Omega$ of constant cohomology class. According to Theorem 14, if E is the total space of a circle bundle over U_φ with Euler class c with the property that $c|_{U\times\{0\}}$ is trivial, then E has a symplectic structure. According to Note 10, if there is a nonzero class b in $H^1(U,\mathbf{R})$ which is in the image of (φ^*-1), then E carries no positive definite Kähler structure.

6. The Structure Theorem

The purpose of this section is to prove the following converse of Theorem 14 by characterizing symplectic manifolds with free S^1-action leaving invariant a symplectic form.

THEOREM 18. *Let (M^{2n}, F_0) be a compact symplectic manifold with a free S^1-action leaving the symplectic form F_0 invariant. Let $\pi: M^{2n} \to B^{2n-1}$ be the S^1-fibration induced by the S^1-action. Then there exist $r > 0$, a compact manifold U, and a diffeomorphism $\varphi: U \to U$ such that:*

(i) *B is the mapping torus of φ; that is,*

$$B \cong U \times \mathbf{R} / \{(x,t) \sim (\varphi(x), t-r)\}.$$

(ii) *There is a 1-parameter family Ω_t of symplectic forms on U with*

$$\varphi^* \Omega_{t-r} = \Omega_t \quad \textit{and} \quad [\Omega_t] = [\Omega_0] + t\xi$$

for some φ-invariant class $\xi \in H^2(U, \mathbf{Z})$.

(iii) *The Euler class of $M \to B$ restricts to $U \times \{0\}$ to give ξ.*

(iv) *Lastly, there is a symplectic form F on M invariant under the S^1-action such that*

$$F = \pi^*(\Omega) + \pi^*(\alpha) \wedge \eta,$$

where η is a connection 1-form for $M \to B$ and α is the pullback of $d\theta/2\pi$ on the circle. Here Ω is the 2-form on B corresponding to the family Ω_t. We can choose U, φ, and η such that F is arbitrarily close to F_0.

Proof. Let $\Phi: S^1 \times M \to M$ be the free circle action. Our first step is to replace F_0 by an arbitrarily close S^1-invariant symplectic form whose cohomology class is rational, that is, in the image $H^2(M, \mathbf{Q}) \hookrightarrow H^2(M, \mathbf{R})$. Clearly, there is an arbitrarily small, closed 2-form δ such that $F_0 + \delta$ represents a rational cohomology class. Let $\hat{\delta}$ be the average of δ over the S^1-action. Because S^1 is connected, $\Phi_t^* \delta$ is a closed form representing the same cohomology class as δ. Therefore $F_0 + \hat{\delta}$ and $F_0 + \delta$ represent the same cohomology class. Clearly, $F = F_0 + \hat{\delta}$ is S^1-invariant. Moreover, F is symplectic, provided that δ is sufficiently small.

From now on F will be an S^1-invariant symplectic form whose cohomology class is rational. Let $\partial/\partial\theta$ be the canonical vector field on S^1; then $\Theta = \Phi_*(\partial/\partial\theta)$ is an everywhere nonzero vector field on M. Define

$$\alpha_0 = \frac{1}{2\pi} \iota_\Theta(F),$$

where ι_Θ denotes the interior product operator. Since F and Θ are S^1-invariant, α_0 is also S^1-invariant. Let $\mathcal{L}_\Theta$ denote the Lie derivative; then, using the well-known formula $d\iota_\Theta + \iota_\Theta d = \mathcal{L}_\Theta$, we see that

$$d\alpha_0 = \frac{1}{2\pi} \mathcal{L}_\Theta(F).$$

But since F is S^1-invariant, we have $\mathcal{L}_\Theta(F) = 0$ and consequently α_0 is a closed 1-form.

Since α_0 is S^1-invariant and vanishes on the tangents to the S^1-action, α_0 can be written uniquely as $\pi^*(\bar{\alpha})$, where $\bar{\alpha}$ is a closed 1-form on the quotient $M/S^1 = B$. We claim that

(a) $\bar{\alpha}$ is nowhere 0 on B, and
(b) $[\bar{\alpha}] \in H^1(B, \mathbf{R})$ is rational.

Condition (a) follows immediately from the fact that F is symplectic. As for (b), let $S^1 \hookrightarrow B$ be a circle in B; then, by Fubini's theorem, we have that

$$\int_{S^1} \bar{\alpha} = \int_T F,$$

where T is the full pre-image of S^1 in M. Since $[F]$ is rational, so is $\int_{S^1} \bar{\alpha}$. This holds for all circles in B, and so we get (b).

Consequently, the periods of $[\bar{\alpha}]$ on integral homology are all multiples of some positive rational number r. Thus the pullback $\tilde{\alpha}$ of $\bar{\alpha}$ to the universal covering $\tilde{B}$ of B is exact. Write $\tilde{\alpha} = d\tilde{\mu}$; then $\tilde{\mu}$ has discrete periods on paths covering loops in B. In fact, for any path $\gamma: [0,1] \to \tilde{B}$ with $\gamma(0)$ and $\gamma(1)$ covering the same point of B, there is an integer $n(\gamma)$ with

$$\tilde{\mu}(\gamma(1)) - \tilde{\mu}(\gamma(0)) = n(\gamma) r.$$

Hence $\tilde{\mu}$ factors to give a map $\mu: B \to \mathbf{R}/(\mathbf{Z} \cdot r)$ whose "differential" is $\bar{\alpha}$; that is, $\mu^*(dt) = \alpha$. This α is never zero, and μ is the projection mapping of a differential fibration of B over S^1. This proves that

$$B \cong U \times \mathbf{R} / \{(x, t) \cong (\varphi(x), t - r)\}$$

for some compact manifold U and some diffeomorphism φ.

Let $\eta' \in \Omega^1(M)$ be a connection 1-form for the principal S^1-bundle $M \to B$. This is an S^1-invariant 1-form whose restriction to any orbit is $d\theta/2\pi$. Since F is an S^1-invariant 2-form and $\iota_\Theta F = \pi^*(\alpha)$, we can write F uniquely as

$$F = \pi^* \Omega' + \pi^*(\alpha) \wedge \eta'$$

for some 2-form Ω' on B. We pull back Ω' to a form $\tilde{\Omega}'$ on $U \times \mathbf{R}$ and write

$$\tilde{\Omega}' = \tilde{\Omega} + \tilde{\omega} \wedge dt,$$

where $\tilde{\Omega}$ and $\tilde{\omega}$ do not involve dt and can be considered as 1-parameter families of forms on U. These two families give rise to forms Ω and ω on B, and we have the decomposition

$$\Omega' = \Omega + \omega \wedge \alpha.$$

If we put $\eta = \eta' - \pi^*(\omega)$, then η is also a connection 1-form for $M \to B$, and we have

$$F = \pi^* \Omega + \pi^*(\alpha) \wedge \eta.$$

The fact that $\tilde{\Omega}$ comes from a form on $B = U_\varphi$ is expressed by the fact that $\varphi^* \tilde{\Omega}|_{U \times \{t-r\}} = \tilde{\Omega}|_{U \times \{t\}}$ for all $t \in \mathbf{R}$. Since $d\eta = \pi^* \nu$, where ν is a closed 2-form representing the Euler class of $M \to B$, and since $dF = 0$, we have

$$d_x \tilde{\Omega} = 0 \quad \text{and} \quad \frac{\partial \tilde{\Omega}}{\partial t} = \tilde{\nu},$$

where $\tilde{\nu}$ is the lift of ν to $U \times \mathbf{R}$. In particular, for each t, the form $\Omega_t = \tilde{\Omega}|_{U \times \{t\}}$ is closed. Also, since F^n is nowhere zero and

$$F^n = n\pi^*\Omega^{n-1} \wedge \pi^*(\alpha) \wedge \eta,$$

we see that $\Omega_t^{n-1} = \tilde{\Omega}^{n-1}|_{U\times\{t\}}$ is nowhere zero for all t. Thus Ω_t is a 1-parameter family of symplectic forms on U.

Lastly, let us consider

$$\left[\frac{\partial \Omega_t}{\partial t}\right] \in H^2(U, \mathbf{R}).$$

It is equal to $[\tilde{\nu}|_{U\times\{t\}}]$. Since $\tilde{\nu}$ is closed, this class is independent of t; call it ξ. Since $\tilde{\nu}$ descends to a form ν on B, ξ is invariant under φ. Since $[\nu]$ is integral, ξ is integral. Clearly, then,

$$[\Omega_t] = [\Omega_0] + t\xi.$$

This completes the proof of Theorem 18. ❑

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