Totally Umbilic Riemannian Foliations

GRANT CAIRNS

1. Introduction

On a Riemannian manifold, a foliation with leaves of dimension $p \geq 2$ is said to be *totally umbilic* if its leaves are totally umbilic submanifolds. An obvious example is that of Euclidean space $\mathbf{E}^n$, minus one point, foliated by concentric spheres. Further examples are provided by *totally geodesic* foliations—that is, foliations whose leaves are totally geodesic submanifolds. Whereas totally geodesic foliations have received a good deal of attention in the literature (see the references in [6]), there are surprisingly few works on totally umbilic foliations (see nevertheless [18], [3], [4], [11]).

Considering the foliation F of $\mathbf{E}^n \setminus \{\star\}$ by concentric spheres, one will notice that it can be made totally geodesic by suitably changing the metric. Indeed, $\mathbf{E}^n \setminus \{\star\}$ is diffeomorphic to $S^{n-1} \times \mathbf{R}$ via the obvious map that sends the leaves of F to the submanifolds $S^{n-1} \times \{pt\}$. A foliation F is said to be *umbilicalisable* [7] (resp. *geodesible*) if there exists a Riemannian metric on the ambient manifold for which F is totally umbilic (resp. totally geodesic). It is not hard to construct umbilicalisable foliations that are not geodesible. One obstruction is that a totally geodesic foliation is necessarily *harmonic;* that is, its leaves are minimal submanifolds. A foliation F is said to be *taut* if there exists a Riemannian metric on the ambient manifold for which F is harmonic. Clearly, a foliation is totally geodesic if and only if it is totally umbilic and harmonic for the same metric. It is less obvious that a taut umbilicalisable foliation is necessarily geodesible. In his thesis [7], Carrière made the following conjecture.

CONJECTURE 1 (Carrière). *Every codimension-*1 *taut umbilicalisable foliation on a compact manifold is geodesible.*

Carrière proved this conjecture for codimension-1 Riemannian foliations. In this paper we consider the case of Riemannian foliations of arbitrary codimension. Recall that a foliation F on a manifold M is *Riemannian* (see [20], [17], [13]) if there exists a Riemannian metric g on the transverse bundle TM/TF for which the leaves are locally equidistant. This amounts to saying that for a codimension-q foliation F defined by local submersions

Received March 21, 1989. Revision received June 19, 1989.
Michigan Math. J. 37 (1990).

$\pi_i: U_i \subset M \to \mathbf{R}^q$, the restriction of g to U_i is the pull-back by π_i of a Riemannian metric on $\mathbf{R}^q$. The hypothesis that a foliation be Riemannian is of course quite restrictive, and in fact the results of this paper are false without it. Moreover, the study of the Riemannian case has little bearing on the general setting of Conjecture 1, which necessitates an investigation of quite different phenomena (see [7]). The point in examining the case of Riemannian foliations is simply that they are now quite well understood (see [17]) and the tautness of these foliations has received a good deal of attention (see the survey article [21]). It is known, for instance, that the basic cohomology of a taut Riemannian foliation verifies Poincaré duality (the basic cohomology is the cohomology of the *basic* forms, which are the differential forms that are locally the pull-backs of forms on the local quotient manifold $\mathbf{R}^q$). Carrière gave an example in his thesis of a Riemannian foliation whose basic cohomology does not satisfy Poincaré duality, and made the following conjecture.

CONJECTURE 2 (Carrière). *A Riemannian foliation on a compact manifold is taut if and only if its basic cohomology verifies Poincaré duality.*

This conjecture has been proven in a number of particular cases (see [21]). In this paper we show that Conjecture 1 holds for Riemannian foliations of arbitrary codimension by showing that Conjecture 2 is true for umbilicalisable foliations. Before stating our result, let us recall yet another set of definitions. A foliation on a Riemannian manifold is said to be *isoparametric* [13] if its leaves are isoparametric submanifolds. This amounts to saying that its mean curvature 1-form is basic (see §2). A foliation is *tense* if it is isoparametric for some choice of Riemannian metric [14]. Finally, for convenience, let us say that a foliation is *cohomologically taut* if its basic cohomology verifies Poincaré duality.

THEOREM 1. *Let F be an umbilicalisable Riemannian foliation on a compact connected manifold M. Then F is tense, and the following conditions are equivalent:*

(i) *F is cohomologically taut;*
(ii) *F is taut;*
(iii) *F is geodesible.*

Moreover, if F does not satisfy these conditions then there is a Riemannian metric on M for which F is totally umbilic and isoparametric, and for which the leaves of F have zero (intrinsic) sectional curvature. Furthermore, there is an open dense subset of M that is saturated by simply connected leaves.

This theorem has a number of corollaries. Let F and M be as in the statement of the above theorem. One has the following.

COROLLARY 1. *If the first betti number of M is zero then F is geodesible.*

COROLLARY 2. *If F has dense leaves then F is geodesible.*

COROLLARY 3. *If F is not geodesible then F has polynomial growth, and so the structural Lie algebra of F is nilpotent.*

The most striking consequence of Theorem 1 occurs in codimension 2 (all codimension-1 Riemannian foliations on compact manifolds are taut). Before presenting our result, let us give an example of an umbilicalisable Riemannian foliation that is not geodesible. Consider a 2-dimensional vector subspace V of $\mathbf{E}^3$. Let Γ be a cocompact discrete subgroup of $\mathbf{E}^3$ and suppose that A is an affine transformation of $\mathbf{E}^3$ of the form $A: x \mapsto B(x)+c$, where $c \in \mathbf{E}^3$ and B is a special linear matrix that preserves both Γ and V and induces in V a similarity transformation with homothetic proportionality constant not equal to 1. The foliation F_V of $\mathbf{E}^3$ defined by V and its translations is Γ-invariant, and hence induces a foliation F_N on the quotient manifold $N = \mathbf{E}^3/\Gamma$. Clearly A induces a diffeomorphism ϕ of N, and since A leaves F_V invariant, ϕ leaves F_N invariant. We now suspend ϕ in the standard manner. Let the group $\mathbf{Z}$ of integers act on the manifold $N \times \mathbf{R}$ in the following way: If $r \in \mathbf{Z}$, $x \in N$, and $t \in \mathbf{R}$, set $r(x,t) = (\phi^r(x), t+r)$. The quotient of $N \times \mathbf{R}$ by this action of $\mathbf{Z}$ is a compact manifold that we denote M_ϕ. The foliation of $N \times \mathbf{R}$ whose leaves are of the form $L_N \times \{\bigstar\}$, where L_N is a leaf of F_N, is invariant under $\mathbf{Z}$ and hence induces a foliation F_ϕ on M_ϕ. By construction, F_ϕ is Riemannian and umbilicalisable. It is easy to show that F_ϕ is not cohomologically taut (see Example (iii) in §3) and hence not geodesible by Theorem 1.

THEOREM 2. *Let F be an umbilicalisable Riemannian foliation on a compact connected manifold M. If F has codimension* 2 *and is not geodesible, then M has dimension* 4 *and there is a finite cover* $(\tilde{M}, \tilde{F})$ *of* (M, F) *and a diffeomorphism from one of the spaces* M_ϕ *to* $\tilde{M}$ *which conjugates* F_ϕ *with* $\tilde{F}$.

The paper is organized as follows. In the next section we recall the facts about Riemannian and umbilicalisable foliations needed for the proofs of our results. In Section 3 we give a number of examples, and Section 4 contains the proofs of our results. Theorem 1 is an application of the theorems of Lelong-Ferrand [15], Obata [19], and Alekseevskii ([1], [2]), according to which the round sphere S^n and the Euclidean space $\mathbf{E}^n$ are the only Riemannian n-manifolds that possess conformal transformations which are not isometries for any choice of metric. Our use of this theorem employs the techniques of Molino's theory of Riemannian foliations [17] and of Kamber and Tondeur's work on the basic cohomology of tense Riemannian foliations [14]. Theorem 2 is proven directly using Theorem 1 and Molino's theory, though the general philosophy comes from Ghys' classification of codimension-1 totally geodesic foliations [12].

In all of the following, unless otherwise stated, F is an umbilicalisable Riemannian foliation, with leaves of dimension $p \geq 2$, on a compact connected manifold M. Since the problems are unaltered by taking finite covers, we assume that M and F are orientable. Everything is supposed C^∞.

I would like to thank Richard Escobales for having aroused my interest in umbilic foliations. This paper was directly inspired by Yves Carrière's thesis, and my thanks go to him for his friendly and pertinent comments.

2. Preliminaries

The Riemannian metric g on the manifold M is *bundle-like* for the foliation F if

$$(L_Z g)(X,Y)=0 \tag{2.1}$$

for all vector fields Z tangent to F and for X and Y perpendicular to F, where L_Z is the Lie derivative with respect of Z. By definition, F is *Riemannian* if there exists a bundle-like metric for F. According to [3], F is *totally umbilic* with respect to g if

$$(L_X g)(Z,W)=\lambda(X)g(Z,W) \tag{2.2}$$

for all vector fields X perpendicular to F and for Z and W tangent to F, where $\lambda(X)$ is a function on M depending only on X. In this case, F is *totally geodesic* with respect to g if $\lambda(X)=0$ for all vector fields X perpendicular to F, that is, if

$$(L_X g)(Z,W)=0 \tag{2.3}$$

for all vector fields X, Z, and W as above.

Let ν be the volume form along the leaves of F determined by g and a choice of orientation of F; thus ν is a p-form on M, where p is the dimension of the leaves of F. The Riemannian metric g determines a natural positive definite symmetric 2-form $\langle\cdot,\cdot\rangle$ on the complex of differential forms on M. The *mean curvature 1-form* κ [14] of F is defined by

$$\kappa(X)=\begin{cases} 0 & \text{if } X \text{ is tangent to } F,\\ \langle L_X\nu,\nu\rangle & \text{if } X \text{ is perpendicular to } F.\end{cases}$$

F is *harmonic* if $\kappa\equiv 0$. F is *isoparametric* if κ is a basic 1-form—that is, if $L_Z\kappa=0$ for all vector fields Z tangent to F.

One will notice that the conditions for F to be totally umbilic, totally geodesic, harmonic, or isoparametric depend only on the metric along the leaves of F and on the orthogonal decomposition $TM\cong TF\oplus TF^{\perp}$ of the tangent bundle. On the other hand, the condition that g be bundle-like depends only on the Riemannian metric on the quotient bundle TM/TF. So if F is Riemannian and umbilicalisable (resp. geodesible, resp. taut, resp. tense) then there exists a bundle-like metric on M with respect to which F is totally umbilic (resp. totally geodesic, resp. harmonic, resp. isoparametric).

Recall that a vector field X on M is *foliate* for F if $L_Z X$ is tangent to F for all vector fields Z tangent to F. One says that F is *transversally parallelizable* if there exists q linearly independent foliate vector fields $X_1,\dots,X_q$ perpendicular to F, where q is the codimension of F.

THEOREM 2.1 [17]. *If g is bundle-like for the foliation F on the compact manifold M then F lifts to a natural foliation F_T^1 on the positive orthonormal transverse frame bundle E_T^1 of F, and one has:*

(i) *the leaves of F_T^1 are the linear holonomy covers of the leaves of F;*
(ii) *F_T^1 is transversally parallelizable.*

THEOREM 2.2 [17]. *If F is transversally parallelizable then the closures of its leaves are the fibres of a locally trivial fibration $\pi_b: M \to W$. The fibration π_b is called the* basic fibration *and W is called the* basic manifold.

Note that if F has a transverse parallelism $\{X_1, \ldots, X_q\}$ then every vector field perpendicular to F can be written as a function linear combination of the foliate vector fields X_i. So in order to establish that F is totally umbilic (resp. totally geodesic) it suffices to verify Equation (2.2) (resp. (2.3)) for the vector fields $X = X_i$, $i = 1, \ldots, q$. Let $\{\phi_t^i\}_{t \in \mathbf{R}}$ be the 1-parameter groups of transformations of M associated to the vector fields X_i. Since X_i is foliate, the maps ϕ_t^i respect F. Equations (2.2) and (2.3) can be reinterpreted in the following way.

LEMMA 2.3 [3]. *F is totally umbilic (resp. totally geodesic) if and only if the maps ϕ_t^i ($t \in \mathbf{R}$, $i = 1, \ldots, q$) induce conformal (resp. isometric) diffeomorphisms between the leaves of F.*

By Theorem 2.1, the above lemma can be applied to the foliation F_T^1. The connection with F is given by the following.

LEMMA 2.4. *F_T^1 is umbilicalisable (resp. geodesible, resp. taut, resp. tense) if and only if F is.*

Proof. We will only treat the umbilicalisability condition, the other cases being perfectly analogous.

First suppose that F is totally umbilic with respect to the Riemannian metric g on M. The Levi-Civita connection [17] ω on the positive orthonormal transverse frame bundle E_T^1 defines a natural decomposition $TE_T^1 \cong V \oplus H$ of the tangent space, where V is the vertical bundle of the natural projection $pr: E_T^1 \to M$ and H is the horizontal bundle associated to ω. One obtains a Riemannian metric g_T^1 on E_T^1 by imposing that this decomposition be orthogonal, by lifting g from M to H, and by transporting to V a bi-invariant metric on the structure group $SO(q, \mathbf{R})$ of the fibration pr. One can easily verify (locally) that F_T^1 is totally umbilic with respect to g_T^1.

Conversely, if F_T^1 is totally umbilic with respect to some Riemannian metric $\hat{g}$ on E_T^1, then averaging $\hat{g}$ by the action of the $SO(q, \mathbf{R})$ on E_T^1 one obtains a $SO(q, \mathbf{R})$-invariant metric $\hat{g}'$ on E_T^1. Consequently $\hat{g}'$ projects to a Riemannian metric g' on M. One easily verifies that, since F_T^1 is $SO(q, \mathbf{R})$-invariant, F_T^1 is also totally umbilic with respect to $\hat{g}'$ and that hence F is totally umbilic with respect to g'. ☐

REMARK. It is equally clear that there exists a Riemannian metric on M for which F is totally umbilic and isoparametric if and only if there exists a similar metric for $F_T^{\perp}$.

3. Examples

We give five examples. The first two are well-known geodesible Riemannian foliations. For more general constructions of geodesible foliations see [6]. Examples (iii) and (iv) give umbilicalisable Riemannian foliations that are not geodesible. The last example, drawn from [7], gives a non-Riemannian umbilicalisable foliation for which the results of Theorem 1 do not hold.

(i) Let G be a connected Lie group and let H be a connected subgroup of the centre of G. Consider a uniform lattice Γ in G. The foliation of G by the cosets of H is clearly Γ-invariant and hence defines a foliation F on the quotient space G/Γ. One can easily verify that F is Riemannian and geodesible.

(ii) Let B and N be two compact Riemannian manifolds and let

$$\phi: \pi_1(B) \to \mathrm{Isom}(N)$$

be a homomorphism from the fundamental group of B to the group of isometries of N. Suspending ϕ, one obtains a fibre bundle over B with typical fibre N and, transverse to the fibre, a natural foliation which is both Riemannian and geodesible.

(iii) We now give an explicit example of the type of foliation F_ϕ considered in the Introduction. The task is to exhibit a matrix B having the required properties.

Consider a cubic polynomial

$$P(\lambda) = \lambda^3 + i\lambda^2 + j\lambda - 1$$

with integer coefficients i and j, and suppose that P has only one real root μ. For example,

$$P(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 1.$$

Let us suppose that $\mu > 0$, $\mu \neq 1$, as in the case of the above polynomial. Rewriting P in the form

$$P(\lambda) = (\lambda - \mu)(\lambda^2 - 2a\mu^{-1/2}\lambda + \mu^{-1}),$$

one obviously has $|a| < 1$. Now consider the matrix

$$B = \begin{pmatrix} \mu & 0 & 0 \\ 0 & a/\sqrt{\mu} & b/\sqrt{\mu} \\ 0 & -b/\sqrt{\mu} & a/\sqrt{\mu} \end{pmatrix},$$

where $b = \sqrt{1-a^2}$. Then P is the characteristic polynomial of A. Clearly B leaves the subspaces $V = \mathbf{R}\{(0,1,0),(0,0,1)\}$ and $W = \mathbf{R}\{(1,0,0)\}$ invariant, and induces in V a similarity transformation. It remains to show that B leaves invariant a uniform lattice in $\mathbf{E}^3$. Pick any vector X in the complement of $V \cup W$ in $\mathbf{E}^3$. One easily sees that the vectors X, BX, and B^2X are linearly independent. Furthermore, by the Cayley-Hamilton theorem,

$$B^3X = X - iBX - jB^2X.$$

So since i and j are integers, the lattice Γ in $\mathbf{E}^3$ generated by the vectors X, BX, and B^2X is clearly B-invariant.

Let (M_ϕ, F_ϕ) be the foliation constructed from B using the method described in the Introduction. It is easy to verify that F_ϕ is Riemannian, tense, and umbilicalisable. To see that F_ϕ is not geodesible, notice that for the obvious choice of Riemannian metric, the mean curvature 1-form κ is nowhere zero. It follows that κ is not an exact form. Indeed, if $\kappa = df$ for some function f on M_ϕ then, since M_ϕ is compact, f must have a maximum value and so κ would be zero at any point where this maximum was attained. Since κ is not exact, it follows from [14] that F is not geodesible (see also Lemma 4.4 below).

(iv) The construction of a nongeodesible totally umbilic Riemannian foliation of codimension 2 given in the Introduction can be obviously generalized to higher codimensions. An example of the codimension-3 case is obtained by repeating the above construction for the matrix

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(v) [7] Let V be a vector subspace of $\mathbf{E}^n$ of dimension $p > 2$, and let $\hat{F}$ be the foliation of $\mathbf{E}^n \setminus \{0\}$ by $V \setminus \{0\}$ and the translations of V. Consider the action of $\mathbf{Z}$ on $\mathbf{E}^n \setminus \{0\}$ generated by a dilation with proportionality constant different than 1. The quotient M of $\mathbf{E}^n \setminus \{0\}$ by this action is diffeomorphic to $S^{n-1} \times S^1$ and since $\hat{F}$ is $\mathbf{Z}$-invariant, it induces a foliation F on M. Clearly F has compact leaves, diffeomorphic to $S^{p-1} \times S^1$, and leaves that are diffeomorphic to $\mathbf{R}^p$. $\hat{F}$ is totally geodesic with respect to the metric g induced on $\mathbf{E}^n \setminus \{0\}$ by the standard metric on $\mathbf{E}^n$. So $\hat{F}$ is totally umbilic with respect to the conformally equivalent metric g' defined by

$$g'_x = \frac{g_x}{\|x\|}, \quad x \in \mathbf{E}^n \setminus \{0\}.$$

Since the action of $\mathbf{Z}$ respects g', one obtains a Riemannian metric on M with respect to which F is totally umbilic. F is not taut by [22], and so F does not verify the results of Theorem 1 since the universal covers of its compact leaves are not conformally flat.

4. Proofs

Proof of Theorem 1. Let F be a foliation satisfying the hypotheses of Theorem 1, and suppose that F is totally umbilic with respect to the bundle-like metric g. As we saw in Section 2, the lift F_T^1 of F to the positive orthonormal transverse frame bundle E_T^1 of F is transversally parallelizable, and F_T^1 is totally umbilic with respect to the lifted metric g_T^1. So we first consider transversally parallelizable foliations.

LEMMA 4.1. *If F is transversally parallelizable and F is not geodesible, then each leaf of F is conformally equivalent to the Euclidean space $\mathbf{E}^p$.*

Proof. By Lemma 2.3, for every foliate vector field X perpendicular to F, the 1-parameter group $\{\phi_t\}_{t\in\mathbf{R}}$ of transformations of M associated to X respects the foliation F and induces conformal diffeomorphisms between its leaves. So the leaves of F are conformally equivalent. If Φ is the set of all the conformal diffeomorphisms induced between the leaves by the maps ϕ_t associated to the different foliate vector fields X perpendicular to F, then Φ acts transitively on the set of leaves and F is totally geodesic with respect to g if and only if all the elements of Φ are isometries (by Lemma 2.3).

Let L be a leaf of F. According to the theorems of Lelong-Ferrand [15], Obata [19], and Alekseevskii ([1], [2]), L is necessarily one of the following three types.

(i) L is inessential; that is, the group of conformal transformations of L acts on L by isometries for some Riemannian metric g_L on L.
(ii) L is conformally equivalent to the round sphere S^p.
(iii) L is conformally equivalent to the Euclidean space $\mathbf{E}^p$.

Suppose that L is of type (i). We claim that F is geodesible. We define a new Riemannian metric g' on M by changing g only along the leaves of F. If L' is another leaf of F, there exists an element ϕ of Φ such that $\phi(L)=L'$. We define a Riemannian metric on L' by pushing forward g_L by ϕ. The metric obtained on L' is independent of the choice of ϕ. Indeed, if $\phi, \psi\in\Phi$ and $\phi(L)=\psi(L)=L'$, then $\phi^{-1}\circ\psi$ is a conformal transformation of L and hence an isometry with respect to g_L. Consequently ϕ and ψ induce the same metric on L'. Proceeding in this manner for all the leaves one obtains a Riemannian metric g_F along the leaves of F. The new Riemannian metric g' on M is then obtained by replacing g along the leaves by g_F. By construction, g' is smooth. Furthermore, the elements of ϕ now clearly act by isometries between the leaves, and hence F is totally geodesic with respect to g'.

It remains to prove that if L is of type (ii) then F is geodesible. But if L is of type (ii) then the leaves of F are all diffeomorphic to S^p and so they are the fibres of a fibre bundle B. The orthogonal vector bundle $F^\perp$ of F is a connection for this bundle with structure group a group of conformal transformations. Because this group deformation retracts to a group H of isometries, B has a connection with structure group H. Thus F is geodesible (see [20]). □

If F_T^1 is not geodesible then, according to Lemma 4.1, the leaves of F_T^1 are conformally equivalent to $\mathbf{E}^p$. We now show that these conformal equivalences can be effected simultaneously by changing the metric g_T^1 on E_T^1.

LEMMA 4.2. *If F is transversally parallelizable and F is not geodesible, then there is a Riemannian metric on M for which F is totally umbilic and for which the leaves of F are isometric to Euclidean space $\mathbf{E}^p$.*

Proof. We start by considering an arbitrary fibre N of the basic fibration $\pi_b: M \to W$. The foliation F_N induced on N by F is clearly totally umbilic for the metric g_N induced on N by g. Let Λ be the (finite-dimensional) vector space of linearly independent foliate vector fields on N perpendicular to F_N. If X is an element of Λ, let $\{\phi_t^X\}_{t\in\mathbf{R}}$ be the 1-parameter group of transformations of N associated to X. Consider the map

$$\begin{aligned}\Psi: N\times\Lambda &\to N\\ :(x,X) &\mapsto \phi_1^X(x).\end{aligned}$$

The maps ϕ_t^X respect the foliation F_N, and locally Ψ projects to the exponential map on the local quotient manifold. The space Λ has a norm $\|\cdot\|$ induced by the Riemannian metric g_N. Because N is compact there exists $\epsilon>0$ such that, for all points x in N, the restriction of the map $\psi_x: X\in\Lambda\mapsto\Psi(x,X)$ to the open ball $B(\epsilon)=\{X\in\Lambda/\|X\|<\epsilon\}$ is injective. If one likes, ϵ is the "transverse injectivity radius" of F_N.

Let L be a leaf of F_N. By Lemma 4.1, L is conformally equivalent to $\mathbf{E}^p$. Let g_L be the flat metric on L induced by this equivalence. We now define a metric g_L along the leaves of F_N by using Ψ. First note that the image of $L\times B(\epsilon)$ under Ψ is a saturated open subset of N. So, because L is dense, $\Psi(L\times B(\epsilon))=N$. Thus, for any leaf L' of F_N, there exists $X\in B(\epsilon)$ such that $\phi_1^X(L)=L'$. We transport g_L from L to L' by the map ϕ_1^X. To see that this is well defined, note that if $\phi_1^X(L)=\phi_1^Y(L)$ where $X, Y\in B(\epsilon)$, then $\phi=(\phi_1^Y)^{-1}\circ\phi_1^X$ is a conformal diffeomorphism of L. We claim that if $X\neq Y$ then ϕ has no fixed point. Indeed, if $\phi(x)=x$ for some $x\in L$ then one would have $\phi_1^X(x)=\phi_1^Y(x)$ and hence $\psi_x(X)=\psi_x(Y)$. So, since ψ_x is injective, one would have $X=Y$. Consequently if ϕ is not the identity map then it is an orientation-preserving conformal diffeomorphism of $L\cong\mathbf{E}^p$ having no fixed point. So ϕ is a translation and hence ϕ_1^X and ϕ_1^Y induce the same metric on L'. We thus obtain a new Riemannian metric g_F along the leaves of F_N. Replacing the original metric along the leaves by g_F one obtains a new metric g_N' on N with respect to which (by construction) F_N is totally geodesic, and its leaves are isometric to $\mathbf{E}^p$.

Having changed the Riemannian metric on an arbitrary fibre N, we now change the Riemannian metric in a saturated open neighbourhood of N in

M. The procedure is essentially the same as that used above for N. If r is the dimension of the basic manifold W, choose r foliate vector fields $X_1, \ldots, X_r$ on M such that at every point on N the vector fields $X_1, \ldots, X_r$ are linearly independent and perpendicular to N. For each $i \in \{i, \ldots, r\}$ let $\{\phi_t^i\}_{t \in \mathbf{R}}$ be the 1-parameter group of transformations of M associated to X_i. Consider the map

$$\Phi : N \times \mathbf{R}^r \to M$$
$$: (x, t_1, \ldots, t_r) \mapsto \phi_{t_1}^1 \phi_{t_2}^2 \cdots \phi_{t_r}^r(x).$$

The vector fields X_i project by π_b to vector fields on W which are linearly independent at $\pi_b(N)$. It follows that for sufficiently small $\epsilon > 0$, Φ induces a diffeomorphism from the open set $N \times \{t \in \mathbf{R}^r / \|t\| < \epsilon\}$ onto its image U, say, in M. Clearly U is saturated with respect to the fibres of π_b, and the maps

$$\phi_t : N \to M$$
$$: x \mapsto \Phi(x, t)$$

send the leaves of F_N conformally to the leaves of F. Now use Φ to replace the Riemannian metric along the leaves of the restriction of F to U by the Riemannian metric induced from the Riemannian metric g_F constructed above on the leaves of F_N. Let g'_U be this new Riemannian metric on U. By construction, g and g'_U differ only along the leaves of F, and on the leaves the two metrics are conformally equivalent. Moreover, with respect to g'_U, the leaves are isometric to $\mathbf{E}^p$.

Now choose a set $\{N_j\}_{j \in J}$ of fibres of π_b such that the corresponding set $\{U_j\}_{j \in J}$ of open neighbourhoods constructed above covers M. For each $j \in J$, let g_j denote the Riemannian metric g'_{U_j} on U_j constructed above. The sets U_j are saturated by the fibres of π_b, and of course their images V_j in W cover W. Choose a locally finite subcover $\{V_k\}_{k \in K}$ of this cover and let $\{f_k\}_{k \in K}$ be a partition of unity subordinate to $\{V_k\}_{k \in K}$. Then $\{f_k \circ \pi_b\}_{k \in K}$ is a partition of unity of M subordinate to $\{U_k\}_{k \in K}$. Gluing together the Riemannian metrics g_k with this partition of unity, one obtains a new Riemannian metric g' on M. Since g' and the original metric g give the same orthogonal decomposition $TM \cong TF \oplus TF^\perp$ of the tangent bundle, and since the metrics along the leaves are conformally equivalent, F is also totally umbilic with respect to g'. By construction, the leaves of F, equipped with the metric induced by g', are isometric to $\mathbf{E}^p$. □

We can now prove that F is tense. In fact, one has the following.

LEMMA 4.3. *There exists a Riemannian metric on M for which F is isoparametric and totally umbilic.*

Proof. By Lemma 2.4 and the Remark that follows it, we may assume that F is transversally parallelizable. Now if F is geodesible there is nothing to prove. So by Lemma 4.2 we may assume that the leaves of F are isometric to $\mathbf{E}^p$. Let X be a foliate vector field perpendicular to the leaves of F, and let

$\{\phi_t\}_{t\in\mathbf{R}}$ be the 1-parameter group of transformations of M associated to X. Let ν be the volume form along the leaves of F determined by the Riemannian metric and a choice of orientation. Since the maps ϕ_t induce conformal diffeomorphisms between the leaves of F, for each $t \in \mathbf{R}$ we have $\phi_t^*(\nu) = \lambda_t \nu$, where the function λ_t is constant on the leaves of F. So $L_X \nu = \mu\nu$, where $\mu = d\lambda_t/dt$ is also a basic function on M. It follows immediately that the mean curvature 1-form of F is basic and so F is isoparametric. □

By Lemma 4.3, we may assume that the mean curvature 1-form κ of F is basic. By [14], κ is thus closed and hence defines a basic cohomology class $[\kappa] \in H_b^1(M, F)$.

LEMMA 4.4. *F is geodesible if* $[\kappa] = 0$.

Proof. Suppose that $\kappa = df$, where f is a basic function. Now replace g by a conformally equivalent Riemannian metric g' such that the volume form ν' along the leaves of F determined by g' is $\nu' = \exp(-f)\nu$, where ν is the volume form along the leaves determined by g. One calculates easily (see [14]) that the mean curvature 1-form of F with respect to g' is zero and so F is harmonic with respect to g'. Then, since F is clearly totally umbilic with respect to g', it is consequently totally geodesic. □

It is well known that if F is taut then F is cohomologically taut (see [21]), and that if F is geodesible then F is obviously taut. The following lemma completes the set of equivalences.

LEMMA 4.5. *If F is cohomologically taut then F is geodesible.*

Proof. By Lemma 4.3, we may assume that F is totally umbilic and isoparametric with respect to the metric g. Thus, according to [14], if F is cohomologically taut then $[\kappa] = 0$ and so F is geodesible by Lemma 4.4. □

To complete the proof of Theorem 1 it remains to consider the case where F is not geodesible. Apply Lemma 4.2 to the foliation F_T^1 and note that the leaves of F_T^1 cover those of F. Taking a metric on E_T^1 for which the leaves of F_T^1 are isometric to $\mathbf{E}^p$ and averaging this metric by the action of the structure group of the fibration $p: E_T^1 \to M$, one obtains the desired metric on M. Furthermore, the leaves of F_T^1 are the holonomy covers of the leaves of F, by Theorem 2.1, and so by [9] M has an open dense subset saturated by simply connected leaves. This completes the proof of Theorem 1. □

Proof of Corollary 1. Corollary 1 follows immediately from Theorem 1 and [5]. Alternatively, one can use Lemma 4.4 and the fact that $H_b^1(M, F)$ injects into $H^1(M)$. □

REMARK. If F is not geodesible then, by Lemma 4.2, there exists a Riemannian metric on E_T^1 for which F_T^1 is totally umbilic and isoparametric and for which the leaves of F_T^1 have zero sectional curvature. Furthermore, as we saw in the proof of Lemma 4.2, the restriction of F to the fibres of the basic fibration $\pi_{Tb}^1 : E_T^1 \to W_T^1$ of F_T^1 is then totally geodesic. So the mean curvature 1-form κ_T^1 of F_T^1 is zero on the fibres of π_{Tb}^1. This has two obvious consequences:

(i) We may choose a metric on M such that the mean curvature 1-form κ of F is zero on the closures of the leaves of F.

(ii) κ_T^1 is the pull-back to E_T^1 of a closed 1-form on W_T^1. So, as in Lemma 4.3, F_T^1 is geodesible if the first betti number of the basic manifold W_T^1 is zero.

Proof of Corollary 2. This follows immediately from Lemma 4.3 and part (i) of the above Remark. □

Proof of Corollary 3. This follows immediately from Theorem 1 and [8]. □

Proof of Theorem 2. The topology of F is largely determined by the structural algebra of F [16]. If F has codimension 2 then this algebra has dimension $s \leq 3$. If $s = 0$, the leaves of F are compact and hence F is taut (see [21]) and therefore geodesible by Theorem 1. If $s = 2$ or 3, then the leaves of F are dense and so F is geodesible by Corollary 2. If $s = 1$, there are two possibilities: Either the closures of the leaves of F define a codimension-1 Riemannian foliation on M, or F has one or two compact leaves and the rest of the leaf closures have codimension 1. We claim that in the second case F is geodesible. Indeed, if $s = 1$, the basic manifold W_T^1 of F_T^1 has dimension 2. The positive orthogonal transverse frame bundle E_T^1 is a S^1-bundle over M, and the action of S^1 on E_T^1 induces an action of S^1 on W_T^1 [17]. The compact leaves of F correspond to the fixed points of this action. So if F has one or two compact leaves, the action of S^1 on W_T^1 has one or two fixed points. But the only 2-manifolds that admit an S^1 action with isolated fixed points are S^2 and the projective space RP^2. So $H^1(W_T^1) = 0$ and consequently F is geodesible by part (ii) of the Remark following the proof of Corollary 2.

We have thus shown that if F is not geodesible, the closures of the leaves of F define a codimension-1 Riemannian foliation F_b on M. Passing to the transverse orientation cover of F_b if necessary, we may assume that F is transversally parallelizable and that the leaves of F_b are the fibres of a locally trivial fibration π_b of M over S^1 (see [17]). Then, by Theorem 1, we may assume that F is isoparametric and that its leaves are isometric to $\mathbf{E}^p$.

Let X be the unit vector field orthogonal to F_b. It is clear that X is foliate both for F and for F_b. Consider the 1-parameter group $\{\phi_t\}$ associated to X. Let N be an arbitrary fibre of π_b and let s be the smallest positive number such that $\phi_s(N) = N$. Then M is clearly diffeomorphic to the suspension of ϕ_s via the map

$$\begin{aligned} f : N \times \mathbf{R}/(x,t) \sim (\phi_s(x), t+s) &\to M \\ : [x,t] &\mapsto \phi_{-t}(x). \end{aligned}$$

We will show that ϕ_s is the map ϕ in the statement of Theorem 2. In particular, we need to show that ϕ_s lifts to a linear map on $\mathbf{E}^p$.

Now, as we saw in the proof of Lemma 4.2, the restriction F_N of F to N is a totally geodesic Riemannian foliation of codimension 1 whose leaves are isometric to $\mathbf{E}^p$. Consequently N has zero sectional curvature. Since, by hypothesis, M is oriented, so too is F_b and consequently N is also oriented. So N is a torus $\mathbf{E}^{p+1}/\Gamma$, where Γ is a group Γ of Euclidean translations of $\mathbf{E}^{p+1}$ with $p+1$ generators. Choose $x \in N$ and let ψ be a translation of N such that $\psi(x) = \phi_s(x)$, where ϕ_s is as in the previous paragraph. Of course the map $\psi^{-1}\phi_s$, $= \rho$ say, respects F_N and induces conformal maps between the leaves of F_N. Since ρ fixes x, ρ induces a map on the set of paths in N based at x and hence defines a diffeomorphism β of the universal cover $\mathbf{E}^{p+1}$ of N. We claim that β is linear. Indeed, let W denote the vector space of Killing vector fields of N. Let Y be an element of W and let $\{\zeta_t\}_{t\in\mathbf{R}}$ be its associated 1-parameter subgroup. Of course ζ_t is an isometry for all t and the 1-parameter subgroup of the pushed-forward vector field $\rho_* Y$ is $\{\rho\circ\zeta_t\circ\rho^{-1}\}_{t\in\mathbf{R}}$. Because ρ induces conformal maps between the leaves of F_N, so for all t, $\rho\circ\zeta_t\circ\rho^{-1}$ induces isometries between the leaves of F_N. It follows that since the leaves of F_N are dense in N, $\rho\circ\zeta_t\circ\rho^{-1}$ is an isometry of N for all t. Hence $\rho_* Y$ is an element of W. Thus the push-forward map ρ_* defines an isomorphism of W. Consequently β is affine and hence linear. Moreover, since ρ is necessarily volume preserving, β is special linear.

Let V denote the vector subspace of $\mathbf{E}^{p+1}$ covering the leaf of F_N passing through x. Then β induces a similarity transformation in V, and clearly F_N is the projection in N of the foliation F_V on $\mathbf{E}^{p+1}$ defined by V and its translates. Notice that the homothetic proportionality constant of the map induced by β in V is not equal to one, for otherwise one could choose a Riemannian metric on M for which the mean curvature 1-form of F would be identically zero and hence F would be geodesible.

In order to complete the proof of Theorem 2 it remains to show that M has dimension 4—in other words, that $p=2$. First notice that β induces a conformal transformation in the hyperspace V, so β has p eigenvalues $\alpha_1, \ldots, \alpha_p$ with the same absolute value, r say, with $r \neq 1$. Clearly β has one other eigenvalue, α_{p+1} say, which is real and not equal to r, as β is special linear. Notice as well that since β preserves a uniform lattice in $\mathbf{E}^{p+1}$, the characteristic polynomial of β has integer coefficients.

We will employ the following result of Epstein.

THEOREM 4.6 [10]. *Let $a_0 + a_1 x + \cdots + a_k x^k$ be an irreducible (over Q) polynomial with integer coefficients and roots $\alpha_1, \ldots, \alpha_k$. Let $|\alpha_1| = \cdots = |\alpha_{k-1}| \neq |\alpha_k|$. Then $k \leq 3$.*

In order to apply this theorem it remains to show that the characteristic polynomial of β is irreducible over Q. First note that, if Z is an eigenvector of β corresponding to the exceptional eigenvalue α_{p+1}, then the leaves of the linear foliation F_Z on N determined by Z are dense in N. Indeed, ρ preserves the leaf L of F_Z passing through x, and so ρ preserves the closure $\bar{L}$ of L. If U is the vector subspace of $\mathbf{E}^{p+1}$ covering $\bar{L}$, then β preserves U; since the diffeomorphism of $\bar{L}$ induced by ρ is volume preserving, the restriction of β to U is special linear. It follows that $U=\mathbf{E}^{p+1}$ and consequently $\bar{L}=N$.

Now, arguing as in [10], let f be a polynomial with integer coefficients and suppose that α_{p+1} is a root of f. Then $f(\beta)Z=0$. The transformation $f(\beta)$ induces a smooth map from N to N that is constant on the leaf L of F_Z and hence, as L is dense in N, constant on N. Thus $f(\beta)=0$, and consequently every eigenvalue of β is a root of f. It follows that the characteristic polynomial of β is irreducible over Q.

We can thus apply Theorem 4.6 to the characteristic polynomial of β, whence $p+1\leq 3$. But by hypothesis $p\geq 2$ and so $p=2$ and M has dimension 4. This completes the proof of Theorem 2. □

References

1. D. V. Alekseevskii, *Groups of conformal transformations of Riemannian spaces,* Math. USSR-Sb. 18 (1972), 285–301.
2. ———, *S^n and $\mathbf{E}^n$ are the only Riemannian spaces that admit an essential conformal transformation* (Russian), Uspekhi Mat. Nauk 28 (1973), 225–226.
3. R. A. Blumenthal and J. J. Hebda, *An analogue of the holonomy bundle for a foliated manifold,* Tôhoku Math. J. 40 (1988), 189–198.
4. ———, *A sufficient condition for the leaves of a totally umbilic foliation to be conformally complete,* preprint.
5. G. Cairns, *Une remarque sur la cohomologie basique d'un feuilletage riemannien,* Séminaire de Géométrie différentielle 1984–1985, 1–7, Université des Sciences et Techniques du Languedoc, Montpellier, 1985.
6. ———, *Feuilletages géodésibles,* thesis, Université des Sciences et Techniques du Languedoc, Montpellier, 1987.
7. Y. Carrière, *Flots riemanniens et feuilletages géodésibles de codimension un,* thesis, Université des Sciences et Techniques de Lille, Lille, 1981.
8. ———, *Feuilletages riemanniens à croissance polynomiale,* Comment. Math. Helv. 63 (1988), 1–20.
9. D. B. A. Epstein, K. C. Millett, and D. Tischler, *Leaves without holonomy,* J. London Math. Soc. 16 (1977), 548–552.
10. D. B. A. Epstein, *Transversely hyperbolic 1-dimensional foliations,* Astérisque 116 (1984), 53–69.
11. R. H. Escobales, Jr. and P. E. Parker, *Geometric consequences of the normal curvature cohomology class in umbilic foliations,* Indiana Univ. Math. J. 37 (1988), 1–19.
12. E. Ghys, *Classification des feuilletages totalement géodésiques de codimension un,* Comment. Math. Helv. 58 (1983), 543–572.
13. D. Gromoll and K. Grove, *The low-dimensional metric foliations of Euclidean spheres,* J. Differential Geom. 28 (1988), 143–156.
14. F. W. Kamber and P. Tondeur, *Foliations and metrics,* Differential Geometry, Prog. Math., 32, Birkhäuser, Boston, 1983.

15. J. Lelong-Ferrand, *Transformations conformes et quasi-conformes des variétés riemanniennes; application à la démonstration d'une conjecture de A. Lichnerowicz,* C. R. Acad. Sci. Paris Sér. I Math. 269 (1969), 583–586.
16. P. Molino, *Géométrie globale des feuilletages riemanniens,* Pro. Kon. Nederland Akad. Ser. A 85 (1982), 45–76.
17. ———, *Riemannian foliations,* Prog. Math., 73, Birkhäuser, Boston, 1988.
18. A. Naveira and A. Rocamora, *A geometric obstruction to the existence of two totally umbilic complementary foliations in compact manifolds,* Lecture Notes in Math., 1139, Springer, Berlin, 1985.
19. M. Obata, *The conjectures on conformal transformations of Riemannian manifolds,* J. Differential Geom. 6 (1971), 247–258.
20. B. L. Reinhart, *Differential geometry of foliations,* Ergeb. Math. Grenzgeb (3), 99, Springer, Berlin, 1983.
21. V. Sergiescu, *Basic cohomology and tautness of Riemannian foliations,* Riemannian Foliations (Pierre Molino), Prog. Math., 73, Birkhäuser, Boston, 1988.
22. D. Sullivan, *A homological characterization of foliations consisting of minimal surfaces,* Comment. Math. Helv. 54 (1979), 218–223.

Department of Mathematics
La Trobe University
Melbourne, Australia 3083