Variations of Pseudoconvex Domains over C^n

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Various function-theoretic quantities can be associated to a domain D over the complex plane C. In [13]–[16] we studied how these quantities vary when D varies over C. An important quantity among them was the *Robin constant* which is defined as follows: Let D be an unramified covering domain over C whose boundary, ∂D, consists of smooth curves. For a fixed point ζ in D, the domain D carries the Green function $g(z)$ for Laplace's equation $\Delta g = 4\partial^2 g/\partial z\partial\bar{z} = 0$ with pole at ζ. The function g is uniquely determined by the following three conditions: $\Delta g = 0$ in D except at ζ, g is continuous up to ∂D and $g = 0$ on ∂D, and g differs from $\log(1/|z-\zeta|)$ by a harmonic function in a neighborhood of ζ. We put

$$\lambda = \lim_{z\to\zeta}\left(g(z) - \log\frac{1}{|z-\zeta|}\right).$$

Following Faber [3], we call λ the Robin constant for (D,ζ). Now we vary the domain D over C for t in the disk $B: |t| < \rho$; that is, we have a variation $t \to D(t)$ $(t \in B)$ with the following properties: $D(0) = D$, each $D(t)$ $(t \in B)$ is an unramified covering domain over C bounded by the smooth curves forming $\partial D(t)$, and each $D(t)$ contains the point ζ. We then have the Robin constant $\lambda(t)$ for $(D(t),\zeta)$. $\lambda(t)$ defines a real-valued function on B. In [15] we obtained the following.

THEOREM I. *If the set* $\mathbf{D} = \{(t,z) \mid t \in B \text{ and } z \in D(t)\}$ *is a pseudoconvex domain over the product space* $B \times C$, *then* $\lambda(t)$ *is a superharmonic function on* B.

The definition of a pseudoconvex domain over $B \times C$ is given in Oka [8, p. 101]. This theorem was motivated by Nishino's beautiful work on value distribution of entire functions of two complex variables (see his survey [7]). Theorem I has been recently applied to the theory of functions by Suzuki [10] and Fujita [4] and also to other areas by Wermer [12], Kaneko [6], and Suzuki [11].

In this paper we study the case when $D(t)$ varies over the complex n-dimensional Euclidean space C^n, where $n \geq 2$. Let D be an unramified covering domain D over C^n bounded by smooth surfaces ∂D. Fix $\zeta \in D$. Then

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D carries the Green function $g(z)$ for Laplace's equation $\Delta g = 4(\partial^2/\partial z_1 \partial \bar{z}_1 + \cdots + \partial^2/\partial z_n \partial \bar{z}_n)g = 0$ with pole at ζ. Since $g(z)$ differs from $1/\|z-\zeta\|^{2n-2}$ by a harmonic function in a neighborhood of ζ, we put

$$\lambda = \lim_{z \to \zeta}\left(g(z) - \frac{1}{\|z-\zeta\|^{2n-2}}\right).$$

By the maximum principle it follows that $\lambda < 0$. We call λ the Robin constant for (D, ζ). Let D vary over C^n with complex parameter t in the disk B so that D becomes an unramified covering domain $D(t)$ with smooth boundary $\partial D(t)$ over C^n with $D(t) \ni \zeta$. We denote by $\lambda(t)$ the Robin constant for $(D(t), \zeta)$. Consequently, $\lambda(t)$ defines a negative real-valued function on B. We shall demonstrate the following theorem.

THEOREM II. *If the set* $\mathbf{D} = \{(t, z) \mid t \in B \text{ and } z \in D(t)\}$ *is a pseudoconvex domain in* $B \times C^n$ $(n \geq 2)$, *then* $\lambda(t)$ *is a superharmonic function on B. Moreover,* $\log(-\lambda(t))$ *is a subharmonic function on B.*

Theorem I and Theorem II are the same in form, but are different in content. This will be shown in Sections 6, 7, and 8 where we give some applications of Theorem II (cf. the applications of Theorem I given in [15] and [16]).

Originally the significance of the Robin constant appeared in the research of the equilibrium distribution of electric charges on a conductor which is placed in the real Euclidean space R^3. In Section 9, after recalling Robin's paper [9] which was published about 100 years ago, we shall study the case when $D(t)$ varies in R^m $(m \geq 3)$ with a real parameter t. Let I: $-\rho < t < \rho$ be an open interval. For each $t \in I$, let $D(t)$ be a domain in R^m bounded by smooth surfaces $\partial D(t)$. Assume that each $D(t)$ contains a fixed point ξ in R^m. Then $D(t)$ carries the Green function $g(t, x)$ for Laplace's equation $\Delta g = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2)g = 0$ with pole at ξ. We put

$$\lambda(t) = \lim_{x \to \xi}\left(g(t, x) - \frac{1}{\|x-\xi\|^{m-2}}\right)$$

and call $\lambda(t)$ the Robin constant for $(D(t), \xi)$. Accordingly, $\lambda(t)$ is a negative real-valued function on I. We shall show the following.

THEOREM III. *If the set* $\mathbf{D} = \{(t, x) \mid t \in I \text{ and } x \in D(t)\}$ *is a convex domain of* $I \times R^m$, *then*

$$\frac{d^2 \log(-\lambda(t))}{dt^2} \geq \frac{1}{m-2}\left|\frac{d \log(-\lambda(t))}{dt}\right|^2$$

for $t \in I$. *Hence both* $\log(-\lambda(t))$ *and* $-\lambda(t)$ *are convex functions on I.*

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The main results in the paper have been announced in [17].

1. Robin Constants

Let R^m be real m-dimensional Euclidean space whose points are given by m real coordinates $x = (x_1, \ldots, x_m)$ with norm $\|x\|^2 = |x_1|^2 + \cdots + |x_m|^2$. Throughout this paper we assume $m \geq 3$. By an unramified covering domain E *over* the space R^m, or, more simply, a domain *over* R^m, we mean a triple (E, R^m, p) such that E is a connected Hausdorff space and p, the projection, is a local homeomorphism from E to R^m. Moreover, by an open set *over* R^m we mean a union of at most countably many domains over R^m without any relation. We use with caution the convenient notation "$x \in E$", which means precisely that x is a point of E such that $p(x) = x$ $(x \in R^m)$. As usual, for a subset $K \subset E$ we denote by ∂K the boundary of K in E. For $K_1 \subset K_2 \subset E$, if K_1 is relatively compact in K_2 then we write $K_1 \subset\subset K_2$. A complex-valued function $u(x)$ defined in a subdomain of E is said to be *harmonic* if u is of class C^2 and satisfies Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2} = 0.$$

DEFINITION 1.1. An open set D over R^m is said to have *smooth boundary* if there exist an open set $\tilde{D}$ over R^m and a real-valued function ψ of class C^∞ in $\tilde{D}$ such that

(a) $\tilde{D} \supset\supset D$;
(b) if we denote by ∂D the boundary of D in $\tilde{D}$, then $D = \{x \in \tilde{D} \mid \psi(x) < 0\}$ and $\partial D = \{x \in \tilde{D} \mid \psi(x) = 0\}$; and
(c) $\left(\dfrac{\partial \psi}{\partial x_i}\right)_{i=1,\ldots,m}(x) \neq 0$

for any x in ∂D.

We call (D, ψ) a double which *defines* the open set D.

If D has smooth boundary ∂D, then ∂D is a union of $(m-1)$-dimensional smooth surfaces. We always assume that ∂D is positively oriented with respect to the domain D.

We now define the Green function g for an open set D over R^m. In what follows, ξ is a fixed point in D. First, consider the case where (D, R^m, p) is a domain with smooth boundary ∂D. According to potential theory, D carries the Green function $g(x)$ with pole at ξ which is uniquely determined by the following conditions:

(1) g is harmonic in D except at ξ;
(2) g is continuous up to ∂D and $g = 0$ on ∂D; and
(3) in a neighborhood of ξ, g differs from $1/\|x-\xi\|^{m-2}$ by a harmonic function.

From (3) we write, for x in a neighborhood of ξ,

$$g(x)=\frac{1}{\|x-\xi\|^{m-2}}+\lambda+h(x),$$

where λ is a constant, $h(x)$ is harmonic, and $h(\xi)=0$. The constant term λ is called the *Robin constant* for (D, ξ). Next, consider the case where (D, R^m, p) is a domain whose boundary is not necessarily smooth. Then we choose any subdomain $\Omega\subset D$ with smooth boundary such that $\xi\in\Omega$ and $\Omega\subset\subset D$. We thus have the Green function g_Ω *of* Ω with pole at ξ *and the* Robin constant λ_Ω for (Ω, ξ). It follows from the maximum principle that

$$g_\Omega(x)<\gamma_{\Omega'}(x)<\frac{1}{\|x-\xi\|^{m-2}}\quad\text{and}\quad -\infty<\lambda_\Omega<\lambda_{\Omega'}<0$$

for any $\Omega\subset\subset\Omega'\subset\subset D$. Here $1/\|x-\xi\|^{m-2}$ represents the harmonic function $1/\|p(x)-p(\xi)\|^{m-2}$ in D (which has the same singularity as $1/\|x-\xi\|^{m-2}$ at all points of $p^{-1}(\xi)$). We obtain the limits $g(x)=\sup\{g_\Omega(x)\,|\,\Omega\subset\subset D\}$ for $x\in D$ and $\lambda=\sup\{\lambda_\Omega\,|\,\Omega\subset\subset D\}$. Consequently, $g(x)$ is harmonic in D except at ξ and can be expressed for x in a neighborhood of ξ in the form

$$g(x)=\frac{1}{\|x-\xi\|^{m-2}}+\lambda+h(x), \tag{1.1}$$

where $h(x)$ is harmonic and $h(\xi)=0$. Moreover,

$$0<g(x)\leq\frac{1}{\|x-\xi\|^{m-2}}\quad\text{and}\quad -\infty<\lambda\leq 0 \tag{1.2}$$

for $x\in D$. The function $g(x)$ in D and the constant term λ will be called the Green function and the Robin constant for (D, ξ). Finally, consider the case where D is an open set, not necessarily connected, over R^m. We denote by D_1 the subdomain (i.e., the connected component) of D which contains ξ. Then we have the Green function $g_1(x)$ and the Robin constant λ_1 for (D_1, ξ). By the Green function $g(x)$ for (D, ξ) we mean the function $g(x)=g_1(x)$ in D_1 and $g(x)=0$ in $D-D_1$. Also, by the Robin constant λ for (D, ξ) we mean $\lambda=\lambda_1$.

Take a ball V: $\|x-\xi\|<r$ such that $V\subset\subset D_1$. If we integrate both sides of (1.1) over the sphere ∂V: $\|x-\xi\|=r$, then

$$\lambda=-\frac{1}{r^{m-2}}+\frac{1}{r^{m-1}\omega_m}\int_{\partial V}g(x)\,dS_x, \tag{1.3}$$

where ω_m denotes the surface area of the $(m-1)$-dimensional unit sphere in R^m, and dS_x is the surface area element of ∂V at x. Therefore λ is determined by the values of $g(x)$ on a sphere centered at the pole ξ.

EXAMPLE 1.1. Let V_r be the ball $\|x\|<r$ in R^m and let $\xi\in V_r$. Then the Green function $g_r(\xi,x)$ for (V_r, ξ) is given by

$$g_r(\xi, x) = \frac{1}{\|x-\xi\|^{m-2}} - \left(\frac{r}{\|\xi\|}\right)^{m-2} \frac{1}{\|x-\xi^*\|^{m-2}},$$

where $\xi^* = (r/\|\xi\|)^2 \xi$ is the symmetric point of ξ with respect to the sphere ∂V_r: $\|x\| = r$. The Robin constant $\lambda_r(\xi)$ for (V_r, ξ) is thus

$$\lambda_r(\xi) = -\left(\frac{r}{r^2 - \|\xi\|^2}\right)^{m-2} < 0.$$

We state the properties of the Green function which we shall need.

PRELIMINARY 1.1. Let D be a domain over R^m with smooth boundary ∂D. For $\xi \in D$ we denote by g $(= g(\xi, x))$ the Green function for (D, ξ). Then g can be extended beyond ∂D to be a function of class C^3 in a domain $\hat{D} - \{\xi\}$ with $\hat{D} \supset\supset D$ such that, for every $x \in \partial D$,

$$\text{(1.4)} \qquad \frac{\partial g}{\partial n_x}(x) = -\left[\left(\frac{\partial g}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial g}{\partial x_m}\right)^2\right]^{1/2} < 0,$$

where n_x denotes the unit outer normal vector to ∂D at the point x, and such that

$$\text{(1.5)} \qquad D = \{x \in \hat{D} \mid g(x) > 0\} \quad \text{and} \quad \partial D = \{x \in \hat{D} \mid g(x) = 0\}.$$

PRELIMINARY 1.2. Let D be a domain over R^m with smooth boundary ∂D. Then any harmonic function $u(x)$ in D which is continuous on $D \cup \partial D$ can be written in the form

$$\text{(1.6)} \qquad u(\xi) = \frac{-1}{(m-2)\omega_m} \int_{\partial D} u(x) \frac{\partial g(\xi, x)}{\partial n_x} \, dS_x$$

for $\xi \in D$. For the special case where D is a ball V_r: $\|x-a\| < r$, we have Poisson's formula

$$\text{(1.7)} \qquad u(\xi) = \frac{1}{r\omega_m} \int_{\partial V_r} u(x) \frac{r^2 - \|\xi - a\|^2}{\|x-\xi\|^{m-2}} \, dS_x.$$

2. Main Theorem

Let C^n be a complex n-dimensional Euclidean space whose points are given by n complex coordinates $z = (z_1, \ldots, z_n)$ with norm $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$. If we put $z_\alpha = x_{2\alpha-1} + i x_{2\alpha}$ $(1 \le \alpha \le n)$ where $x_{2\alpha-1}, x_{2\alpha}$ are real and $i^2 = -1$, and if we put $x = (x_1, x_2, \ldots, x_{2n})$, then C^n is equal to the space R^{2n} of $2n$ real coordinates x such that $\|x\| = \|z\|$. We shall use the following.

NOTATION 2.1. For a complex-valued function $u(z)$ we set

$$\frac{\partial u}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial u}{\partial x_{2\alpha-1}} - i \frac{\partial u}{\partial x_{2\alpha}}\right); \qquad \frac{\partial u}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial u}{\partial x_{2\alpha-1}} + i \frac{\partial u}{\partial x_{2\alpha}}\right);$$

$$\operatorname{Grad}_{(z)} u = \left(\frac{\partial u}{\partial z_1}, \dots, \frac{\partial u}{\partial z_n}\right); \qquad \overline{\operatorname{Grad}}_{(z)} u = \left(\frac{\partial u}{\partial \bar{z}_1}, \dots, \frac{\partial u}{\partial \bar{z}_n}\right);$$

$$\operatorname{Grad}_{(x)} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_{2n}}\right);$$

$$\Delta_{(z)} u = \sum_{\alpha=1}^{n} \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\alpha}; \qquad \Delta_{(x)} u = \sum_{i=1}^{2n} \frac{\partial^2 u}{\partial x_i^2}.$$

We note that

$$\overline{\operatorname{Grad}}_{(z)} u = \tfrac{1}{2} \operatorname{Grad}_{(x)} u \quad \text{and} \quad \Delta_{(z)} u = \tfrac{1}{4} \Delta_{(x)} u.$$

Throughout this paper we assume $n \geq 2$. Let B be a region of the complex t plane and consider the product space $B \times C^n$ of B and C^n. We denote by p_B and by p_n the canonical projections of $B \times C^n$ to B and to C^n. Let $\mathbf{D}$ be a domain over $B \times C^n$, that is, an unramified covering domain over $B \times C^n$ with projection p. Suppose $p_B \circ p(\mathbf{D}) = B$. Given $t \in B$, we put $D(t) = (p_B \circ p)^{-1}(t)$, which is called the *fiber* of $\mathbf{D}$ at t. The fiber $D(t)$ becomes an open set over C^n defined by the triple $(D(t), C^n, p_n \circ p)$. We regard the domain $\mathbf{D}$ over $B \times C^n$ as a variation of open sets $D(t)$ over C^n with complex parameter $t \in B$, and write

$$\mathbf{D}: t \to D(t) \quad (t \in B).$$

Let $\zeta: t \to \zeta(t)$ $(t \in B)$ be a holomorphic section of $\mathbf{D}$ defined on B; that is, ζ is a holomorphic mapping of B into $\mathbf{D}$ such that $p_B \circ p \circ \zeta$ is the identity mapping on B. Since $\zeta(t) \in D(t)$, the open set $D(t)$ (over R^{2n}) carries the Green function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta(t))$. Therefore $g(t, z)$ may be expressed in a neighborhood of $\zeta(t)$ in the form

$$g(t, z) = \frac{1}{\|z - \zeta(t)\|^{2n-2}} + \lambda(t) + h(t, z),$$

where $h(t, z)$ is harmonic with respect to z and $h(t, \zeta(t)) = 0$. Thus the Robin constant $\lambda(t)$ defines a nonpositive real-valued function on B. Under these circumstances our main theorem may be stated as follows.

MAIN THEOREM. *If* $\mathbf{D}$ *is a pseudoconvex domain over* $B \times C^n$, *then* $\log(-\lambda(t))$ *is a subharmonic function on* B.

Since the pseudoconvexity of $\mathbf{D}$ induces that of each fiber $D(t)$, our variation $\mathbf{D}: t \to D(t)$ $(t \in B)$ is necessarily a variation of pseudoconvex open sets over C^n with parameter t. We shall prove the Main Theorem in Sections 3, 4, and 5. We now show a simple example suggesting why it should be true.

EXAMPLE 2.1. Let B be a region of C and let correspond to each $t \in B$ a ball $D(t)$: $\|z\| < r(t)$ in C^n. Consider a holomorphic mapping $z = \zeta(t)$ of B into C^n such that $\zeta(t) \in D(t)$ for $t \in B$. We put $\mathbf{D} = \{(t, z) \in B \times C^n \mid t \in B$ and

$z\in D(t)\}$. Suppose that **D** is a pseudoconvex domain in $B\times C^n$, or equivalently that $\log r(t)$ is a superharmonic function on B. By Example 1.1, the Robin constant $\lambda(t)$ for $(D(t), \zeta(t))$ can be written in the form

$$\lambda(t)=-\left(\frac{r(t)}{r(t)^2-\|\zeta(t)\|^2}\right)^{2n-2}<0$$

for $t\in B$. Hence

$$\log(-\lambda(t))=(2n-2)\left\{-\log r(t)+\sum_{k=0}^{\infty}\left(\frac{\|\zeta(t)\|}{r(t)}\right)^{2k}\right\}.$$

Since $\log(\|\zeta(t)\|/r(t))^{2k}$ is subharmonic for t in B, so is $(\|\zeta(t)\|/r(t))^{2k}$. It follows that $\log(-\lambda(t))$ is a subharmonic function on B.

3. Inequalities

In order to prove the Main Theorem we begin with some inequalities. Let B be a region in C and let **D** be a domain over $B\times C^n$ with projection p which admits a holomorphic section $\zeta: t\to\zeta(t)$ of **D** defined on B. In this section we impose the following restrictions on **D** and ζ.

CONDITION 3.1. The holomorphic section ζ is constant; that is, the mapping $p_n\circ p\circ\zeta(t)$ of t $(\in B)$ is a constant $(\in C^n)$. We thus write $\zeta(t)=\zeta$ for all $t\in B$.

CONDITION 3.2. There exist another domain $\tilde{\mathbf{D}}$ over $B\times C^n$ and a real-valued function ψ of class C^∞ in $\tilde{\mathbf{D}}$ which satisfy:

(a) $\tilde{\mathbf{D}}\supset\mathbf{D}$ and $\tilde{D}(t)\supset\supset D(t)$ for each $t\in B$ (we denote by $\partial\mathbf{D}$ and $\partial D(t)$ the boundary of **D** in $\tilde{\mathbf{D}}$ and of $D(t)$ in $\tilde{D}(t)$, respectively);

(b) $\mathbf{D}=\{(t,z)\in\tilde{\mathbf{D}}\mid\psi(t,z)<0\}$ and $\partial\mathbf{D}=\{(t,z)\in\tilde{\mathbf{D}}\mid\psi(t,z)=0\}$;

(c) $\left(\frac{\partial\psi}{\partial t},\frac{\partial\psi}{\partial z_1},\dots,\frac{\partial\psi}{\partial z_n}\right)\neq 0$

for any $(t,z)\in\partial\mathbf{D}$.

CONDITION 3.3. For each $t\in B$, the fiber $D(t)$ is connected, and the double $(\tilde{D}(t),\psi(t,z))$ defines the domain $D(t)$: namely, $\mathrm{Grad}_{(z)}\,\psi(t,z)\neq 0$ for any $z\in\partial D(t)$.

We denote by $g(t,z)$ and $\lambda(t)$ the Green function and the Robin constant for $(D(t),\zeta)$, where ζ is the point mentioned in Condition 3.1. Hence, in a neighborhood of ζ, $g(t,z)$ can be written in the form

$$g(t,z)=\frac{1}{\|z-\zeta\|^{2n-2}}+\lambda(t)+h(t,z), \tag{3.1}$$

where $h(t,z)$ is harmonic with respect to z and $h(t,\zeta)=0$.

DEFINITION 3.1. Let $\mathbf{D}$ be a domain over $B\times C^n$. If there exists a double $(\tilde{\mathbf{D}},\psi)$ which satisfies (a), (b), and (c) of Condition 3.2, then we say that $\mathbf{D}$ has *smooth* boundary $\partial\mathbf{D}$ over $B\times C^n$, and that the double $(\tilde{\mathbf{D}},\psi)$ *defines* the domain $\mathbf{D}$.

NOTATION 3.1. For each subset $\mathbf{K}$ of $\tilde{\mathbf{D}}$, each open set B_o of B, and each point t of B, we put

$$\mathbf{K}_{B_o}=\mathbf{K}\cap(p_B\circ p)^{-1}(B_o) \quad\text{and}\quad K(t)=\mathbf{K}\cap(p_B\circ p)^{-1}(t). \tag{3.2}$$

In particular, $\mathbf{D}_{B_o}$ becomes an open set over $B_o\times C^n$ such that $D_{B_o}(t)=D(t)$ for $t\in B_o$.

Conditions 3.2 and 3.3 imply that $(\partial\mathbf{D})(t)=\partial D(t)$ for $t\in B$. For every $\epsilon>0$, we put $\mathbf{D}_\epsilon=\{(t,z)\in\tilde{\mathbf{D}}\mid\psi(t,z)<\epsilon\}$. Then (a) implies that $\mathbf{D}\subset\mathbf{D}_\epsilon\subset\tilde{\mathbf{D}}$ and $D(t)\subset\subset D_\epsilon(t)\subset\tilde{D}(t)$ for $t\in B$. Now let $t_o\in B$. In view of Conditions 3.2 and 3.3 we can find a sufficiently small $\epsilon>0$ and a disk $B_o\colon|t-t_o|<\rho$ in B with the following properties: $(D_\epsilon)_{B_o}(t)\subset\subset\tilde{D}(t_o)$ for each $t\in B_o$, and there exists a C^∞-diffeomorphism of $(\mathbf{D}_\epsilon)_{B_o}$ onto the product $B_o\times D_\epsilon(t_o)$ of the form

$$T\colon(t,z)\to(t,\varphi(t,z))\quad\text{with } \varphi(t,D(t))=D(t_o)\text{ for } t\in B_o.$$

In other words, the variation

$$\mathbf{D}_{B_o}\cup\partial\mathbf{D}_{B_o}\colon t\to D(t)\cup\partial D(t)\quad(t\in B_o)$$

is diffeomorphically equivalent to the trivial one: $t\to D(t_o)\cup\partial D(t_o)$ $(t\in B_o)$. In these circumstances it is clear that $g(t,z)$ is of class C^3 with respect to (t,z) in $\mathbf{D}_{B_o}-\zeta(B_o)$ and can be extended beyond $\partial\mathbf{D}_{B_o}$ in $\tilde{\mathbf{D}}_{B_o}$. We summarize these in the next proposition.

PROPOSITION 3.1. *Suppose that the triple $(\mathbf{D},B\times C^n,p)$ satisfies Conditions 3.1, 3.2, and 3.3. Then, given $t_o\in B$, we can find a disk B_o of center t_o in B with the following property: there exist a domain $\hat{\mathbf{D}}$ over $B_o\times C^n$ and a function $\hat{g}(t,z)$ defined in $\hat{\mathbf{D}}$ which satisfy*

(1) $\mathbf{D}_{B_o}\subset\hat{\mathbf{D}}\subset\tilde{\mathbf{D}}_{B_o}$ *and* $D(t)\subset\subset\hat{D}(t)$ *for each* $t\in B_o$;
(2) $\hat{g}(t,z)$ *is of class* C^3 *with respect to* (t,z) *in* $\hat{\mathbf{D}}$ *except at the pole* $\zeta(B_o)$, *and* $\hat{g}(t,z)=g(t,z)$ *in* $\mathbf{D}_{B_o}$;
(3) *for each* $t\in B_o$, $\mathrm{Grad}_{(z)}\,\hat{g}(t,z)\neq0$ *for all* $z\in\partial D(t)$; *and*
(4) $\mathbf{D}_{B_o}=\{(t,z)\in\hat{\mathbf{D}}\mid\hat{g}(t,z)>0\}$ *and* $\partial\mathbf{D}_{B_o}=\{(t,z)\in\hat{\mathbf{D}}\mid\hat{g}(t,z)=0\}$.

We will write $g(t,z)$ for $\hat{g}(t,z)$ in $\hat{\mathbf{D}}$. Under the same conditions as in Proposition 3.1, take a point (t,z) of $\partial\mathbf{D}$ and let $n_{t,z}$ be the unit outer normal vector to the real $(2n+1)$-dimensional surface $\partial\mathbf{D}$ at the point (t,z). By Condition 3.2(c) on ψ and Proposition 3.1(3), we have

$$n_{t,z}=\left(\frac{\partial u}{\partial\bar{t}},\frac{\partial u}{\partial\bar{z}_1},\dots,\frac{\partial u}{\partial\bar{z}_n}\right)\Big/\left[\left|\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial u}{\partial z_1}\right|^2+\cdots+\left|\frac{\partial u}{\partial z_n}\right|^2\right]^{1/2}$$

at (t,z), where $u=\psi$ or $u=-g$. It follows that

$$(3.3)\qquad \frac{\partial g}{\partial t}\Big/\frac{\partial \psi}{\partial t}=\frac{\partial g}{\partial z_1}\Big/\frac{\partial \psi}{\partial z_1}=\cdots=\frac{\partial g}{\partial z_n}\Big/\frac{\partial \psi}{\partial z_n}=-\frac{\|\mathrm{Grad}_{(z)}\, g\|}{\|\mathrm{Grad}_{(z)}\, \psi\|}<0$$

for all $(t,z)\in\partial\mathbf{D}$. In the case where $(\partial g/\partial t)(t,z)$ or $(\partial g/\partial z_\alpha)(t,z)$ is 0, then $(\partial\psi/\partial t)(t,z)$ or $(\partial\psi/\partial z_\alpha)(t,z)$ is 0, and we omit this ratio in (3.3). Proposition 3.1(2) together with (1.3) implies

$$(3.4)\qquad \lambda(t) \text{ is of class } C^3 \text{ on } B.$$

We can thus differentiate each side of (3.1) with respect to t and $\bar t$, obtaining

$$\frac{\partial g}{\partial t}(t,z)=\frac{\partial\lambda}{\partial t}(t)+\frac{\partial h}{\partial t}(t,z)\quad\text{and}\quad \frac{\partial^2 g}{\partial t\partial\bar t}(t,z)=\frac{\partial^2\lambda}{\partial t\partial\bar t}(t)+\frac{\partial^2 h}{\partial t\partial\bar t}(t,z)$$

for $z\neq\zeta$. Moreover, since $h(t,\zeta)=0$ for $t\in B$, we have

$$\frac{\partial h}{\partial t}(t,\zeta)=\frac{\partial^2 h}{\partial t\partial\bar t}(t,\zeta)=0.$$

If we put

$$\frac{\partial g}{\partial t}(t,\zeta)=\frac{\partial\lambda(t)}{\partial t}\quad\text{and}\quad\frac{\partial^2 g}{\partial t\partial\bar t}(t,\zeta)=\frac{\partial^2\lambda(t)}{\partial t\partial\bar t}$$

then, for each $t\in B$, we see that both $(\partial g/\partial t)(t,z)$ and $(\partial^2 g/\partial t\partial\bar t)(t,z)$ are harmonic functions of z in the whole domain $D(t)$ (although $g(t,z)$ has a singularity at ζ). Also, these functions are continuous for z in $D(t)\cup\partial D(t)$. Formula (1.6) applied to $u=\partial g/\partial t$ or $u=\partial^2 g/\partial t\partial\bar t$ in $D=D(t)$ thus yields the following.

PROPOSITION 3.2. *If the triple* $(\mathbf{D}, B\times C^n, p)$ *satisfies Conditions 3.1, 3.2, and 3.3, then*

$$\frac{\partial\lambda(t)}{\partial t}=\frac{-1}{2(n-1)\omega_{2n}}\int_{\partial D(t)}\frac{\partial g(t,z)}{\partial t}\,\frac{\partial g(t,z)}{\partial n_z}\,dS_z$$

and

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar t}=\frac{-1}{2(n-1)\omega_{2n}}\int_{\partial D(t)}\frac{\partial^2 g(t,z)}{\partial t\partial\bar t}\,\frac{\partial g(t,z)}{\partial n_z}\,dS_z$$

for all $t\in B$.

This proof seems somewhat artificial. In order to understand Proposition 3.2 better, we consider a more restrictive case where $\psi(t,z)$ of Condition 3.2 is real analytic with respect to (t,z) in $\tilde{\mathbf{D}}$, and give an intuitive proof of the proposition. Let $t_o\in B$. Since $\partial D(t)$ $(t\in B)$ is real analytic, it is known that $g(t,z)$ can be extended beyond $\partial D(t)$ to be a harmonic function of z in $\hat D(t)-\{\zeta\}$, where $\hat D(t)\supset\supset D(t)$. Moreover, in our case, we can find a disk $B_o: |t-t_o|<\rho$ in B such that $\hat D(t)\supset\supset D(t_o)$ for each $t\in B_o$. Given $t\in B_o$, set $u(t,z)=g(t,z)-g(t_o,z)$ for $z\in\hat D(t)\cap\hat D(t_o)$. Then, from (3.1), $u(t,z)$ is regular at ζ and assumes the value $\lambda(t)-\lambda(t_o)$ there. Consequently, $u(t,z)$

is a harmonic function for z in a neighborhood of $D(t_o)\cup\partial D(t_o)$ satisfying $u(t,\zeta)=\lambda(t)-\lambda(t_o)$. If we apply formula (1.6) to $u(t,z)$ in $D(t_o)$, we obtain

$$\lambda(t)-\lambda(t_o)=\frac{-1}{2(n-1)\omega_{2n}}\int_{\partial D(t_o)} g(t,z)\frac{\partial g(t_o,z)}{\partial n_z}\,dS_z$$

for all $t\in B_o$. This is known as *Hadamard's variation formula,* and was proved in his paper [5, p. 519]. We differentiate each side of this formula with respect to t and $\bar{t}$ and then put $t=t_o$ in order to prove Proposition 3.2.

We emphasize that, although the Main Theorem is true for any holomorphic section $\zeta: t\to\zeta(t)$ on B, the Condition 3.1 that the section ζ is constant is indispensable for Hadamard's variation formula and the formulas in Proposition 3.2.

PROPOSITION 3.3. *Suppose that the triple* $(\mathbf{D}, B\times C^n, p)$ *satisfies Conditions 3.1, 3.2, and 3.3. Then, for each* $t\in B$ *and* α $(1\le\alpha\le n)$,

$$\frac{\partial g}{\partial\bar{z}_\alpha}dS_z=\frac{i^n}{2^{n-1}}\|\mathrm{Grad}_{(z)}\,g\|dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_\alpha\wedge\widehat{d\bar{z}_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$$

for all $z\in\partial D(t)$, *where* $\hat{A}$ *denotes the absence of* A.

Proof. Let $t\in B$ and let $z\in\partial D(t)$. We consider the unit outer normal vector n_z to the real $(2n-1)$-dimensional surface $\partial D(t)$ at the point z. Let $\vec{x}_{2\alpha-1}$ denote a unit vector in the direction of the positive $x_{2\alpha-1}$ axis. Since $n_z=-\mathrm{Grad}_{(x)}\,g(t,z)/\|\mathrm{Grad}_{(x)}\,g(t,z)\|$, if $\langle\, ,\rangle$ denotes the usual Euclidean inner product on R^{2n} then it follows that, for z along $\partial D(t)$,

$$\begin{aligned}\text{the projection of } dS_z \text{ to the space } &(x_1,\ldots,\widehat{x_{2\alpha-1}},\ldots,x_{2n})\\ &=dx_1\wedge dx_2\wedge\cdots\wedge\widehat{dx_{2\alpha-1}}\wedge\cdots\wedge dx_{2n}\\ &=\langle\vec{x}_{2\alpha-1},n_z\rangle\,dS_z\\ &=\left(-\frac{\partial g}{\partial x_{2\alpha-1}}\Big/\|\mathrm{Grad}_{(x)}\,g\|\right)dS_z.\end{aligned}$$

We thus have

$$\frac{\partial g}{\partial x_{2\alpha-1}}dS_z=-\|\mathrm{Grad}_{(x)}\,g\|dx_1\wedge dx_2\wedge\cdots\wedge\widehat{dx_{2\alpha-1}}\wedge\cdots\wedge dx_{2n}.$$

By observing the orientation of $\partial D(t)$, we similarly obtain

$$\frac{\partial g}{\partial x_{2\alpha}}dS_z=-\|\mathrm{Grad}_{(x)}\,g\|dx_1\wedge dx_2\wedge\cdots\wedge\widehat{dx_{2\alpha}}\wedge\cdots\wedge dx_{2n}.$$

In terms of Notation 2.1, they can be written in the compact form stated in Proposition 3.3. □

In addition to Conditions 3.1, 3.2, and 3.3, we impose on $(\mathbf{D}, B\times C^n, p)$ the following function-theoretic restriction.

CONDITION 3.4. The domain $\mathbf{D}$ is pseudoconvex over $B\times C^n$.

The pseudoconvexity of $\mathbf{D}$ induces the following.

PROPOSITION 3.4. *Suppose that the triple* $(\mathbf{D}, B\times C^n, p)$ *satisfies Conditions 3.1–3.4. Then*

$$\frac{\partial^2 g}{\partial t\partial\bar t}\le 2\,\mathrm{Re}\left\{\frac{\partial g}{\partial t}\sum_{\alpha=1}^{n}\frac{\partial^2 g}{\partial\bar t\partial z_\alpha}\frac{\partial g}{\partial\bar z_\alpha}\right\}\Big/\|\mathrm{Grad}_{(z)}\,g\|^2 \tag{3.5}$$

for all $(t,z)\in\partial\mathbf{D}$.

Proof. Let t_o be any point of B. We find a disk $B_o\subset\subset B$ with center t_o such that the pair $(\mathbf{D}_{B_o}, -g)$ satisfies properties (1)–(4) in Proposition 3.1. Since the domain $\mathbf{D}_{B_o}$ is pseudoconvex over $B\times C^n$, $-g(t,z)$ must satisfy the following condition, which is known as *Levi's condition* (see [2, p. 54]): For each $(t,z)\in\partial\mathbf{D}_{B_o}$,

$$\sum_{\alpha,\beta=0}^{n}\frac{\partial^2(-g)}{\partial z_\alpha\partial\bar z_\beta}(t,z)\zeta_\alpha\bar\zeta_\beta\ge 0$$

for all $(\zeta_0,\zeta_1,\dots,\zeta_n)\in C^{n+1}$ such that

$$\sum_{\alpha=0}^{n}\frac{\partial g}{\partial z_\alpha}(t,z)\zeta_\alpha=0,$$

where z_0 represents the variable t in B.

Now, given β $(1\le\beta\le n)$, we choose $\zeta_\alpha=0$ $(1\le\alpha\le n,\ \alpha\ne\beta)$. Then Levi's condition implies that, for $(t,z)\in\partial\mathbf{D}_{B_o}$,

$$\frac{\partial^2 g}{\partial t\partial\bar t}|\zeta_0|^2+2\,\mathrm{Re}\left\{\frac{\partial^2 g}{\partial\bar t\partial z_\beta}\bar\zeta_0\zeta_\beta\right\}+\frac{\partial^2 g}{\partial z_\beta\partial\bar z_\beta}|\zeta_\beta|^2\le 0$$

for $(\zeta_0,\zeta_\beta)\in C^2$ satisfying $(\partial g/\partial t)\zeta_o+(\partial g/\partial z_\beta)\zeta_\beta=0$. Elimination of ζ_o and ζ_β leads us to the well-known formula

$$\frac{\partial^2 g}{\partial t\partial\bar t}\left|\frac{\partial g}{\partial z_\beta}\right|^2-2\,\mathrm{Re}\left\{\frac{\partial^2 g}{\partial\bar t\partial z_\beta}\frac{\partial g}{\partial t}\frac{\partial g}{\partial\bar z_\beta}\right\}+\frac{\partial^2 g}{\partial z_\beta\partial\bar z_\beta}\left|\frac{\partial g}{\partial t}\right|^2\le 0 \tag{3.6}$$

for $(t,z)\in\partial\mathbf{D}_{B_o}$. Summation of each side for $\beta=1,\dots,n$ gives a symmetric inequality

$$\frac{\partial^2 g}{\partial t\partial\bar t}\|\mathrm{Grad}_{(z)}\,g\|^2-2\,\mathrm{Re}\left\{\frac{\partial g}{\partial t}\sum_{\beta=1}^{n}\frac{\partial^2 g}{\partial\bar t\partial z_\beta}\frac{\partial g}{\partial\bar z_\beta}\right\}+\left|\frac{\partial g}{\partial t}\right|^2\Delta_{(z)}g\le 0 \tag{3.7}$$

for $(t,z)\in\partial\mathbf{D}_{B_o}$. On the other hand, $g(t,z)$ is of class C^3 near $\partial\mathbf{D}_{B_o}$ in $\hat{\mathbf{D}}$ and is harmonic with respect to z in $\mathbf{D}_{B_o}-\zeta(B_o)$. Therefore $\Delta_{(z)}g=0$ on $\partial\mathbf{D}_{B_o}$. It follows from (1.4) that inequality (3.7) can be written in the form

$$\frac{\partial^2 g}{\partial t\partial\bar t}\le 2\,\mathrm{Re}\left\{\frac{\partial g}{\partial t}\sum_{\alpha=1}^{n}\frac{\partial^2 g}{\partial\bar t\partial z_\alpha}\frac{\partial g}{\partial\bar z_\alpha}\right\}\Big/\|\mathrm{Grad}_{(z)}g\|^2$$

for $(t,z)\in\partial\mathbf{D}_{B_o}$. Proposition 3.4 is thus proved. □

We are now ready to prove the fundamental inequality which will be of frequent use in what follows.

LEMMA 3.1. *Suppose that the triple* $(\mathbf{D}, B\times C^n, p)$ *satisfies Conditions 3.1–3.4. Then*

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}\le\frac{-4}{(n-1)\omega_{2n}}\iint_{D(t)}\left(\sum_{\alpha=1}^{n}\left|\frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha}\right|^2\right)dV\le 0$$

for all $t\in B$, *where* $dV=dx_1\wedge dx_2\wedge\cdots\wedge dx_{2n-1}\wedge dx_{2n}$ *denotes the volume element of* R^{2n}.

Proof. Given $t\in B$, we have from (1.4) that

$$d\Omega_z=\frac{-1}{2(n-1)\omega_{2n}}\frac{\partial g(t,z)}{\partial n_z}dS_z>0$$

for $z\in\partial D(t)$. After multiplying both sides of inequality (3.5) by $d\Omega_z$, we integrate with respect to z over $\partial D(t)$:

$$\frac{-1}{2(n-1)\omega_{2n}}\int_{\partial D(t)}\frac{\partial^2 g}{\partial t\partial\bar{t}}\frac{\partial g}{\partial n_z}dS_z$$
$$\le\frac{-1}{(n-1)\omega_{2n}}\operatorname{Re}\left\{\sum_{\alpha=1}^{n}\int_{\partial D(t)}\left[\left(\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial\bar{t}\partial z_\alpha}\frac{\partial g}{\partial\bar{z}_\alpha}\frac{\partial g}{\partial n_z}\right)\Big/\|\mathrm{Grad}_{(z)}\,g\|^2\right]dS_z\right\}.$$

According to Proposition 3.2 the left-hand side is equal to $\partial^2\lambda(t)/\partial t\partial\bar{t}$. It follows from Proposition 3.3 that

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}$$
$$\le\frac{1}{(n-1)\omega_{2n}}\operatorname{Re}\left\{\sum_{\alpha=1}^{n}\frac{i^n}{2^{n-2}}\int_{\partial D(t)}\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial\bar{t}\partial z_\alpha}dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_\alpha\wedge\widehat{d\bar{z}_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n\right\}.$$

As already noted, $\partial g/\partial t$ is of class C^2 for z on $D(t)\cup\partial D(t)$. Hence, if we apply Green's formula to the integral I of the right-hand side, then

$$I=\sum_{\alpha=1}^{n}\int_{\partial D(t)}\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial\bar{t}\partial z_\alpha}dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_\alpha\wedge\widehat{d\bar{z}_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$$
$$=\sum_{\alpha=1}^{n}\iint_{D(t)}d\left(\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial\bar{t}\partial z_\alpha}dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_\alpha\wedge\widehat{d\bar{z}_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n\right)$$
$$=-\sum_{\alpha=1}^{n}\iint_{D(t)}\left(\left|\frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha}\right|^2+\frac{\partial g}{\partial t}\frac{\partial^3 g}{\partial\bar{t}\partial z_\alpha\partial\bar{z}_\alpha}\right)dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n.$$

Because $\partial g/\partial\bar{t}$ is harmonic for z in $D(t)$: $\sum_{\alpha=1}^{n}\partial^3 g/\partial\bar{t}\partial z_\alpha\partial\bar{z}_\alpha=0$, it follows from $dz_\alpha\wedge d\bar{z}_\alpha=-2i\,dx_{2\alpha-1}\wedge dx_{2\alpha}$ that

$$I=-(-2i)^n\iint_{D(t)}\left(\sum_{\alpha=1}^{n}\left|\frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha}\right|^2\right)dx_1\wedge dx_2\wedge\cdots\wedge dx_{2n}.$$

Consequently,

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}} \leq \frac{-4}{(n-1)\omega_{2n}} \iint_{D(t)} \left(\sum_{\alpha=1}^{n} \left| \frac{\partial^2 g}{\partial t \partial \bar{z}_\alpha} \right|^2 \right) dV \leq 0.$$

Lemma 3.1 is proved. □

For the proof of Lemma 3.1 we used the pseudoconvexity of **D**, but we did not need the plurisubharmonicity of $\psi(t,z)$ in the larger set $\tilde{\mathbf{D}}$. This fact will be important when we treat the extended case where $D(t)$ varies in a complex manifold **M** (see [18]).

4. Differentiability

Consider a triple $(\mathbf{D}, B\times C^n, p)$ which satisfies the following Conditions 4.1–4.4.

CONDITION 4.1. There exists a constant section $\zeta : t \to \zeta$ of **D** defined for $t \in B$.

Each fiber $D(t)$ then carries the Green function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$.

CONDITION 4.2. There exists a double $(\tilde{\mathbf{D}}, \psi)$ defining the domain **D** such that $\psi(t,z)$ is real analytic with respect to (t,z) in $\tilde{\mathbf{D}}$.

We consider the subset Γ of **D** defined by

$$\Gamma = \left\{ (t,z) \in \tilde{\mathbf{D}} \,|\, \psi = \frac{\partial\psi}{\partial z_1} = \cdots = \frac{\partial\psi}{\partial z_n} = 0 \text{ at } (t,z) \right\}$$

and the projection of Γ to B:

$$\gamma = p_B \circ p(\Gamma).$$

The set γ is thus determined by $2n+1$ real analytic equations in the real $(2n+2)$-dimensional space $\tilde{\mathbf{D}}$. Following (3.2), we use the notations $\Gamma_{B_o} = \Gamma \cap (p_B\circ p)^{-1}(B_o)$ and $\Gamma(t) = \Gamma \cap (p_B \circ p)^{-1}(t)$ for any open set $B_o \subset B$ and $t \in B$.

CONDITION 4.3.

(a) For each open set $B_o \subset\subset B$, the set Γ_{B_o} consists of a finite number of real one-dimensional nonsingular curves in $\tilde{\mathbf{D}}_{B_o}$, except perhaps for a finite number of singular points.

(b) The set $\Gamma(t)$ for $t \in \gamma$ consists of a finite number of points: $\Gamma(t) = \{z^{(1)}(t), \ldots, z^{(q)}(t)\}$, where q (≥ 1) may depend on $t \in \gamma$.

The set $\gamma \cap B_o$ then consists of a finite number of smooth curves in B_o, except perhaps for a finite number of singular points.

CONDITION 4.4. The function $\psi(t,z)$ of Condition 4.2 is plurisubharmonic with respect to (t,z) in $\tilde{\mathbf{D}}$.

Conditions 3.1 and 4.1 are identical. Condition 4.2 is stronger than Condition 3.2. Condition 4.3 is weaker than Condition 3.3, in which Γ was empty. Although Condition 4.2 and 4.3 yield $(\partial \mathbf{D})(t)=\partial D(t)$ for all $t\in B$, the variation $\mathbf{D}: t\to D(t)$ $(t\in B)$ is no longer locally diffeomorphically (at times, even topologically) equivalent to the trivial one. So neither $g(t,z)$ nor $\lambda(t)$ need be of class C^2 for every $(t,z)\in \mathbf{D}-\zeta(B)$ or for each $t\in B$. However, we are able to show the following differentiability of $\lambda(t)$.

LEMMA 4.1. *Suppose that the triple $(\mathbf{D}, B\times C^n, p)$ satisfies Conditions 4.1–4.4. Then $\lambda(t)$ is a function of class C^1 on B.*

We proved in [14, §4] the same differentiability lemma in the case of a domain $\mathbf{D}$ over $B\times C$, namely, in the case of a variation of Riemann surfaces. It was based on the fact that any harmonic function of one complex variable is locally the real part of a holomorphic function. In the present case where $n\geq 2$ this is no longer true, and we give a different proof of the lemma. It will be divided into several short steps.

Proof of Lemma 4.1. Throughout these steps we set

$$B^*=B-\gamma, \qquad B_o^*=B_o-\gamma,$$

and

$$\mathbf{D}_{B_o}=\mathbf{D}\cap(p_B\circ p)^{-1}(B_o)$$

for an open set B_o in B.

1st Step. (1) The function $g(t,z)$ is of class C^3 for (t,z) on

$$(\mathbf{D}_{B^*}-\zeta(B^*))\cup\partial\mathbf{D}_{B^*};$$

(2) The function $\lambda(t)$ is superharmonic and of class C^3 on B^*.

Indeed, fix $t_o\in B^*$ and take a disk B_o of center t_o such that $B_o\subset\subset B^*$. Then Condition 4.3 implies that the open set $\mathbf{D}_{B_o}$ over $B_o\times C^n$ consists of a finite number of domains $\mathbf{D}_j$ $(j=1,\dots,m)$ over $B_o\times C^n$ such that $(\mathbf{D}_j\cup\partial\mathbf{D}_j)\cap(\mathbf{D}_i\cup\partial\mathbf{D}_i)=\emptyset$ for $j\neq i$. One of them, say $\mathbf{D}_1$, has constant section ζ defined on B_o, where ζ is the section mentioned in Condition 4.1. By definition of the Green function for an open set, for each $t\in B_o$, $g(t,z)$ is the Green function for $(D_1(t),\zeta)$ in $D_1(t)$ and is defined to be 0 in $D_j(t)$ $(2\leq j\leq m)$. Also, $\lambda(t)$ means the Robin constant for $(D_1(t),\zeta)$. So it is clear that $g(t,z)$ is of class C^3 for (t,z) on $\bigcup_{j=2}^m(\mathbf{D}_j\cup\partial\mathbf{D}_j)$. The domain $\mathbf{D}_1$ over $B_o\times C^n$ with section ζ satisfies Conditions 3.1–3.4. It follows from Proposition 3.1 that $g(t,z)$ is of class C^3 for (t,z) on $\mathbf{D}_1\cup\partial\mathbf{D}_1-\zeta(B_o)$. By (3.4) and Lemma 3.1, $\lambda(t)$ is of class C^3 and superharmonic on B_o. The 1st step is thus proved.

2nd Step. Let B_o be a region of B such that $B_o\subset\subset B$. Let U_o: $\|z-\zeta\|<r_o$ be a ball with center at the pole ζ such that $B_o\times U_o\subset\subset\mathbf{D}$. Then there exists

a constant $c>0$ (depending on B_o and U_o) such that the following inequalities hold:

(4.1) $$g(t,z)\le -c\psi(t,z)$$

for all $z\in D(t)-U_o\cup\partial U_o$ and $t\in B_o$;

(4.2) $$\|\mathrm{Grad}_{(z)}\, g(t,z)\|\le c\|\mathrm{Grad}_{(z)}\,\psi(t,z)\|$$

for all $z\in\partial D(t)$ and $t\in B_o^*$.

In fact, if we put

$$m=\inf\{-\psi(t,z)\,|\,(t,z)\in B_o\times U_o\}\quad\text{and}\quad c=\frac{1}{mr_o^{2n-2}},$$

then $m>0$ and $c>0$. We shall verify that this constant c satisfies (4.1) and (4.2). To this end, let t be an arbitrary point in B_o. Inequality (1.2) implies from the maximum principle that

(4.3) $$0<g(t,z)<\frac{1}{r_o^{2n-2}}$$

for all $z\in D(t)-U_o$. Let Ω be any open set in $D(t)$ with smooth boundary $\partial\Omega$ and such that $U_o\subset\subset\Omega\subset\subset D(t)$. We denote by $g_\Omega(t,z)$ the Green function for (Ω,ζ). Construct the function

$$v_\Omega(t,z)=c\psi(t,z)+g_\Omega(t,z)$$

on $\Omega-U_o\cup\partial U_o$. Condition 4.4 implies that the restriction of ψ to $D(t)$ is plurisubharmonic for z in $D(t)$. Consequently, $v_\Omega(t,z)$ is subharmonic for z in $\Omega-U_o\cup\partial U_o$; that is, $\Delta_{(z)}v_\Omega(t,z)\ge 0$. Since g_Ω satisfies inequality (4.3) for $z\in\Omega-U_o$, $v_\Omega(t,z)<0$ on $\partial(\Omega-U_o)=\partial\Omega\cup\partial U_o$, and it follows that $v_\Omega(t,z)<0$ in $\Omega-U_o\cup\partial U_o$. Because $g_\Omega(t,z)\nearrow g(t,z)$ as $\Omega\to D(t)$, we have $c\psi(t,z)+g(t,z)\le 0$ on $D(t)-U_o\cup\partial U_o$. Hence the constant c satisfies (4.1).

Fix $t\in B_o^*$. Then $g(t,z)$ can be extended of class C^3 beyond $\partial D(t)$ in $\tilde{D}(t)$. Since $g(t,z)=\psi(t,z)=0$ on $\partial D(t)$ it follows from (4.1) that

$$\|\mathrm{Grad}_{(z)}\,g(t,z)\|=-\frac{1}{2}\frac{\partial g}{\partial n_z}(t,z)\le\frac{c}{2}\frac{\partial\psi(t,z)}{\partial n_z}=c\|\mathrm{Grad}_{(z)}\,\psi(t,z)\|$$

for all $z\in\partial D(t)$. Hence our c satisfies (4.2) and the 2nd step is proved.

Before proceeding to the following steps we shall state two preliminary results. Let $B_o: |t-t_o|<\rho_o$ be a disk in the complex t plane, and let G be an open set over C^n. Let $\psi_1(t,z)$ and $\psi_2(t,z)$ be real-valued, real analytic functions with respect to (t,z) in $B_o\times G$. We put

$$\mathbf{E}_j=\{(t,z)\in B_o\times G\,|\,\psi_j(t,z)<0\}\quad(j=1,2);$$
$$E_j(t)=\{z\in G\,|\,(t,z)\in\mathbf{E}_j\}\quad(t\in B_o);$$
$$\mathbf{E}=\mathbf{E}_1\cap\mathbf{E}_2\quad\text{and}\quad E(t)=E_1(t)\cap E_2(t).$$

We denote by $\partial\mathbf{E}$ or $\partial\mathbf{E}_j$ the boundary of $\mathbf{E}$ or $\mathbf{E}_j$ in $B_o\times G$. For $t\in B_o$, we denote by $\partial E(t)$ or $\partial E_j(t)$ the boundary of $E(t)$ or $E_j(t)$ in G. Assume that:

(a) for each $t \in B_o$, $\mathrm{Grad}_{(z)}\, \psi_j(t,z) \neq 0$ for all $z \in \partial E_j(t)$ $(j=1,2)$;
(b) $\partial E_j(t) \cap \partial E(t) \neq \emptyset$ $(j=1,2)$ for all $t \in B_o$;
(c) for each $t \in B_o$, $\partial E_1(t)$ and $\partial E_2(t)$ intersect transversally in G;
(d) $E(t) \subset\subset G$ for each $t \in B_o$.

For the sake of simplicity we say that such a set $\mathbf{E}$ is an open set *with corners* in $B_o \times G$. Also we say that the double $(B_o \times G, \{\psi_1, \psi_2\})$ *defines* the open set $\mathbf{E}$. In this case, each $E(t)$ $(t \in B_o)$ is bounded by a finite number of smooth surfaces such that whenever two of these surfaces intersect, they intersect transversally. Moreover, for any region $B_1 \subset\subset B_o$, the variation

$$\mathbf{E} \cup \partial \mathbf{E} \colon t \to E(t) \cup \partial E(t) \quad (t \in B_1)$$

is diffeomorphically equivalent to the trivial one $t \to E(t_o) \cup \partial E(t_o)$ $(t \in B_1)$, where t_o is a fixed point in B_1. Using this notation, we have the following.

PRELIMINARY 4.1. Let $\mathbf{E}$ be an open set with corners in $B_o \times G$. Assume that there exists a point $\zeta \in G$ such that $B_o \times \{\zeta\} \subset \mathbf{E}$. For $t \in B_o$, we denote by $g(t,z)$ the Green function for $(E(t), \zeta)$. Then $g(t,z)$ is continuous with respect to (t,z) in $\mathbf{E}$ except for the pole $B_o \times \{\zeta\}$.

PRELIMINARY 4.2. Let $\mathbf{E}$ be an open set with corners defined by the double $(B_o \times G, \{\psi_1, \psi_2\})$. Assume that $u(t,z)$ is a continuous function with respect to (t,z) in $\mathbf{E} \cup \partial \mathbf{E}$ such that, for any fixed $t \in B_o$, $u(t,z)$ is harmonic for z in $E(t)$ and vanishes on $\partial E(t) \cap \partial E_1(t)$. Then $(\partial u / \partial z_\alpha)(t,z)$ $(1 \le \alpha \le n)$ is continuous with respect to $(t,z) \in \mathbf{E} \cup \{\partial \mathbf{E} \cap (\partial \mathbf{E}_1 - \partial \mathbf{E}_2)\}$.

These can be proved without difficulty by following the concrete construction of Green function by means of the theory of Fredholm's integral equations. Let us return to the proof of Lemma 4.1.

3rd Step. (1) The function $g(t,z)$ is continuous with respect to (t,z) on $(\mathbf{D} - \zeta(B)) \cup \partial \mathbf{D}$ and vanishes on $\partial \mathbf{D}$; (2) The function $\lambda(t)$ is continuous for $t \in B$.

In fact, by the 1st step, it remains to prove the 3rd step for $t \in \gamma$. We assume $t_o \in \gamma$. By Condition 4.3(b), the boundary surfaces $\partial D(t_o)$ have the singular points $\Gamma(t_o) = \{z^{(1)}(t_o), \ldots, z^{(q)}(t_o)\}$. First, let $(t_o, z_o) \in \partial \mathbf{D}$. Take a disk B_o of center t_o and a ball U_o with center at the pole ζ such that $B_o \times U_o \subset\subset \mathbf{D}$. By the 2nd step we can find a constant $c > 0$ satisfying (4.1). Therefore, if $(t,z) \in \mathbf{D}$ tends to (t_o, z_o), then

$$0 \le \varlimsup_{(t,z) \to (t_o,z_o)} g(t,z) \le -c \varlimsup_{(t,z) \to (t_o,z_o)} \psi(t,z) = -c\psi(t_o,z_o) = 0.$$

This means that $g(t_o,z_o) = 0$ and $g(t,z)$ is continuous at (t_o,z_o). Next, let $(t_o,z_o) \in \mathbf{D} - \zeta(B)$. Fix balls $U_o \colon \|z - \zeta\| < r_o$ and $V_o \colon \|z - z_o\| < s_o$ such that $U_o \cup V_o \subset\subset D(t_o)$ and $(U_o \cup \partial U_o) \cap (V_o \cup \partial V_o) = \emptyset$. We also take an open set G of $\tilde{D}(t_o)$ such that $D(t_o) \subset\subset G \subset\subset \tilde{D}(t_o)$ and such that the boundary ∂G of G in $\tilde{D}(t_o)$ is smooth. We use the following notation: Given $\rho > 0$ and $\eta > 0$, we put

$$B_\rho=\{t\in B\,|\,|t-t_o|<\rho\}\quad\text{and}\quad W_\eta=\bigcup_{k=1}^{q} W_\eta^{(k)},$$

where $W_\eta^{(k)}=\{z\in\tilde{D}(t_o)\,|\,\|z-z^{(k)}(t_o)\|<\eta\}$ $(1\le k\le q)$.

By Condition 4.3(a) and (b), we can choose small numbers $\eta_1>0$ and $\rho_1>0$ such that:

(i) $W_{\eta_1}\subset\subset G$ and $(W_{\eta_1}\cup\partial W_{\eta_1})\cap(U_o\cup\partial U_o\cup V_o\cup\partial V_o)=\emptyset$;
(ii) $U_o\cup V_o\subset\subset D(t)\subset\subset G$ for each $t\in B_{\rho_1}$; and
(iii) given a number η such that $0<\eta<\eta_1$, we can find a number $\rho(\eta)$ such that $0<\rho(\eta)<\rho_1$ and such that the set
$\mathbf{E}=\mathbf{D}_{B_{\rho(\eta)}}-B_{\rho(\eta)}\times(W_\eta\cup\partial W_\eta)$
is an open set with corners in $B_{\rho(\eta)}\times G$.

To achieve (iii), it is sufficient to take, as a double defining $\mathbf{E}$, $(B_{\rho(\eta)}\times G, \{\psi_1,\psi_2\})$, where $\psi_1=\psi$ and $\psi_2=\prod_{k=1}^{q}\psi_{2,k}$ with

$$\psi_{2,k}(t,z)=\eta^2-\|z-z^{(k)}(t_o)\|^2.$$

Now fix η $(0<\eta<\eta_1)$. For each $t\in B_{\rho_1}$ we denote by $g_\eta(t,z)$ the Green function for $(D(t)-W_\eta\cup\partial W_\eta,\zeta)$. We construct the harmonic function $u_\eta(z)$ defined on $G-W_\eta\cup\partial W_\eta$ whose boundary values are

$$u_\eta(z)=\begin{cases}1/r_o^{2n-2} & \text{on } \partial W_\eta,\\ 0 & \text{on } \partial G.\end{cases}$$

It is clear that as $\eta\to 0$, $u_\eta(z)\searrow 0$ uniformly on any compact set in $G-\{z^{(k)}(t_o)\}_{k=1}^{q}$ and, in particular, on $V_o\cup\partial V_o$. On the other hand, from the maximum principle and (4.3), we have, for each $t\in B_{\rho_1}$,

$$0<g(t,z)-g_\eta(t,z)<u_\eta(z)$$

for all $z\in D(t)-W_\eta\cup\partial W_\eta$. It follows from (i) and (ii) that

$$|g(t,z)-g(t_o,z_o)|\le u_\eta(z)+u_\eta(z_o)+|g_\eta(t,z)-g_\eta(t_o,z_o)|$$

for $(t,z)\in B_{\rho_1}\times V_o$. Given $\epsilon>0$, we take a number η_o such that $0<\eta_o<\eta_1$ and such that $0<u_{\eta_o}(z)<\epsilon/3$ for $z\in V_o\cup\partial V_o$. Preliminary 4.1 together with (iii) implies that $g_{\eta_o}(t,z)$ is continuous with respect to (t,z) in $\mathbf{E}=\mathbf{D}_{B_{\rho(\eta_o)}}-B_{\rho(\eta_o)}\times(W_{\eta_o}\cup\partial W_{\eta_o})$. Because $\mathbf{E}\ni(t_o,z_o)$, we can find a neighborhood $\mathbf{V}$ of (t_o,z_o) in $\mathbf{E}$ such that $|g_{\eta_o}(t,z)-g_{\eta_o}(t_o,z_o)|<\epsilon/3$ for $(t,z)\in\mathbf{V}$. It follows that $|g(t,z)-g(t_o,z_o)|<\epsilon$ for $(t,z)\in\mathbf{V}$. Consequently, (1) of the 3rd step is proved. From expression (1.3), (2) of the 3rd step follows.

4th Step. The derivative $(\partial g/\partial z_\alpha)(t,z)$ $(1\le\alpha\le n)$ is continuous with respect to (t,z) in $(\mathbf{D}-\zeta(B))\cup(\partial\mathbf{D}-\Gamma)$. Precisely, $(\partial g/\partial z_\alpha)(t,z)$, which is certainly defined in $\mathbf{D}-\zeta(B)$, can be continuously extended to $\partial\mathbf{D}-\Gamma$.

Indeed, first suppose $(t_o,z_o)\in\mathbf{D}-\zeta(B)$. We take B_o: $|t-t_o|<\rho$ and V: $\|z-z_o\|<r$ such that $B_o\times V\subset\subset\mathbf{D}-\zeta(B)$. Then expression (1.7) combined with (1) of the 3rd step implies that $(\partial g/\partial z_\alpha)(t,z)$ is continuous for (t,z) in $B_o\times V$. Next suppose $(t_o,z_o)\in\partial\mathbf{D}-\Gamma$. Then we find B_o: $|t-t_o|<\rho$ and

V: $\|z-z_o\|<r$ such that $B_o\times V\subset\subset\tilde{\mathbf{D}}-\zeta(B)$ and $\mathrm{Grad}_{(z)}\,\psi(t,z)\neq 0$ for all $(t,z)\in\partial\mathbf{D}\cap(B_o\times V)$. Therefore $\mathbf{E}=\mathbf{D}\cap(B_o\times V)$ is an open set with corners. It follows from Preliminary 4.2 together with (1) of the 3rd step that $(\partial g/\partial z_\alpha)(t,z)$ is continuous on $(\mathbf{D}\cup\partial\mathbf{D})\cap(B_o\times V)$. Consequently, the 4th step is proved.

From (1) of the 1st step, the derivative $(\partial g/\partial t)(t,z)$ exists for any $(t,z)\in\mathbf{D}_{B^*}\cup\partial\mathbf{D}_{B^*}$. Thus we consider its restriction to the boundary $\partial\mathbf{D}_{B^*}$ and put

$$u(t,z)=\frac{\partial g}{\partial t}(t,z)\quad\text{for}\quad(t,z)\in\partial\mathbf{D}_{B^*}.$$

With this terminology we shall state the following.

5th Step. (1) The function $u(t,z)$ defined on $\partial\mathbf{D}_{B^*}$ can be uniquely extended to a continuous function $\hat{u}(t,z)$ on $\partial\mathbf{D}-\Gamma$; (2) Let B_o be a region such that $B_o\subset\subset B$; then there exists a constant $K>0$ (depending on B_o) such that

$$|\hat{u}(t,z)|\leq K\quad\text{and}\quad\left|\hat{u}(t,z)\cdot\frac{\partial g}{\partial n_z}(t,z)\right|\leq K \tag{4.4}$$

for all $(t,z)\in\partial\mathbf{D}_{B_o}-\Gamma$, where n_z denotes the unit outer normal vector to the $(2n-1)$-dimensional surface $\partial D(t)$ at the point z.

In fact, by (1) of the 1st step, $u(t,z)$ is continuous for $(t,z)\in\partial\mathbf{D}_{B^*}$. By Condition 4.3, $\partial\mathbf{D}_{B^*}$ is dense in $\partial\mathbf{D}-\Gamma$. Thus, to prove (1) of the 5th step it suffices to verify the following fact: Let $(t_o,z_o)\in\partial\mathbf{D}-\Gamma$ with $t_o\in\gamma$, and let $(t,z)\in\partial\mathbf{D}_{B^*}$ tend to (t_o,z_o). Then the limit of $u(t,z)$ exists. Indeed, because $\partial D(t_o)$ is nonsingular at z_o, we can find a neighborhood $B_o\times V_o$ of (t_o,z_o) in $\tilde{\mathbf{D}}$ where B_o: $|t-t_o|<\rho$ and V_o: $\|z-z_o\|<r$ such that $\mathrm{Grad}_{(z)}\,\psi(t,z)\neq 0$ for all $(t,z)\in B_o\times V_o$. On the other hand, equality (3.3) implies that

$$u(t,z)=\frac{\partial g}{\partial t}(t,z)=-\left(\frac{\partial\psi}{\partial t}(t,z)\right)\frac{\|\mathrm{Grad}_{(z)}\,g(t,z)\|}{\|\mathrm{Grad}_{(z)}\,\psi(t,z)\|} \tag{4.5}$$

for all $(t,z)\in\partial\mathbf{D}_{B^*}$. By the 4th step, $(\partial g/\partial z_\alpha)(t,z)$ $(1\leq\alpha\leq n)$ is a continuous function for (t,z) on $(\mathbf{D}-\zeta(B))\cup(\partial\mathbf{D}-\Gamma)$. By Condition 4.2, $(\partial\psi/\partial t)(t,z)/\|\mathrm{Grad}_{(z)}\,\psi(t,z)\|$ is continuous for (t,z) in $B_o\times V_o$. It follows that

$$\lim_{(t,z)\to(t_o,z_o)}u(t,z)=-\left(\frac{\partial\psi}{\partial t}(t_o,z_o)\right)\frac{\|\mathrm{Grad}_{(z)}\,g(t_o,z_o)\|}{\|\mathrm{Grad}_{(z)}\,\psi(t_o,z_o)\|},$$

where $(t,z)\in(\partial\mathbf{D}_{B^*})\cap(B_o\times V_o)$. This proves (1) of the 5th step.

For the proof of (2), let B_o be a region such that $B_o\subset\subset B$. By definition of $\hat{u}(t,z)$ for $t\in\gamma$ and by the 4th step, it suffices to prove the existence of a constant K such that

$$\left|\frac{\partial g}{\partial t}(t,z)\right|\leq K\quad\text{and}\quad\left|\frac{\partial g}{\partial t}(t,z)\cdot\frac{\partial g}{\partial n_z}(t,z)\right|\leq K \tag{4.4$'$}$$

for all $(t,z)\in\partial\mathbf{D}_{B_o^*}$. Since $B_o\subset\subset B$, we can find a ball U_o: $\|z-\zeta\|<r_o$ such that $B_o\times U_o\subset\subset\mathbf{D}$. By the 2nd step, we can choose a constant $c>0$ (depending on B_o and U_o) which satisfies

$$(4.6) \qquad \|\mathrm{Grad}_{(z)}\, g(t,z)\| \le c \|\mathrm{Grad}_{(z)}\, \psi(t,z)\|$$

for all $(t,z) \in \partial \mathbf{D}_{B_o^*}$. It follows from (4.5) that

$$\left|\frac{\partial g}{\partial t}\right| \le c \left|\frac{\partial \psi}{\partial t}\right| \quad \text{and} \quad \left|\frac{\partial g}{\partial t} \cdot \frac{\partial g}{\partial n_z}\right| \le c^2 \left|\frac{\partial \psi}{\partial t}\right| \cdot \|\mathrm{Grad}_{(z)}\, \psi\|$$

for all $(t,z) \in \partial \mathbf{D}_{B_o^*}$. Since $\psi(t,z)$ is real analytic for (t,z) in $\tilde{\mathbf{D}}$ and since $\mathbf{D}_{B_o} \subset\subset \tilde{\mathbf{D}}$, we can find a number $M>0$ such that

$$\left|\frac{\partial \psi}{\partial t}\right|, \left|\frac{\partial \psi}{\partial z_\alpha}\right| \le M \quad \text{for } (t,z) \in \mathbf{D}_{B_o} \cup \partial \mathbf{D}_{B_o} \text{ and } 1 \le \alpha \le n.$$

Consequently, if we put $K = \max\{cM, \sqrt{n}c^2M^2\}$, then K satisfies inequality (4.4′). Thus (2) is proved.

It must be noted that (1) of the 5th step implies neither the existence nor the continuity of $(\partial g/\partial t)(t,z)$ as a function with respect to (t,z) on $\mathbf{D} \cup (\partial \mathbf{D} - \Gamma)$ at $(t_o, z_o) \in \partial \mathbf{D} - \Gamma$ with $t_o \in \gamma$.

6th Step. The function $\lambda(t)$ is of class C^1 on B.

In fact, by (2) of the 1st step, $\lambda(t)$ is of class C^2 on B^*. It suffices for the 6th step to prove the following.

PROPERTY (A). Let t_o be any fixed point of γ. Then, given $\epsilon>0$, there exists a disk $B_o \subset\subset B$ of center t_o such that

$$\left|\frac{\partial \lambda}{\partial t}(t) - \frac{\partial \lambda}{\partial t}(t')\right| < \epsilon \quad \text{for all } t, t' \in B_o^* = B_o - \gamma.$$

For, assume that Property (A) is true for all $t_o \in \gamma$. Since $B - B^* = \gamma$ consists of real analytic curves, we see that $\partial\lambda/\partial t$ on B^* is uniquely extended to be a continuous function $\lambda_1(t)$ on all of B. Because $\lambda(t)$ is of class C^2 on B^*, Stokes' formula implies

$$\int_C (\lambda_1\, dt + \bar{\lambda}_1\, d\bar{t}) = 0$$

for any closed curve C in B such that the domain bounded by C is contained in B. Since, by (2) of the 3rd step, $\lambda(t)$ is continuous on B, it follows that

$$\lambda(t) = \lambda(\tau_o) + \int_{\tau_o}^{t} (\lambda_1\, dt + \bar{\lambda}_1\, d\bar{t}),$$

where τ_o is a fixed point in B^*. Hence $\partial\lambda/\partial t$ exists, even at $t_o \in \gamma$, and is equal to $\lambda_1(t_o)$. The 6th step is thus true.

Now, given $a \in C^n$ and $\eta>0$, we consider the ball $V(a,\eta)$: $\|z-a\| < \eta$ in C^n. Since $\psi(t,z)$ is real analytic with respect to (t,z) in $\tilde{\mathbf{D}}$ and since $\mathbf{D}_{B_o} \subset\subset \tilde{\mathbf{D}}$ for any $B_o \subset\subset B$, the following fact is clear: Let B_o be a region such that $B_o \subset\subset B$. Then, given $\delta>0$, there exists a number $\eta>0$ such that

$$(4.7) \qquad \int_{(\partial D(t) - \Gamma(t)) \cap V(a,\eta)} dS_z < \delta$$

for all $(t,a) \in B_o \times C^n$.

To show Property (A), let $t_o \in \gamma$. The surface $\partial D(t_o)$ then has the singular points $\Gamma(t_o) = \{z^{(k)}(t_o)\}_{k=1,\dots,q}$. We choose a disk $B_o \subset\subset B$ of center t_o and a ball $U_o \subset\subset D(t_o)$ with center at the pole ζ such that $B_o \times U_o \subset\subset \mathbf{D}$. From (2) of the 5th step, we can find a constant $K > 0$ (depending on B_o) which satisfies inequalities (4.4) for $(t, z) \in \partial \mathbf{D}_{B_o} - \Gamma$.

Let $\epsilon > 0$ be given. We put $\delta = 2(n-1)\omega_{2n}/(3qK)$. With this $\delta > 0$, we can find a number $\eta > 0$ for which (4.7) holds for all $(t, z) \in B_o \times C^n$. If we put

$$W_\eta = \bigcup_{k=1}^{q} W_\eta^{(k)} \quad \text{where } W_\eta^{(k)} = V(z^{(k)}(t_o), \eta) \quad (1 \le k \le q),$$

then $\partial D(t_o) - W_\eta \cup \partial W_\eta$ consists of only nonsingular points. Take an open set G with $D(t_o) - W_\eta \cup \partial W_\eta \subset\subset G \subset\subset \tilde{D}(t_o)$. As already noted in the 3rd step, we can find a disk $B_1 : |t - t_o| < \rho(\eta)$ in B_o such that

$$\mathbf{E} = \mathbf{D}_{B_1} - B_1 \times (W_\eta \cup \partial W_\eta)$$

is an open set with corners in $B_1 \times G$. Thus, the surfaces $\partial D(t) - W_\eta \cup \partial W_\eta$, together with their unit outer normal vectors, approach those of $\partial D(t_o) - W_\eta \cup \partial W_\eta$ in a continuous way as $t \in B_1$ tends to t_o. Also, by (1) of the 5th step, $\hat{u}(t, z)$ is uniformly continuous on $\bigcup_{t \in B_1}(t, \partial D(t) - W_\eta \cup \partial W_\eta)$, a relatively compact subset of $\partial \mathbf{D} - \Gamma$. It follows from the 4th step that

$$\lim_{\substack{t \to t_o \\ t \in B_1}} \int_{\partial D(t) - W_\eta \cup \partial W_\eta} \hat{u}(t, z) \cdot \frac{\partial g}{\partial n_z}(t, z)\, dS_z = \int_{\partial D(t_o) - W_\eta \cup \partial W_\eta} \hat{u}(t_o, z) \cdot \frac{\partial g}{\partial n_z}(t_o, z)\, dS_z.$$

Hence there exists a small disk $B_2 : |t - t_o| < \rho_2$ in B_1 such that

$$\left| \int_{\partial D(t) - W_\eta \cup \partial W_\eta} \hat{u}(t, z) \cdot \frac{\partial g}{\partial n_z}(t, z)\, dS_z - \int_{\partial D(t') - W_\eta \cup \partial W_\eta} \hat{u}(t', z) \cdot \frac{\partial g}{\partial n_z}(t', z)\, dS_z \right| \tag{4.8}$$
$$< \frac{2(n-1)\omega_{2n}\epsilon}{3}$$

for $t, t' \in B_2$. Since $B_2 \subset B_1 \subset B_o \subset\subset B$, the inequalities (4.4) for $(t, z) \in \partial \mathbf{D}_{B_2}$, (4.7) for $(t, a) \in B_2 \times C^n$, and (4.8) for $t, t' \in B_2$ remain valid. Therefore, in view of Proposition 3.2 we see that, for every $t, t' \in B_2^*$,

$$\left| \frac{\partial \lambda}{\partial t}(t) - \frac{\partial \lambda}{\partial t}(t') \right|$$
$$= \left| \frac{-1}{2(n-1)\omega_{2n}} \left\{ \int_{\partial D(t)} \left(\frac{\partial g}{\partial t} \cdot \frac{\partial g}{\partial n_z} \right)(t, z)\, dS_z - \int_{\partial D(t')} \left(\frac{\partial g}{\partial t} \cdot \frac{\partial g}{\partial n_z} \right)(t', z)\, dS_z \right\} \right|$$
$$\le \frac{1}{2(n-1)\omega_{2n}} \left\{ \left| \int_{\partial D(t) - W_\eta \cup \partial W_\eta} \left(u \cdot \frac{\partial g}{\partial n_z} \right)(t, z)\, dS_z \right.\right.$$
$$\left. - \int_{\partial D(t') - W_\eta \cup \partial W_\eta} \left(u \cdot \frac{\partial g}{\partial n_z} \right)(t', z)\, dS_z \right| +$$

$$+\sum_{k=1}^{g}\left(\int_{\partial D(t)\cap W_\eta^{(k)}} K\,dS_z+\int_{\partial D(t')\cap W_\eta^{(k)}} K\,dS_z\right)\Bigg\}$$
$$\leq \frac{1}{2(n-1)\omega_{2n}}\left\{\frac{2(n-1)\omega_{2n}\epsilon}{3}+2Kq\delta\right\}=\epsilon.$$

We thus have Property (A). Lemma 4.1 is completely proved. □

REMARK 4.1. By means of the above proof we conclude that, at each $t_o\in\gamma$, the improper integral

$$\int_{\partial D(t_o)-\Gamma(t_o)} \hat{u}(t_o,z)\cdot\frac{\partial g}{\partial n_z}(t_o,z)\,dS_z$$

exists and is equal to $-2(n-1)\omega_{2n}(\partial\lambda/\partial t)(t_o)$.

COROLLARY 4.1. *Under the same conditions as in Lemma 4.1, the function $\lambda(t)$ is superharmonic and of class C^1 on B.*

Proof. By (1) of the 1st step, $\lambda(t)$ is superharmonic and of class C^2 on $B^*=B-\gamma$. Lemma 4.1 implies that $\lambda(t)$ is of class C^1 on all of B. Since the set of singular points of γ is locally finite in B, it thus suffices to prove, at each nonsingular point t_o of γ, the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\lambda(t_o+re^{i\theta})\leq\lambda(t_o)$$

for every sufficiently small $r>0$. Let B_r: $|t-t_o|<r$ in B. If r is small enough, then $B_r\subset\subset B$ and B_r is divided by γ into two regions (B_r', B_r''). Since $\lambda(t)$ is of class C^1 on $B_r\cup\partial B_r$, it follows from Green's formula that

$$r\frac{\partial}{\partial r}\left\{\int_0^{2\pi}\lambda(t_o+re^{i\theta})\,d\theta\right\}=\int_{\partial B_r}\frac{\partial\lambda}{\partial n_t}\,ds_t=\int_{\partial B_r'\cup\partial B_r''}\frac{\partial\lambda}{\partial n_t}\,ds_t$$
$$=i\iint_{B_r'\cup B_r''}\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(t)\,dt\,d\bar{t}\leq 0.$$

Consequently, if we put $L(r)=\int_0^{2\pi}\lambda(t_o+re^{i\theta})\,d\theta$, then $L(r)$ is a decreasing function of r for $r>0$ sufficiently close to 0. Because $L(r)\to 2\pi\lambda(t_o)$ as $r\to 0$, we obtain the desired inequality. Corollary 4.1 is proved. □

5. Hartog's Transformations

In Sections 3 and 4 we treated variations of an open set D with smooth boundary over C^n. In this section we study variations of D without smooth boundary. Let $(\mathbf{D}, B\times C^n, p)$ be a triple satisfying Condition 4.1. For each $t\in B$, we consider the Green function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t),\zeta)$, where ζ is the point mentioned in Condition 4.1. Assume that $\mathbf{D}$ is a pseudoconvex domain over $B\times C^n$ (which may be infinitely many sheeted). Following Oka [8, p. 143], we construct a sequence of subdomains $\{\mathbf{D}_p\}$ of $\mathbf{D}$ such that $\mathbf{D}_1\subset\subset\mathbf{D}_2\subset\subset\cdots$, $\bigcup_{p=1}^{\infty}\mathbf{D}_p=\mathbf{D}$, and such that $\mathbf{D}_p$ is pseudoconvex over C^{n+1} with smooth boundary $\partial\mathbf{D}_p$. Each $\mathbf{D}_p$ carries a real analytic plurisubharmonic function $\psi_p(t,z)$ such that $\psi_p(t,z)\to+\infty$ as $(t,z)\to\partial\mathbf{D}_p$ (see, e.g., Theorem 8.1 of this paper). Now, let $\{B_p\}$ be a sequence of subregions of B such that $B_1\subset\subset B_2\subset\subset\cdots$ and $\bigcup_{p=1}^{\infty}B_p=B$. We choose a subsequence $\mathbf{D}_{j(p)}$ of $\mathbf{D}_p$ such that $\mathbf{D}_{j(p)}\supset\supset\zeta(B_{p+1})$. We relabel $j(p)=p$ $(p=1,2,\ldots)$. For each $t\in B_{p+1}$, we consider the Robin constant $\lambda_p(t)$ for $(D_p(t),\zeta)$. Since $\psi_p(t,z)$ is real analytic in $\mathbf{D}_p$, there exists a large number $\alpha_p>0$ with the following property: If we put $\mathbf{D}_p^*=\{(t,z)\in\mathbf{D}_p\mid\psi_p(t,z)<\alpha_p\}$, then $\mathbf{D}_p\supset\supset\mathbf{D}_p^*\supset\supset\mathbf{D}_{p-1}$ and the triple $(\mathbf{D}^*_{p,B_p},B_p\times C^n,p)$, where $\mathbf{D}^*_{p,B_p}=\bigcup_{t\in B_p}(t,D_p^*(t))$, satisfies Conditions 4.1–4.4. For each $t\in B_p$, let $\lambda_p^*(t)$ denote the Robin constant for $(D_p^*(t),\zeta)$. Corollary 4.1 implies that $\lambda_p^*(t)$ is a superharmonic function of class C^1 on B_p. On the other hand, because $D_{p-1}(t)\subset\subset D_p^*(t)\subset\subset D_p(t)\subset\subset D(t)$ for $t\in B_p$, we have $\lambda_{p-1}(t)<\lambda_p^*(t)<\lambda_p(t)<\lambda(t)$ for $t\in B_p$. Since $\lambda_p(t)\nearrow\lambda(t)$ as $p\to+\infty$ for $t\in B$, it follows that $\lambda(t)$ is a superharmonic function on B. We have thus proved the following.

THEOREM 5.1. *Let* $\mathbf{D}$ *be a domain over* $B\times C^n$ *satisfying Condition 4.1. If* $\mathbf{D}$ *is a pseudoconvex domain over* $B\times C^n$, *then* $\lambda(t)$ *is a superharmonic function on* B.

In order to derive the Main Theorem from Theorem 5.1, we need the following elementary property of the space C^n.

PROPOSITION 5.1. *Consider an affine transformation of* C^n *of the form* $w=\varphi(z)=a(Az)+b$, *where* $a\neq0$, $a\in C$, $b\in C^n$, *and* A *is an* $n\times n$ *unitary matrix* ($\bar{A}^tA=E_n=n\times n$ *identity matrix*). *For an open set* D *over* C^n *and* $\zeta\in D$, *we put* $D^*=\varphi(D)$ *and* $\zeta^*=\varphi(\zeta)$. *Then:*

(1) *If we denote by* $g(z)$ *and* λ (*resp.* $g^*(w)$ *and* λ^*) *the Green function and the Robin constant for* (D,ζ) (*resp.* (D^*,ζ^*)), *then*

$$g^*(w)=\frac{g(z)}{|a|^{2n-2}}\quad\text{and}\quad\lambda^*=\frac{\lambda}{|a|^{2n-2}}\tag{5.1}$$

for $z\in D$, *where* $w=\varphi(z)$.

(2) *Both* $g(z)$ *and* λ *are independent of the choice of complex coordinates.*

Proof. We may assume that D has smooth boundary ∂D. Since harmonicity is invariant under φ or φ^{-1}, the function $G(w)=g\circ\varphi^{-1}(w)$ is harmonic in $D^*-\{\zeta^*\}$. Moreover $G(w)$ vanishes on ∂D^* and can be expressed in a neighborhood of ζ^* in the form

$$G(w)=\frac{|a|^{2n-2}}{\|w-\zeta^*\|^{2n-2}}+\lambda+H(w),$$

where $H(w)$ is harmonic and $H(\zeta^*)=0$. It follows that $g^*(w)=G(w)/|a|^{2n-2}$ and $\lambda^*=\lambda/|a|^{2n-2}$. Hence (1) is proved. If we take $a=1$ and $b=0$, then (2) follows and Proposition 5.1 is proved. □

Theorem 5.1 and Proposition 5.1 now having been established, the Main Theorem is proved by a standard method as follows.

Proof of Main Theorem. Let $\mathbf{D}$ be a pseudoconvex domain over $B\times C^n$ with holomorphic section $\zeta: t \to \zeta(t)$ $(t\in B)$. For $t\in B$, we denote by $\lambda(t)$ the Robin constant for $(D(t), \zeta(t))$. Now take any open disk $B_o: |t-t_o|<r_o$ such that $B_o \subset\subset B$. Let $f(t)$ be any holomorphic function on a neighborhood of $B_o\cup\partial B_o$ such that $f(t)\neq 0$. We choose a single-valued branch f_1 of $f^{1/(2n-2)}$ on B_o and consider the following transformation, known as a *Hartogs'* transformation, of $\mathbf{D}_{B_o}$ onto $\mathbf{D}^* = T(\mathbf{D}_{B_o})$:

$$T: \begin{cases} t=t, \\ w=f_1(t)(z-\zeta(t)). \end{cases}$$

The domain $\mathbf{D}^*$ becomes then a pseudoconvex domain over $B_o\times C^n$ with constant zero section O. For $t\in B_o$, we denote by $\lambda^*(t)$ the Robin constant for $(D^*(t), O)$. From Theorem 5.1, $\lambda^*(t)$ is a superharmonic function on B_o. On the other hand, equation (5.1) implies that $\lambda^*(t)=\lambda(t)/|f(t)|$ for $t\in B_o$. It follows from the inequality $\lambda(t)\leq 0$ that

$$\log(-\lambda(t_o))-u(t_o)\leq \max_{|t-t_o|=r_o}\{\log(-\lambda(t))-u(t)\},$$

where $u(t)=\log|f(t)|$. Since $f(t)$ is an arbitrary holomorphic function on $B_o\cup\partial B_o$ with $f(t)\neq 0$, $\log(-\lambda(t))$ is thus subharmonic on B_o. (Note that $\log(-\lambda(t))$ may be identically $-\infty$ on B_o.) It follows that $\lambda(t)$ is superharmonic on B_o. □

6. Fiber Uniformity

From the beginning of this paper we have treated variations of open sets $D(t)$ over C^n $(n\geq 2)$ with complex parameter t. In Sections 6, 7, and 8, we will give some applications of the Main Theorem and Lemma 3.1. These results will be compared with those in the case when $D(t)$ varies over C (given in [15] and [16]).

Let D be a domain over C^n and let $\zeta\in D$. We denote by $g(z)$ and λ the Green function and the Robin constant for (D,ζ). In view of (1.2) we have $0<g(z)\leq 1/\|z-\zeta\|^{2n-2}$ for $z\in D$ and $\lambda\leq 0$. We easily see by the maximum principle that

$$(6.1)\qquad g(z)=\frac{1}{\|z-\zeta\|^{2n-2}} \text{ in } D \quad \text{if and only if } \lambda=0.$$

Moreover, whether $\lambda=0$ or $\lambda<0$ does not depend on the choice of the pole $\zeta\in D$. We thus introduce the following.

DEFINITION 6.1. A domain D over C^n with $\lambda=0$ (resp. <0) is said to be *parabolic* (resp. *hyperbolic*).

Now let B be a region of C and consider a domain $\mathbf{D}$ over $B\times C^n$. The fiber $D(t)$ of $\mathbf{D}$ at $t\in B$ is thus an open set over C^n; that is, $D(t)$ consists of at most countably many domains over C^n. We have the following result on fiber uniformity.

THEOREM 6.1. *Let $(\mathbf{D}, B\times C^n, p)$ be a triple with $p_B\circ p(\mathbf{D})=B$ and let $K=\{t\in B\,|\,D(t)$ has at least one parabolic connected component$\}$. Suppose that $\mathbf{D}$ is a pseudoconvex domain over $B\times C^n$. If K is of positive logarithmic capacity in C, then $K=B$ and each connected component of $D(t)$ is parabolic for all $t\in B$.*

Proof. We denote by $D^*(t)$ one of the connected components of $D(t)$ for $t\in K$. Suppose that K is of positive logarithmic capacity. Then we can find a point $(t_o, \zeta_o)\in\mathbf{D}$ with the following property: There exists a disk B_o: $|t-t_o|<\rho$ such that $B_o\times\{\zeta_o\}\subset\subset\mathbf{D}$ and such that the subset $K_o=\{t\in B_o\cap K\,|\, D^*(t)$ contains the point $\zeta_o\}$ of K is of positive logarithmic capacity. If, for $t\in B_o$, we consider the Robin constant $\lambda(t)$ for $(D(t), \zeta_o)$, then $\lambda(t)=0$ for $t\in K_o$. According to the Main Theorem, $\log(-\lambda(t))$ is a subharmonic function on B_o. It follows that $\log(-\lambda(t))\equiv-\infty$ on B_o. Consequently, for each $t\in B_o$, the connected component $D_1(t)$ of $D(t)$ which contains ζ_o is parabolic, so that $K\supset B_o$. By repeating the same process at a point (t_1, ζ_1) where $t_1\in\partial B_o$ and $\zeta_1\in D_1(t_1)$, instead of the point (t_o, ζ_o), we can eventually show that $K=B$. The connectedness of $\mathbf{D}$ over $B\times C^n$ then implies the latter assertion. Theorem 6.1 is proved. □

Following Ahlfors and Sario [1, Chap. IV] we briefly recall the notion of the Robin constant of a domain D over C. Let D be a domain over C and let $\zeta\in D$. Choose a subdomain Ω of D with smooth boundary $\partial\Omega$ such that $\zeta\in\Omega\subset\subset D$. Then Ω carries the Green function $g_\Omega(z)$ with pole at ζ. Hence $g_\Omega(z)$ can be written in a neighborhood of ζ in the form

$$g_\Omega(z)=\log\frac{1}{|z-\zeta|}+\lambda_\Omega+h_\Omega(z),$$

where λ_Ω is a constant, $h_\Omega(z)$ is harmonic, and $h_\Omega(\zeta)=0$. Since $g_\Omega<g_{\Omega'}$ and $\lambda_\Omega<\lambda_{\Omega'}$ for $\Omega\subset\subset\Omega'\subset\subset D$, the limits $g(z)=\sup\{g_\Omega(z)\,|\,\Omega\subset\subset D\}$ and $\lambda=\sup\{\lambda_\Omega\,|\,\Omega\subset\subset D\}$ exist. We may have $g(z)\equiv+\infty$ in D and $\lambda=+\infty$. Then $g(z)$ and λ are called the Green function and the Robin constant for (D, ζ). If $\lambda=+\infty$ (resp. $<+\infty$), the domain D is said to be parabolic (resp. hyperbolic). Using this notation, we obtained in [15, p. 72] the same Theorem as Theorem 6.1 with C^n replaced by C. Although both theorems are quite similar in form, there is some difference in content between them. In the case of a domain D over C, harmonicity is invariant under any analytic transformation $w=\varphi(z)$. It follows that

$$\lambda_{\varphi(D)}(\varphi(\zeta))=\lambda_D(\zeta)+\log\left|\frac{\partial\varphi}{\partial z}(\zeta)\right|,$$

where $\lambda_D(\zeta)$ and $\lambda_{\varphi(D)}(\varphi(\zeta))$ denote the Robin constant for (D, ζ) and for $(\varphi(D), \varphi(\zeta))$, respectively. Therefore parabolicity is invariant under analytic transformations. In the case $n \geq 2$ this does not remain valid, as is seen in the first of the following examples.

EXAMPLE 6.1. From the well-known Fatou–Bieberbach example, the whole space C^n $(n \geq 2)$ can be transformed by an analytic transformation φ onto a domain $\varphi(D)$ of C^n such that $C^n - \varphi(D) \cup \partial\varphi(D)$ is nonvoid. Consequently, $\varphi(D)$ is hyperbolic while C^n is parabolic (see Example 1.1).

EXAMPLE 6.2. Let $w = P(z)$ be an algebraic function of $z = (z_1, \ldots, z_n)$ $(n \geq 1)$ which is not a polynomial. For $n \geq 2$ (resp. $= 1$) it determines a ramified Riemann domain R over C^n with a finite number of branch surfaces (points in the case $n = 1$) S. We put $R^* = R - S$, which is at least a two-sheeted unramified covering domain over C^n. In the case $n = 1$, the domain R^* is parabolic. On the contrary, in the case $n \geq 2$, R^* is always hyperbolic, because statement (6.1) implies that *every parabolic domain is schlicht*.

EXAMPLE 6.3. A typical example of a parabolic domain for $n \geq 1$ is a domain D of C^n such that $C^n - D$ is a polar set, namely, such that there exists a subharmonic function $s(z)$ in D with $\lim_{z \to C^n - D} s(z) = -\infty$.

7. Rigidity

For other applications we need the following rigidity lemma.

LEMMA 7.1. *Let $(\mathbf{D}, B \times C^n, p)$ be a triple satisfying the same conditions as in Lemma 3.1. If $(\partial^2\lambda/\partial t\partial\bar{t})(t_o) = 0$ for some $t_o \in B$, then $(\partial g/\partial t)(t_o, z) = 0$ for all $z \in D(t_o)$.*

Proof. Suppose that $(\partial^2\lambda/\partial t\partial\bar{t})(t_o) = 0$. Then Lemma 3.1 implies

$$0 = \frac{\partial^2\lambda}{\partial t\partial\bar{t}}(t_o) \leq \frac{-4}{(n-1)\omega_{2n}} \iint_{D(t_o)} \left(\sum_{\alpha=1}^{n} \left| \frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha}(t_o, z) \right|^2 \right) dV \leq 0. \tag{7.1}$$

Hence inequalities (3.5) and (3.6) in the proof of Proposition 3.4 must reduce to equalities:

$$\frac{\partial^2 g}{\partial t\partial\bar{t}} = 2\,\mathrm{Re}\left\{ \frac{\partial g}{\partial\bar{t}} \sum_{\alpha=1}^{n} \frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha} \frac{\partial g}{\partial z_\alpha} \right\} \Big/ \|\mathrm{Grad}_{(z)}\, g\|^2; \tag{3.5'}$$

$$\frac{\partial^2 g}{\partial t\partial\bar{t}} \left| \frac{\partial g}{\partial z_\beta} \right|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial g}{\partial\bar{t}} \frac{\partial^2 g}{\partial t\partial\bar{z}_\beta} \frac{\partial g}{\partial z_\beta} \right\} + \frac{\partial^2 g}{\partial z_\beta\partial\bar{z}_\beta} \left| \frac{\partial g}{\partial t} \right|^2 = 0 \tag{3.6'}$$

for each β $(1 \leq \beta \leq n)$ and each point (t_o, z) such that $z \in \partial D(t_o)$. Inequality (7.1) yields

$$\frac{\partial^2 g}{\partial t\partial\bar{z}_\alpha}(t_o, z) = 0 \quad (1 \leq \alpha \leq n) \tag{7.2}$$

for $z \in D(t_o)$; that is, $(\partial g/\partial t)(t_o, z)$ is a holomorphic function of z in $D(t_o)$. By continuity, $(\partial^2 g/\partial t \partial \bar{z}_\alpha)(t_o, z) = 0$ for $z \in \partial D(t_o)$. It follows from (3.5′) that $(\partial^2 g/\partial t \partial \bar{t})(t_o, z) = 0$ for $z \in \partial D(t_o)$. Consequently, (3.6′) and (7.2) yield

$$\frac{\partial^2 g}{\partial z_\beta \partial \bar{z}_\beta}(t_o, z) \cdot \left| \frac{\partial g}{\partial t}(t_o, z) \right|^2 = 0 \quad (1 \le \beta \le n) \tag{7.3}$$

for $z \in \partial D(t_o)$. On the other hand, since $D(t_o)$ is bounded, there uniquely exists a sphere $\pi : \|z - \zeta\| = R$ with center at the pole ζ which is tangent to the smooth surface $\partial D(t_o)$ from the outside. Take one of the points of contact, say $z_o \in \pi \cap \partial D(t_o)$. If we choose a point $a \in C^n$ with $\|a\| = 1$ such that $\sum_{\alpha=1}^n a_\alpha \overline{(z_{o\alpha} - \zeta_\alpha)} = R$ where $\zeta = (\zeta_1, \dots, \zeta_n)$ and $z_o = (z_{o1}, \dots, z_{on})$, then

$$\frac{\partial^2 g(t_o, z_o + a\tau)}{\partial \tau \partial \bar{\tau}}(0) < 0.$$

By Proposition 5.1(2) we may assume $a = (1, 0, \dots, 0)$, so that

$$\frac{\partial^2 g}{\partial z_1 \partial \bar{z}_1}(t_o, z_o) < 0.$$

Hence $(\partial^2 g/\partial z_1 \partial \bar{z}_1)(t_o, z)$ does not vanish in a neighborhood V of z_o in $\tilde{D}(t_o)$. Expression (7.3) implies that $(\partial g/\partial t)(t_o, z) = 0$ for all $z \in V \cap \partial D(t_o)$. Because $(\partial g/\partial t)(t_o, z)$ is holomorphic for z in $D(t_o)$ and is continuous on $D(t_o) \cup \partial D(t_o)$, it follows from the uniqueness theorem that $(\partial g/\partial t)(t_o, z) = 0$ for all $z \in D(t_o)$. Lemma 7.1 is proved. □

In the proof, the point of contact z_o is called a strictly pseudoconvex boundary point of $D(t_o)$.

COROLLARY 7.1. *Under the same conditions as in Lemma 3.1, if $\lambda(t)$ is a harmonic function on B then* **D** *is identical with the product $B \times D(t_o)$, where t_o is a fixed point in B.*

Proof. Assume that $\lambda(t)$ is harmonic on B. By Lemma 7.1, the function $g(t, z)$ in **D** does not depend on $t \in B$. Consequently, if for any number $\epsilon > 0$ we put $\mathbf{D}_\epsilon = \{(t, z) \in \mathbf{D} \mid g(t, z) < \epsilon\}$, then $\mathbf{D}_\epsilon = B \times D_\epsilon(t_o)$, where t_o is a fixed point in B. Since $D_\epsilon(t) \nearrow D(t)$ for $\epsilon \to 0$, we have $\mathbf{D} = B \times D(t_o)$. Corollary 7.1 is proved. □

In the case of a domain D over C, neither the same rigidity as Lemma 7.1 nor the similar result to Corollary 7.1 is valid. However, the following fact was proved in [16, Thm. 1]: Let $(\mathbf{D}, B \times C^n, p)$ be a triple which satisfies the same conditions as in Lemma 3.1 with C^n replaced by C. Let q denote the Euler characteristic of the fiber $D(t)$ $(t \in B)$. Suppose that there exist at least $-q+2$ holomorphic sections of **D** defined on B, $\zeta_i : t \to \zeta_i(t)$ $(t \in B,\ 1 \le i \le -q+2)$ such that the Robin constants $\lambda_i(t)$ for $(D(t), \zeta_i(t))$ are harmonic functions on B. Then **D** is analytically equivalent to the product $B \times D(t_o)$ by a transformation $(t, z) \to (t, \varphi(t, z))$, where $\varphi(t, z)$ is holomorphic with respect to (t, z) in **D** and where t_o is a fixed point in B.

8. Strictly Plurisubharmonic Functions

Let D be an arbitrary domain over C^n ($n \geq 2$) and $\zeta \in D$. We denote by $G(\zeta, z)$ and $\Lambda(\zeta)$ the Green function and the Robin constant for (D, ζ). The function $G(\zeta, z)$ can be expressed in a neighborhood of ζ in the form

$$G(\zeta, z) = \frac{1}{\|z-\zeta\|^{2n-2}} + \Lambda(\zeta) + H(\zeta, z), \tag{8.1}$$

where $H(\zeta, z)$ is harmonic for z in D and $H(\zeta, \zeta) = 0$. The Robin constant $\Lambda(\zeta)$ defines a real-valued function in D. In the case where D is parabolic, we have $\Lambda(\zeta) \equiv 0$ in D, so there is no interest.

(1) Consider the case where D is hyperbolic, that is, $-\infty < \Lambda(\zeta) < 0$ for $\zeta \in D$. Then $\Lambda(\zeta)$ is real analytic in D.

In fact, let ζ_o be in D and take a ball V: $\|z-\zeta_o\| < r$ such that $V \subset\subset D$. Consider the function $v(\zeta, z)$ in $V \times V$:

$$v(\zeta, z) = \begin{cases} G(\zeta, z) - 1/\|z-\zeta\|^{2n-2} & (\zeta \neq z), \\ \Lambda(\zeta) & (\zeta = z). \end{cases}$$

According to the well-known symmetry property of G (viz., $G(\zeta, z) = G(z, \zeta)$ in $D \times D$), we have $v(\zeta, z) = v(z, \zeta)$ in $V \times V$. Hence $v(\zeta, z)$ is harmonic for ζ as well as z in V. Moreover, inequality (1.2) yields $v(\zeta, z) \leq 0$ in $V \times V$. Fubini's theorem together with (1.7) then gives

$$\Lambda(\zeta) = \frac{1}{(r\omega_{2n})^2} \int\int_{\partial V \times \partial V} v(z, w) \frac{(r^2 - \|z-\zeta\|^2)(r^2 - \|w-\zeta^2\|)}{\|z-\zeta\|^{2n}\|w-\zeta\|^{2n}} \, dS_z \, dS_w$$

for all $\zeta \in V$. It follows that $\Lambda(\zeta)$ is real analytic in V. Assertion (1) is proved.

(2) Consider the case where D is a relatively compact domain over C^n for which Dirichlet's problem can be solved. Then $-\Lambda(\zeta)$ is an *exhaustion* function in D; that is,

$$\lim_{\zeta \to \partial D} \Lambda(\zeta) = -\infty.$$

In fact, let $\tilde{D}$ be a domain over C^n such that $\tilde{D} \supset\supset D \cup \partial D$, and let $\zeta_o \in \partial D$. Let $M > 0$ be given. We choose a small ball W: $\|z-\zeta_o\| < r$ in $\tilde{D}$ such that $1/\|z-\zeta\|^{2n-2} > M$ for all $z, \zeta \in W$. Since we can solve Dirichlet's problem in D, we can construct the bounded harmonic function $u_{-M}(z)$ in D whose boundary values are $-M$ on $\partial D \cap W$ and 0 on $\partial D - W \cup \partial W$. Then $-M < u_{-M}(z) < 0$ in D. Now fix a point $\zeta \in D \cap W$. Consider the following function $s(z)$ in D:

$$s(z) = \begin{cases} u_{-M}(z) - (G(\zeta, z) - 1/\|z-\zeta\|^{2n-2}) & (z \neq \zeta), \\ u_{-M}(\zeta) - \Lambda(\zeta) & (z = \zeta). \end{cases}$$

By virtue of expression (8.1), $s(z)$ is a continuous superharmonic function in D. Inequality (1.2) implies that $s(z) \geq -M$ in D and that $\underline{\lim}_{z \to \zeta} s(z) \geq 0$ for any $\zeta \in \partial D$. It follows from the maximum principle that $s(z) \geq 0$ in D. In particular, $s(\zeta) \geq 0$. We thus conclude that $u_{-M}(\zeta) \geq \Lambda(\zeta)$ for all $\zeta \in D \cap W$.

Consequently,

$$\varlimsup_{z\to\zeta_o}\Lambda(\zeta)\le\varlimsup_{z\to\zeta_o}u_{-M}(\zeta)=-M,$$

which means that $\lim_{\zeta\to\zeta_o}\Lambda(\zeta)=-\infty$. Assertion (2) is proved.

(3) Consider the case where D is a hyperbolic pseudoconvex domain (which may be infinitely many sheeted) over C^n. Then $-\Lambda(\zeta)$ and $\log(-\Lambda(\zeta))$ are real analytic plurisubharmonic functions in D.

It suffices to prove the plurisubharmonicity of $\log(-\Lambda(\zeta))$. Take $\zeta_o\in D$ and consider a complex line through ζ_o: $\zeta=\zeta(t)=\zeta_o+at$, where $a\in C^n$ with $\|a\|=1$ and where $t\in C$. We choose a small disk $B:|t|<\rho$ such that $\zeta(t)\in D$ for $t\in B$. Then $\mathbf{D}=B\times D$ is a pseudoconvex domain over $B\times C^n$ with holomorphic section $\zeta: t\to\zeta(t)$ $(t\in B)$. It follows from the Main Theorem that $\log\{-\Lambda(\zeta(t))\}$ is subharmonic on B, and assertion (3) is proved.

(4) Consider the case where D is a bounded pseudoconvex domain over C^n with smooth boundary ∂D. Then $-\Lambda(\zeta)$ is a real analytic strictly plurisubharmonic function in D.

To this end it suffices to prove that, for each $\zeta_o\in D$ and $a\in C^n$ with $\|a\|=1$, the inequality

$$\sum_{\alpha,\beta=1}^{n}\frac{\partial^2(-\Lambda)}{\partial\zeta_\alpha\partial\bar\zeta_\beta}(\zeta_o)a_\alpha\bar a_\beta>0 \tag{8.2}$$

holds. Take a disk $B:|t|<\rho$ such that $\zeta_o+at\in D$ for $t\in B$. We transform the product $B\times D$ by the transformation

$$T_1:(t,z)\to(t,w)=(t,z-at)$$

and put $\mathbf{D}_1=T_1(B\times D)$, so that $\mathbf{D}_1$ is a domain over $B\times C^n$ and has a constant section $\zeta_o: t\to\zeta_o$ $(t\in B)$. Since D is a pseudoconvex domain over C^n with smooth boundary, the domain $\mathbf{D}_1$, with section ζ_o, satisfies all the conditions of Lemma 3.1. Therefore, if we denote by $g_1(t,w)$ and $\lambda_1(t)$ the Green function and the Robin constant for $(D_1(t),\zeta_o)$, then

$$\frac{\partial^2\lambda_1}{\partial t\partial\bar t}(0)\le\frac{-4}{(n-1)\omega_{2n}}\iint_{D_1(0)}\left(\sum_{\alpha=1}^{n}\left|\frac{\partial^2 g_1}{\partial t\partial\bar w_\alpha}(0,w)\right|^2\right)dV. \tag{8.3}$$

By virtue of (5.1) we have

$$g_1(t,w)=G(\zeta_o+at,z)\quad\text{and}\quad\lambda_1(t)=\Lambda(\zeta_o+at) \tag{8.4}$$

for $t\in B$ and $z\in D$, where $w=z-at$. Hence

$$\frac{\partial^2\lambda_1}{\partial t\partial\bar t}(0)=\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\Lambda}{\partial\zeta_\alpha\partial\bar\zeta_\beta}(\zeta_o)a_\alpha\bar a_\beta.$$

Differentiation of the first equation of (8.4) with respect to t and each $\bar w_\alpha$ $(1\le\alpha\le n)$ shows that

$$\frac{\partial g_1}{\partial t}(0,w)=\sum_{\beta=1}^{n}a_\beta\left(\frac{\partial G}{\partial\zeta_\beta}+\frac{\partial G}{\partial z_\beta}\right)(\zeta_o,z)$$

and

$$\frac{\partial^2 g_1}{\partial t \partial \bar{w}_\alpha}(0, w) = \sum_{\beta=1}^{n} a_\beta \frac{\partial}{\partial \bar{z}_\alpha}\left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta}\right)(\zeta_o, z)$$

for $w = z \in D$ $(= D_1(0))$. It follows from (8.3) that

$$\begin{aligned} &\sum_{\alpha,\beta=1}^{n} \frac{\partial^2(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_o) a_\alpha \bar{a}_\beta \\ &\geq \frac{4}{(n-1)\omega_{2n}} \iint_D \left(\sum_{\alpha=1}^{n} \left| \sum_{\beta=1}^{n} a_\beta \frac{\partial}{\partial \bar{z}_\alpha}\left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta}\right)(\zeta_o, z)\right|^2 \right) dV \geq 0. \end{aligned} \tag{8.5}$$

We prove inequality (8.2) by contradiction. Assume that, for some $\zeta_o \in D$ and $a \in C^n$ with $\|a\| = 1$, we have $(\partial^2 \lambda_1 / \partial t \partial \bar{t})(0) = 0$. Then Lemma 7.1 yields $(\partial g_1 / \partial t)(0, w) = 0$ for $w \in D_1(0) \cup \partial D_1(0)$, so that

$$\sum_{\beta=1}^{n} a_\beta \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta}\right)(\zeta_o, z) = 0$$

for $z \in D \cup \partial D$. Since $G(\zeta, z) = 0$ for $(\zeta, z) \in D \times \partial D$, we have

$$\left(\frac{\partial G}{\partial \zeta_\beta}\right)(z_o, z) = 0 \quad (1 \leq \beta \leq n) \quad \text{for } z \in \partial D.$$

Consequently,

$$\sum_{\beta=1}^{n} a_\beta \frac{\partial G}{\partial z_\beta}(\zeta_o, z) = 0 \quad \text{for } z \in \partial D. \tag{8.6}$$

It follows from Preliminary 1.1 that the function $\sum_{\beta=1}^{n} a_\beta (\partial G / \partial z_\beta)(\zeta_o, z)$ is divisible by $G(\zeta_o, z)$ at every point of ∂D. Precisely, let z_o be any point of ∂D. Then there exists a neighborhood V of z_o which is contained in a domain $\tilde{D}$ $(\supset\supset D \cup \partial D)$, and a complex-valued function $c(z)$ of class C^1 in V such that

$$\sum_{\beta=1}^{n} a_\beta \frac{\partial G}{\partial z_\beta}(\zeta_o, z) = c(z) G(\zeta_o, z) \tag{8.7}$$

for $z \in V$. Fix $z_o \in \partial D$ and put $z(t) = z_o + at$ for $t \in C$. Then

$$z(t) \in \partial D \quad \text{for all } t \in C. \tag{8.8}$$

For let $K = \{t \in C \mid z(t) \in \partial D\}$. It is clear that K is closed in C. Moreover, K must be open in C. Indeed, let $t_o \in K$. We have $G(\zeta_o, z(t_o)) = 0$. Take a small disk $B_o: |t - t_o| < \rho_o$ such that $z(t) \subset V$ for all $t \in B_o$. Then we have the *real-valued* function $f(t) = G(\zeta_o, z(t))$ defined on B_o. Equation (8.7) implies that $f(t)$ satisfies the following partial differential equation on B_o:

$$\frac{\partial f(t)}{\partial t} = c(z_o + at) f(t) \quad \text{with } f(t_o) = 0.$$

By a uniqueness theorem, $f(t)$ is identically zero on B_o, so that $G(\zeta_o, z(t)) = 0$ for $t \in B_o$. It follows from (1.5) that $z(t) \in \partial D$ for $t \in B_o$. Consequently, K is open in C. Since $K \ni 0$, we have $K = C$. Hence (8.8) is proved.

Statement (8.8) contradicts the fact that D is bounded over C^n. We thus have (8.2) and assertion (4) is proved.

(5) Consider the same case as in (4). Then $\log(-\Lambda(\zeta))$ is a real analytic strictly plurisubharmonic function in D.

To this end, it remains to prove that, for each $\zeta_o \in D$ and $a \in C^n$ with $\|a\| = 1$, we have the inequality

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_o) a_\alpha \bar{a}_\beta > 0. \tag{8.9}$$

We use the same notation B, T_1, $\mathbf{D}_1 = T_1(B \times D)$, $g_1(t, w)$, and $\lambda_1(t)$ used to prove (8.2). We consider the Taylor development of $\log(-\lambda_1(t))$ at $t = 0$:

$$\log(-\lambda_1(t)) = \operatorname{Re}\{c_o + c_1 t\} + k(t),$$

where

$$c_o = \log(-\lambda_1(0)) \text{ and } c_1 = 2\frac{\partial \log(-\lambda_1)}{\partial t}(0).$$

Then $k(t)$ is a real analytic function on B such that

$$k(0) = \frac{\partial k}{\partial t}(0) = 0$$

and

$$\frac{\partial^2 k}{\partial t \partial \bar{t}}(0) = \frac{\partial^2 \log(-\lambda_1)}{\partial t \partial \bar{t}}(0) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_o) a_\alpha \bar{a}_\beta.$$

We use the following Hartogs' transformation:

$$T_2 : (t, w) \to (t, W) = (t, e^{\varphi(t)/(2n-2)}(w - \zeta_o)) \quad \text{where } \varphi(t) = c_o + c_1 t,$$

which maps $\mathbf{D}_1$ onto $\mathbf{D}_2 = T_2(\mathbf{D}_1)$. The image $\mathbf{D}_2$ becomes a domain over $B \times C^n$ with constant section $O : t \to O$ $(t \in B)$. Because D is a pseudoconvex domain over C^n with smooth boundary, the domain $\mathbf{D}_2$, with section O, fulfills all the conditions of Lemma 3.1. If we denote by $g_2(t, W)$ and $\lambda_2(t)$ the Green function and the Robin constant for $(D_2(t), O)$, then

$$\frac{\partial^2 \lambda_2}{\partial t \partial \bar{t}}(0) \le \frac{-4}{(n-1)\omega_{2n}} \iint_{D_2(0)} \left(\sum_{\alpha=1}^{n} \left| \frac{\partial^2 g_2}{\partial t \partial \bar{W}_\alpha}(0, W) \right|^2 \right) dV. \tag{8.10}$$

In view of (5.1) we have

$$g_2(t, W) = e^{-\operatorname{Re}\varphi(t)} g_1(t, w) \quad \text{and} \quad \lambda_2(t) = e^{-\operatorname{Re}\varphi(t)} \lambda_1(t)$$

for $w \in D_1(t)$ and $t \in B$, where $W = e^{\varphi(t)/(2n-2)}(w - \zeta_o)$. Hence

$$\frac{\partial^2 \lambda_2}{\partial t \partial \bar{t}}(0) = -\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_o) a_\alpha \bar{a}_\beta.$$

On the other hand, $g_2(t, W)$ can be expressed in terms of $G(\zeta, z)$ as follows:

$$g_2(t, W) = e^{-\operatorname{Re}\varphi(t)} G(\zeta_o + at, We^{-\varphi(t)/(2n-2)} + \zeta_o + at)$$

for $(t, W) \in \mathbf{D}_2$. Precisely,

$$g_2(t, W) = e^{-\mathrm{Re}\,\varphi(t)} G(\zeta_1, \ldots, \zeta_n, z_1, \ldots, z_n),$$

where

$$\zeta_\beta = \zeta_{o\beta} + a_\beta t \quad \text{and} \quad z_\beta = W_\beta e^{-\varphi(t)/(2n-2)} + \zeta_{o\beta} + a_\beta t \quad (1 \le \beta \le n).$$

Differentiation of this equation with respect to t shows that

$$\frac{\partial g_2}{\partial t}(0, W) = \frac{-1}{\Lambda(\zeta_o)} H(a, \zeta_o, z),$$

where

$$H(a, \zeta_o, z) = \left[-\frac{c_1}{2}\left(G + \frac{1}{n-1} \sum_{\beta=1}^n (z_\beta - \zeta_{o\beta}) \frac{\partial G}{\partial z_\beta} \right) + \sum_{\beta=1}^n a_\beta \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right) \right]_{(\zeta_o, z)}.$$

Moreover, for each α $(1 \le \alpha \le n)$,

$$\frac{\partial^2 g_2}{\partial t \partial \overline{W}_\alpha}(0, W) = \frac{1}{(-\Lambda(\zeta_o))^{1+1/(2n-2)}} \frac{\partial H(a, \zeta_o, z)}{\partial \bar{z}_\alpha}$$

for $W \in D_2(0)$ and $z \in D$, where

$$(0, W) = T_2 \circ T_1(0, z) = (0, (-\Lambda(\zeta_o))^{1/(2n-2)}(z - \zeta_o)).$$

It follows from (8.10) that

$$\begin{aligned} &\sum_{\alpha, \beta=1}^n \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_o) a_\alpha \bar{a}_\beta \\ &\quad \ge \frac{4}{(n-1)\omega_{2n}(-\Lambda(\zeta_o))} \iint_D \left(\sum_{\alpha=1}^n \left| \frac{\partial H(a, \zeta_o, z)}{\partial \bar{z}_\alpha} \right|^2 \right) dV \ge 0. \end{aligned} \tag{8.11}$$

We prove inequality (8.9) by contradiction. Assume that, for some $\zeta_o \in D$ and $a \in C^n$ with $\|a\| = 1$, we have $(\partial^2 \lambda_2 / \partial t \partial \bar{t})(0) = 0$. Then Lemma 7.1 implies $H(a, \zeta_o, z) = 0$ for all $z \in D \cup \partial D$. Since $G(\zeta_o, z) = (\partial G / \partial \zeta_\beta)(\zeta_o, z) = 0$ $(1 \le \beta \le n)$ for $z \in \partial D$, we have

$$\sum_{\beta=1}^n \left(a_\beta - \frac{c_1}{2(n-1)}(z_\beta - \zeta_{o\beta}) \right) \frac{\partial G}{\partial z_\beta}(\zeta_o, z) = 0 \quad \text{for } z \in \partial D. \tag{8.12}$$

Let $z_o = (z_{o1}, \ldots, z_{on}) \in \partial D$ such that $p(z_o) \ne p(\zeta_o)$ in C^n. Consider the system of differential equations on C:

$$\frac{dz_\beta}{dt} = a_\beta - \frac{c_1}{2(n-1)}(z_\beta - \zeta_{o\beta}) \quad (1 \le \beta \le n)$$

with initial values $z_\beta(0) = z_{o\beta}$. Their solutions $z_\beta = z_\beta(t)$ are given in all of C as follows:

$$z_\beta(t) = \begin{cases} z_{o\beta} + a_\beta t & \text{for } c_1 = 0, \\ A_\beta + (z_{o\beta} - A_\beta) e^{kt} & \text{for } c_1 \ne 0, \end{cases}$$

where

$$k = \frac{-c_1}{2n-2} \quad \text{and} \quad A_\beta = \zeta_{o\beta} - \frac{a_\beta}{k}.$$

In both cases we put $z(t)=(z_\beta(t))_{\beta=1,\dots,n}$ for $t\in C$. Then equality (8.12) implies that $z(t)\in\partial D$ for all $t\in C$, by the same way that equality (8.6) implied statement (8.8). This also contradicts the boundedness of D over C^n. We thus have (8.9). Assertion (5) is proved.

From (2), (4), and (5) we have proved the following.

THEOREM 8.1. *Suppose that D is a bounded pseudoconvex domain over C^n with smooth boundary. Let $\zeta\in D$ and denote by $\Lambda(\zeta)$ the Robin constant for (D,ζ). Then $-\Lambda(\zeta)$ and $\log(-\Lambda(\zeta))$ are real analytic, strictly plurisubharmonic exhaustion functions in D.*

COROLLARY 8.1. *Under the same circumstances as in Theorem 8.1, the quadratic form*

$$ds^2=\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\log(-\Lambda(z))}{\partial z_\alpha\partial\bar z_\beta}dz_\alpha\otimes d\bar z_\beta$$

is a Kähler metric in D.

In [19] we make a study of this metric, for which expressions (8.5) and (8.11) are useful.

9. Variations of Domains in R^m

Until now we have dealt with variations of the Robin constant λ of a domain D when D varies over C^n. In this section we study the case where D varies in the real Euclidean space R^m $(m\geq 3)$. Robin's original paper [9] is concerned with electromagnetism in R^3; we briefly discuss the role of the Robin constant in it. Let (S) be a conductor bounded by a smooth surface S and let $\{M_j\}_{j=1,\dots,p}$ be a finite number of charged particles. Assume that (S) has total charge q and that each M_j has charge $q_j>0$. Let (S) be placed in the space R^3 and let M_j be placed outside of (S). We denote by x^j the position coordinates of M_j. Since (S) is a conductor, the charge on (S) will be redistributed on the surface S as a charge distribution

$$\mu: x\to\mu(x)\quad (x\in S)$$

such that the electric field $e(x)=(e_i(x))_{i=1,2,3}$ in R^3 induced by μ is identically zero on $(S)-S$. If we construct the Newtonian potential

$$v(x)=\int_S\mu(y)\frac{1}{\|x-y\|}dS_y+\sum_{j=1}^{p}\frac{q_j}{\|x-x_j\|},$$

then Coulomb's law implies that

$$e(x)=\left(\frac{\partial v}{\partial x_i}\right)_{i=1,2,3}\qquad\text{for } x\in R^3-S\cup\{x^j\}_{j=1,\dots,p}.$$

Because $e(x)\equiv 0$ on $(S)-S$, the potential $v(x)$ reduces to a constant $k>0$ on (S):

$$v(x) \equiv k \quad \text{for } x \in (S).$$

We call μ (resp. $v(x)$) the *equilibrium* distribution (resp. potential). It was a problem posed by Poisson (1811) to find an integral representation for μ. Robin (1886) showed that μ must satisfy the following integral equation:

$$\mu(x) = \frac{1}{2\pi} \int_S \mu(y) \frac{\langle x-y, n_y \rangle}{\|x-y\|^2}\, dS_y + \frac{1}{2\pi} \sum_{j=1}^{p} \frac{q_j \langle x-x^j, n_x \rangle}{\|x-x^j\|^2} \tag{9.1}$$

for $x \in S$, where n_y denotes the unit outer normal vector to S at the point y and where in general $\langle a, b \rangle$ denotes the cosine of the angle between vectors a and b. Further, he showed that (9.1) could be solved in the following case: In terms of polar coordinates (ρ, θ, φ) of R^3, consider a sphere $\rho = r$ and a surface $\rho = r(1+n(\theta, \varphi))$, where $n(\theta, \varphi)$ is a real analytic function of (θ, φ). Given $\alpha > 0$, let S_α denote the surface $\rho = r(1+\alpha n(\theta, \varphi))$. Then there exists an $\alpha_o > 0$ such that (9.1) is solvable for every S_α such that $0 \le \alpha \le \alpha_o$. It is well known that the resolution of (9.1) for general S needed the theory of Fredholm's integral equations developed in 1906.

From now on we restrict ourselves to the case where $q = 1$ and where there are no particles M_j. Hence

$$v(x) = \int_S \mu(y) \frac{1}{\|x-y\|}\, dS_y \quad \text{and} \quad \int_S \mu(y)\, dS_y = 1. \tag{9.2}$$

In this case, $v(x)$ is a continuous function in R^3 such that: (i) $v(x)$ reduces to a constant k on (S); (ii) $v(x)$ is harmonic in $R^3 - (S)$; (iii) $v(x) - 1/\|x\| = O(1/\|x\|^2)$ in a neighborhood of ∞. We put

$$\|e\|^2 = \iint_{R^3} \|e(x)\|^2\, dV_x$$

and call $\|e\|^2$ the *total energy* of the electric field $e(x)$. By Green's formula we find that $\|e\|^2 = \omega_3 k$. In electromagnetism, the reciprocal of k becomes an important quantity called the *capacity* c of the conductor (S):

$$c = \frac{1}{k} = \frac{\omega_3}{\|e\|^2}. \tag{9.3}$$

Fix $x_o \in (S) - S$. We let π denote the sphere $\|x - x_o\| = 1$. For $x \in R^3 \cup \{\infty\}$, the symmetric point x^* of x with respect to π is defined by

$$x^* = x_o + \frac{x - x_o}{\|x - x_o\|^2} \quad (x \ne x_o, \infty); \quad x_o^* = \infty \text{ and } \infty^* = x_o.$$

We note that $(x^*)^* = x$ for all $x \in R^3 \cup \{\infty\}$. For each $A \subset R^3 \cup \{\infty\}$, we put $A^* = \{x^* \in R^3 \cup \{\infty\} \mid x \in A\}$. We now consider the following domain $D \subset R^3$:

$$D = (R^3 \cup \{\infty\} - (S))^*.$$

So D is the bounded domain surrounded by the surface S^*. Since D contains x_o, we have the Green function $g(x)$ and the Robin constant λ for (D, x_o). We shall show the following.

PROPOSITION 9.1. *$c=-\lambda$ and $\|e\|^2=\omega_3/-\lambda$.*

Proof. Let $v(x)$ be the equilibrium potential defined by (9.2) such that $v(x)\equiv k$ on (S). We restrict the function $v(x)-k$ (defined on R^3) to the domain $R^3-(S)$ and construct the Kelvin transformation of $v(x)-k$ with respect to the sphere π as follows:

$$K(x)=\frac{1}{\|x-x_o\|}(v(x^*)-k) \quad \text{for } x\in D,$$

where D is the domain bounded by the surface S^*. Then $K(x)$ is harmonic in $D-\{x_o\}$, continuous up to ∂D, and vanishes on ∂D. From (9.2) it can be expressed in a neighborhood of x_o in the form

$$K(x)=\frac{-k}{\|x-x_o\|}+1+H(x),$$

where $H(x)$ is harmonic and $H(x_o)=0$. It follows that $g(x)=-K(x)/k$, and hence $\lambda=-1/k$. By relation (9.3) we get Proposition 9.1. □

Although $g(x)$ and thus λ are defined with respect to $x_o\in(S)-S$, Proposition 9.1 yields the following.

REMARK 9.1. The Robin constant λ does not depend on the choice of $x_o\in(S)-S$.

We now assume that the conductor $(S(t))$ with total charge $+1$ varies in R^3 with a real parameter t. Then the capacity $c(t)$ of $(S(t))$ or the total energy $\|e(t)\|^2$ of the electric field $e(t,x)$ induced by the equilibrium distribution $\mu(t,x)$ also varies with t. According to Proposition 9.1 we can investigate the variation of $c(t)$ or $\|e(t)\|^2$ by observing the variation of the Robin constant $\lambda(t)$ for $(D(t),x_o)$, where $D(t)=(R^3-(S(t)))^*$. This problem will be discussed in Theorem 9.3.

We return to the general space R^m $(m\geq 3)$. Let I be an open interval in the real t line. Let $\mathbf{D}$ be a domain of the product space $I\times R^m$. For each $t\in I$, we put $D(t)=\{x\in R^m\,|\,(t,x)\in\mathbf{D}\}$, which we call the *fiber* of $\mathbf{D}$ at t. As usual, $\mathbf{D}$ may be regarded as a variation of open sets $D(t)$ in R^m with real parameter $t\in I$. We write

$$\mathbf{D}: t\to D(t) \quad (t\in I).$$

Assume that $\mathbf{D}$ satisfies the following conditions.

CONDITION 9.1. There exists a point $\xi\in R^m$ such that $\mathbf{D}\supset I\times\{\xi\}$.

CONDITION 9.2. The domain $\mathbf{D}$ is convex in $I\times R^m$; that is:

(1) there exists a double $(\tilde{\mathbf{D}}, \psi(t,x))$ defining the domain $\mathbf{D}$;
(2) for each $t\in I$, the double $(\tilde{D}(t), \psi(t,x))$ defines $D(t)$; and
(3) for each $P\in\tilde{\mathbf{D}}$,

$$\sum_{i,j=0}^{m} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(P) a_i a_j \geq 0$$

for any $a \in R^{m+1}$ with $\|a\| = 1$ such that $\sum_{i=0}^{m} (\partial\psi/\partial x_i)(P) a_i = 0$, where x_o represents the variable t.

By Condition 9.2(3), each $D(t)$ $(t \in I)$ is connected. Because $D(t)$ contains the point ξ mentioned in Condition 9.1, we have the Green function $g(t,x)$ and the Robin constant $\lambda(t)$ for $(D(t), \xi)$. Hence $g(t,x)$ can be written in $D(t)$ in the form

$$g(t,x) = \frac{1}{\|x-\xi\|^{m-2}} + \lambda(t) + h(t,x), \tag{9.4}$$

where $h(t,x)$ is harmonic for x in $D(t)$ and $h(t,\xi) = 0$. By condition 9.2(1) and (2), the variation $\mathbf{D} \cup \partial\mathbf{D}: t \to D(t) \cup \partial D(t)$ $(t \in I)$ is diffeomorphically equivalent to the trivial one. It follows that $g(t,x)$ and $\lambda(t)$ are of class C^3 on $\mathbf{D} \cup \partial\mathbf{D} - I \times \{\xi\}$ and on I. The next lemma is analogous to Lemma 3.1.

LEMMA 9.1. *Assume that the domain $\mathbf{D} \subset I \times R^m$ satisfies Conditions 9.1 and 9.2. Then*

$$\frac{d^2\lambda(t)}{dt^2} \leq \frac{-2}{(m-2)\omega_m} \iint_{D(t)} \left(\sum_{i=1}^{m} \left(\frac{\partial^2 g}{\partial t \partial x_i} \right)^2 \right) dV$$

for all $t \in I$.

Proof. The pseudoconvexity of $\mathbf{D}$ in Section 3 yielded inequality (3.6). Analogously, the convexity of $\mathbf{D}$ in this section yields that, for $i = 1, \ldots, m$,

$$\frac{\partial^2 g}{\partial t^2}\left(\frac{\partial g}{\partial x_i}\right)^2 - 2\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial t \partial x_i}\frac{\partial g}{\partial x_i} + \frac{\partial^2 g}{\partial x_i^2}\left(\frac{\partial g}{\partial t}\right)^2 \leq 0$$

for $(t,x) \in \partial\mathbf{D}$. If we sum up each side for $i = 1, \ldots, m$, then we have

$$\frac{\partial^2 g}{\partial t^2} \leq \left(2\frac{\partial g}{\partial t} \sum_{i=1}^{m} \frac{\partial^2 g}{\partial t \partial x_i}\frac{\partial g}{\partial x_i} \right) \Big/ \|\mathrm{Grad}_{(x)}\, g\|^2$$

for $(t,x) \in \partial\mathbf{D}$. On the other hand, Proposition 3.2 implies

$$\begin{aligned} \frac{d\lambda(t)}{dt} &= \frac{-1}{(m-2)\omega_m} \int_{\partial D(t)} \frac{\partial g}{\partial t}\frac{\partial g}{\partial n_x}\, dS_x; \\ \frac{d^2\lambda(t)}{dt^2} &= \frac{-1}{(m-2)\omega_m} \int_{\partial D(t)} \frac{\partial^2 g}{\partial t^2}\frac{\partial g}{\partial n_x}\, dS_x \end{aligned} \tag{9.5}$$

for $t \in I$. It follows that

$$\begin{aligned} \frac{d^2\lambda(t)}{dt^2} &\leq \frac{-2}{(m-2)\omega_m} \sum_{i=1}^{m} \int_{\partial D(t)} \left\{ \left(\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial t \partial x_i}\frac{\partial g}{\partial x_i} \right) \Big/ \|\mathrm{Grad}_{(x)}\, g\|^2 \right\} \frac{\partial g}{\partial n_x}\, dS_x \\ &= I(t). \end{aligned}$$

Also, the proof of Proposition 3.3 gives, for $i = 1, \ldots, m$,

$$\frac{\partial g}{\partial x_i} dS_x = \frac{\partial g}{\partial n_x}(-1)^{i-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

along $\partial D(t)$. By substituting this in $I(t)$ we obtain

$$I(t) = \frac{-2}{(m-2)\omega_m} \sum_{i=1}^{m} \int_{\partial D(t)} \frac{\partial g}{\partial t} \frac{\partial^2 g}{\partial t \partial x_i}(-1)^{i-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m.$$

By (9.4), $(\partial g/\partial t)(t, x)$ is harmonic for all $x \in D(t)$; thus it follows from Green's formula that

$$\begin{aligned} I(t) &= \frac{-2}{(m-2)\omega_m} \iint_{D(t)} \left\{ \sum_{i=1}^{m} \left(\frac{\partial^2 g}{\partial t \partial x_i} \right)^2 + \frac{\partial g}{\partial t} \Delta_{(x)} \frac{\partial g}{\partial t} \right\} dV \\ &= \frac{-2}{(m-2)\omega_m} \iint_{D(t)} \left\{ \sum_{i=1}^{m} \left(\frac{\partial^2 g}{\partial t \partial x_i} \right)^2 \right\} dV. \end{aligned} \tag{9.6}$$

Lemma 9.1 is proved. □

In the case where **D** was a pseudoconvex domain over $B \times C^n$, Lemma 3.1 implied that $\log(-\lambda(t))$ was a subharmonic function on B. This was based on the fact that pseudoconvexity is invariant under any Hartogs' transformation $T: (t, z) \to (t, \varphi(t)(z - \zeta))$, where $\varphi(t)$ is holomorphic and nonvanishing in B. The proof seems rather qualitative. In the present case where **D** is a convex domain in $I \times R^m$, the transformation of Hartogs' type $T: (t, x) \to (t, \varphi(t)(x - \xi))$, where $\varphi(t)$ is differentiable and nonvanishing in I, does not necessarily preserve convexity. Thus Lemma 9.1 does not directly imply that $\log(-\lambda(t))$ is a convex function on I. However, a quantitative inequality will lead us further.

LEMMA 9.2. *Assume that the domain* **D** *of* $I \times R^m$ *satisfies Conditions 9.1 and 9.2. Then*

$$\frac{d^2 \log(-\lambda(t))}{dt^2} \geq \frac{1}{m-2} \left| \frac{d \log(-\lambda(t))}{dt} \right|^2$$

for all $t \in I$.

Proof. We may suppose that the point ξ mentioned in Condition 9.1 is the origin O in R^m and that the interval I contains the origin 0 in R. Then it suffices to prove Lemma 9.2 for $t = 0$. Let T be any transformation of Hartogs' type:

$$T: (t, x) \to (t, y) = (t, \varphi(t) x),$$

where $\varphi(t)$ is of class C^∞ and $\varphi(t) > 0$ for all $t \in I$. If we put $\mathbf{D}^* = T(\mathbf{D})$, then $\mathbf{D}^*$ becomes a domain in $I \times R^m$ such that $\mathbf{D}^* \supset I \times \{O\}$. In general, $\mathbf{D}^*$ is not convex. For each $t \in I$ we denote by $G(t, y)$ and $\Lambda(t)$ the Green function and the Robin constant for $(D^*(t), O)$. By (5.1) we have

$$G(t, y) = \frac{g(t, x)}{\varphi(t)^{m-2}} \quad \text{and} \quad \Lambda(t) = \frac{\lambda(t)}{\varphi(t)^{m-2}} \tag{9.7}$$

for $x \in D(t)$ and $t \in I$, where $y = \varphi(t)x$. We first show the inequality

$$\frac{d^2\Lambda}{dt^2} - 2\frac{d\log\varphi}{dt} \cdot \frac{d\Lambda}{dt} - (m-2)\varphi\frac{d^2(1/\varphi)}{dt^2} \cdot \Lambda \le 0 \tag{9.8}$$

for $t \in I$.

Let t_o be any point of I. For any $x_o \in \partial D(t_o)$ we let τ denote the tangent plane to the surface $\partial\mathbf{D}$ at the point (t_o, x_o). Note that τ has dimension m as a subset of $I \times R^m$. Since $\mathbf{D}$ is convex, we have $\tau \subset I \times R^m - \mathbf{D}$. If we put $T(t_o, x_o) = (t_o, y_o)$ and $\tau^* = T(\tau)$, then τ^* is an m-dimensional surface in $I \times R^m$ tangent to $\partial\mathbf{D}^*$ at the point $(t_o, y_o) = (t_o, \varphi(t_o)x_o)$ such that $\tau^* \subset I \times R^m - \mathbf{D}$. If we write

$$\tau: a(t-t_o) + \sum_{j=1}^{m} b_j(x_j - x_{oj}) = 0,$$

where $(a, b_1, \dots, b_m)$ is the unit outer normal vector to $\partial\mathbf{D}$ at (t_o, x_o), then τ^* can be expressed in the form

$$\tau^*: a(t-t_o) + \sum_{j=1}^{m} b_j\left(\frac{y_j}{\varphi(t)} - x_{oj}\right) = 0$$

or, equivalently,

$$\tau^*: a\varphi(t)(t-t_o) - (\varphi(t) - \varphi(t_o)) \sum_{j=1}^{m} b_j x_{oj} + \sum_{j=1}^{m} b_j(y_j - y_{oj}) = 0.$$

Since $\partial\mathbf{D}^*$ and τ^* have the same outer normal at (t_o, y_o), it follows that

$$\frac{\partial G}{\partial t} \bigg/ \left(a\varphi - \varphi' \sum_{j=1}^{m} b_j x_{oj}\right) = \frac{1}{b_1}\frac{\partial G}{\partial y_1} = \cdots = \frac{1}{b_m}\frac{\partial G}{\partial y_m}$$

at (t_o, y_o), where $\varphi'(t_o) = (d\varphi/dt)(t_o)$. This implies directly that, for $i = 1, \dots, m$,

$$\left(\sum_{j=1}^{m} b_j x_{oj}\right)\frac{1}{b_i}\frac{\partial G}{\partial y_i} = \sum_{j=1}^{m} \frac{\partial G}{\partial y_j} x_{oj} \quad \text{and}$$
$$a\varphi\frac{1}{b_i}\frac{\partial G}{\partial y_i} = \frac{\partial G}{\partial t} + \varphi' \sum_{j=1}^{m} \frac{\partial G}{\partial y_j} x_{oj} \tag{9.9}$$

at (t_o, y_o). Note that the right-hand sides are independent of i. For each i $(1 \le i \le m)$, let E_i be the (t, y_i)-plane defined by the equations

$$y_1 = y_{o1}, \dots, y_i = y_{oi}, \dots, y_m = y_{om}.$$

If we put $\tau_i^* = \tau^* \cap E_i$, then τ_i^* is a one-dimensional curve passing through (t_o, y_o). It can be written in the form

$$\tau_i^*: y_i = L_i(t) = y_{oi} - \frac{1}{b_i}\left\{a\varphi(t)(t-t_o) - (\varphi(t) - \varphi(t_o)) \sum_{j=1}^{m} b_j x_{oj}\right\}$$

for $t \in I$. Since both $\partial\mathbf{D}^*$ and $\partial D^*(t)$ $(t \in I)$ are smooth, $G(t, y)$ can be extended of class C^3 beyond $\partial\mathbf{D}^*$. Precisely, there exists a neighborhood $\mathbf{V}$ of $\partial\mathbf{D}^*$ in $I \times R^m$ such that $G(t, y)$ is of class C^3 in $\mathbf{V}$ and such that

(9.10) $G(t,y)=0$ on $\partial\mathbf{D}^*$ and $G(t,y)<0$ in $\mathbf{V}-(\mathbf{D}^*\cup\partial\mathbf{D}^*)$.

We consider the restriction $G_i(t)$ of $G(t,y)$ to $\tau_i^*\cap\mathbf{V}$: $G_i(t)=G(t,y_{o1},\dots,L_i(t),\dots,y_{om})$. Hence $G_i(t)$ is a real-valued function on an open interval $I_o\subset I$ which contains t_o. Since τ_i^* is tangent to $\partial\mathbf{D}^*$ from the outside of $\mathbf{D}^*$ at (t_o,y_o), it follows from (9.10) that the function $G_i(t)$ assumes its maximum value 0 on I_o at the point t_o. Hence $G_i'(t_o)=0$ and $G_i''(t_o)\le 0$. By calculating these derivatives, we have

$$\frac{\partial G}{\partial t}+\frac{\partial G}{\partial y_i}L_i'=0 \tag{9.11}$$

and

$$\frac{\partial^2 G}{\partial t^2}+2\frac{\partial^2 G}{\partial t\partial y_i}L_i'+\frac{\partial^2 G}{\partial y_i^2}(L_i')^2+\frac{\partial G}{\partial y_i}L_i''\le 0 \tag{9.12}$$

at (t_o,y_o), where

$$L_i'(t_o)=-\frac{1}{b_i}\left\{a\varphi(t_o)-\varphi'(t_o)\sum_{j=1}^m b_jx_{oj}\right\}$$

and

$$L_i''(t_o)=-\frac{1}{b_i}\left\{2a\varphi'(t_o)-\varphi''(t_o)\sum_{j=1}^m b_jx_{oj}\right\}.$$

Let us eliminate a and $b_1,\dots,b_m$ from inequality (9.12). First, assume that $(\partial G/\partial y_i)(t_o,y_o)\neq 0$. Then by (9.11) we obtain $L_i'=-((\partial G/\partial t)/(\partial G/\partial y_i))$ at (t_o,y_o). Expression (9.9) yields

$$\begin{aligned}\frac{\partial G}{\partial y_i}L_i''&=-\frac{2\varphi'}{\varphi}\left(a\varphi\frac{1}{b_i}\frac{\partial G}{\partial y_i}\right)+\varphi''\left(\sum_{j=1}^m b_jx_{oj}\right)\frac{1}{b_i}\frac{\partial G}{\partial y_i}\\&=-\frac{2\varphi'}{\varphi}\frac{\partial G}{\partial t}-\left(\frac{2\varphi'^2}{\varphi}-\varphi''\right)\sum_{j=1}^m\frac{\partial G}{\partial y_j}x_{oj}\end{aligned}$$

at (t_o,y_o). If we substitute these and $x_{oj}=y_{oj}/\varphi(t_o)$ into (9.12), then

$$\left\{\frac{\partial^2 G}{\partial t^2}-\frac{2\varphi'}{\varphi}\frac{\partial G}{\partial t}+\frac{\varphi\varphi''-2\varphi'^2}{\varphi^2}\sum_{j=1}^m\frac{\partial G}{\partial y_j}y_{oj}\right\}\left(\frac{\partial G}{\partial y_i}\right)^2+\frac{\partial^2 G}{\partial y_i^2}\left(\frac{\partial G}{\partial t}\right)^2 \le 2\frac{\partial^2 G}{\partial t\partial y_i}\frac{\partial G}{\partial t}\frac{\partial G}{\partial y_i} \tag{9.13}$$

at (t_o,y_o). Next assume that $(\partial G/\partial y_i)(t_o,y_o)=0$. Then, by (9.11), we get $(\partial G/\partial t)(t_o,y_o)=0$. Hence, (9.13) is also true in this case. Summation of each side for $i=1,\dots,m$ shows that

$$\left\{\frac{\partial^2 G}{\partial t^2}-2(\log\varphi)'\frac{\partial G}{\partial t}-\varphi\left(\frac{1}{\varphi}\right)''\sum_{j=1}^m\frac{\partial G}{\partial y_j}y_{oj}\right\}\|\mathrm{Grad}_{(y)}G\|^2+\left(\frac{\partial G}{\partial t}\right)^2\Delta_{(y)}G \le 2\frac{\partial G}{\partial t}\sum_{i=1}^m\frac{\partial^2 G}{\partial t\partial y_i}\frac{\partial G}{\partial y_i}$$

at (t_o, y_o). Since $G(t,y)$ is harmonic for y, and since y_o is an arbitrary point of $\partial D^*(t_o)$, it follows that

$$\frac{\partial^2 G}{\partial t^2} - 2(\log\varphi)'\frac{\partial G}{\partial t} - \varphi\left(\frac{1}{\varphi}\right)''\sum_{j=1}^{m}\frac{\partial G}{\partial y_j}y_j \leq \left(2\frac{\partial G}{\partial t}\sum_{i=1}^{m}\frac{\partial^2 G}{\partial t\partial y_i}\frac{\partial G}{\partial y_i}\right)\Big/\|\mathrm{Grad}_{(y)}G\|^2$$

for $t=t_o$ and all $y\in\partial D^*(t_o)$. After multiplying both sides by

$$d\Omega_y = \frac{-1}{(m-2)\omega_m}\frac{\partial G(t_o,y)}{\partial n_y}dS_y > 0$$

for $y\in\partial D^*(t_o)$, we integrate over $\partial D^*(t_o)$:

$$\int_{\partial D^*(t_o)}\frac{\partial^2 G}{\partial t^2}(t_o,y)\,d\Omega_y - 2(\log\varphi)'(t_o)\int_{\partial D^*(t_o)}\frac{\partial G}{\partial t}(t_o,y)\,d\Omega_y - \varphi(t_o)\left(\frac{1}{\varphi}\right)''(t_o)\int_{\partial D^*(t_o)}\left\{\sum_{i=1}^{m}\frac{\partial G}{\partial y_i}(t_o,y)y_i\right\}d\Omega_y \leq 2\int_{\partial D^*(t_o)}\left\{\left(\sum_{i=1}^{m}\frac{\partial G}{\partial t}\frac{\partial^2 G}{\partial t\partial y_i}\frac{\partial G}{\partial y_i}\right)\Big/\|\mathrm{Grad}_{(y)}G\|^2\right\}_{(t_o,y)}d\Omega_y.$$

Identities (9.5) and (9.6) remain valid when we replace t, x, $D(t)$, $\partial D(t)$, $g(t,x)$, and $\lambda(t)$ by t, y, $D^*(t)$, $\partial D^*(t)$, $G(t,y)$, and $\Lambda(t)$. It follows that

$$\Lambda''(t_o) - 2(\log\varphi)'(t_o)\cdot\Lambda'(t_o) - \varphi(t_o)\left(\frac{1}{\varphi}\right)''(t_o)\int_{\partial D^*(t_o)}\left\{\sum_{i=1}^{m}\frac{\partial G}{\partial y_i}(t_o,y)y_i\right\}d\Omega_y \leq \frac{-2}{(m-2)\omega_m}\iint_{D^*(t_o)}\left\{\sum_{i=1}^{m}\left(\frac{\partial^2 G}{\partial t\partial y_i}(t_o,y)\right)^2\right\}dV \leq 0.$$

Because $G(t_o,y)=0$ for $y\in\partial D^*(t_o)$, the integral in the third term on the left-hand side can be written in the form

$$I = (m-2)\int_{\partial D^*(t_o)}\left\{G(t_o,y) + \frac{1}{m-2}\sum_{i=1}^{m}\frac{\partial G}{\partial y_i}(t_o,y)y_i\right\}d\Omega_y.$$

We observe that the integrand of the right-hand side is harmonic for y in the whole domain $D^*(t_o)$, and that it attains the value $\Lambda(t_o)$ at $y=0$ (cf. the first term of $H(a,\zeta_o,z)$ in (8.11)). It follows that $I=(m-2)\Lambda(t_o)$. Consequently,

$$\Lambda'' - 2(\log\varphi)'\cdot\Lambda' - (m-2)\varphi\left(\frac{1}{\varphi}\right)''\cdot\Lambda \leq 0$$

at $t=t_o$. Inequality (9.8) is thus proved. □

Next we prove the inequality in Lemma 9.2. Consider the following transformation of Hartogs' type: $(t,x)\to(t,y)=(t,\varphi(t)x)$, where

$$\varphi(t) = \exp\left\{\frac{1}{m-2}(\alpha(t-t_o)+\beta)\right\}$$

with $\alpha=(\log(-\lambda))'(t_o)$ and $\beta=\log(-\lambda)(t_o)$ for $t\in I$. Then

$$\varphi(t_o)\left(\frac{1}{\varphi}\right)''(t_o)=\left(\frac{\alpha}{m-2}\right)^2.$$

By virtue of (9.7), we have $\Lambda(t)=\lambda(t)\exp\{-(\alpha(t-t_o)+\beta)\}$ for $t\in I$. Therefore $\Lambda(t_o)=-1$, $\Lambda'(t_o)=0$, and $\Lambda''(t_o)=-(\log(-\lambda))''(t_o)$. Substituting these in (9.8), we find

$$(\log(-\lambda))''(t_o)\geq\frac{1}{m-2}\{(\log(-\lambda))'(t_o)\}^2.$$

Lemma 9.2 is proved. □

Now that Lemma 9.1 and 9.2 are established, we can obtain the following results, which parallel those in the preceding sections.

THEOREM 9.1. *Let* $\mathbf{D}$ *be a convex domain of* $I\times R^m$. *Let* $\xi: t\to\xi(t)$ $(t\in I)$ *be a section of* $\mathbf{D}$ *defined on* I *of the form* $\xi(t)=at+b$, *where* $a, b\in R^m$. *For each* $t\in I$ *we denote by* $\lambda(t)$ *the Robin constant for* $(D(t),\xi(t))$. *Then* $\log(-\lambda(t))$ *is a convex function on* I.

Proof. Consider the transformation $T_1:(t,x)\to(t,y)=(t,x-(at+b))$ which maps ξ to the constant zero section O. If we put $\mathbf{D}_1=T_1(\mathbf{D})$ and denote by $\lambda_1(t)$ the Robin constant for $(D_1(t),O)$, then $\lambda_1(t)=\lambda(t)$ for $t\in I$. Since $\mathbf{D}_1$ is convex in $I\times R^m$, it follows from Lemma 9.2 that $\log(-\lambda_1(t))$ is convex on I, and thus so is $\log(-\lambda(t))$. Theorem 9.1 is proved. □

LEMMA 9.3 (Rigidity Lemma). *Under the same hypothesis as in Lemma 9.1, if* $(d^2\lambda/dt^2)(t_o)=0$ *for some* $t_o\in I$, *then* $(\partial g/\partial t)(t_o,x)=0$ for all $x\in D(t_o)$.

COROLLARY 9.1. *Under the same circumstances as in Lemma 9.1, if* $\lambda(t)=\alpha t+\beta$ $(t\in I)$ *for some* $\alpha,\beta\in R$, *then* $\mathbf{D}$ *is identical with the product* $I\times D(t_o)$, *where* t_o *is a fixed point in* I.

The proofs of Lemma 9.3 and Corollary 9.1 are similar to those of Lemma 7.1 and Corollary 7.1.

THEOREM 9.2. *Let* D *be a convex domain of* R^m *with smooth boundary* ∂D. *For each* $\xi\in D$ *we denote by* $\Lambda(\xi)$ *the Robin constant for* (D,ξ). *Then* $\log(-\Lambda(\xi))$ *and* $-\Lambda(\xi)$ *are real analytic exhaustion functions in* D *having positive definite Hessian matrix.*

Proof. Lemma 9.3 yields the statements concerning $-\Lambda(\xi)$ in Theorem 9.2 in the same manner that Lemma 7.1 yielded those concerning $-\Lambda(\xi)$ in Theorem 8.1. It remains to verify that $\log(-\Lambda(\xi))$ has a positive Hessian in D, that is,

$$H(a,\xi)=\sum_{i,j=1}^{m}\frac{\partial^2\log(-\Lambda(\xi))}{\partial\xi_i\partial\xi_j}a_ia_j>0$$

for all $\xi \in D$ and $a \in R^m$ with $\|a\| = 1$. To this end, fix ξ in D and take a number $\rho > 0$ such that $\xi + at \in D$ for all $t \in I = (-\rho, \rho)$. We transform the product $I \times D$ by the transformation $T_1: (t, x) \to (t, y) = (t, x - \xi - at)$, and put $\mathbf{D}_1 = T_1(I \times D)$. Then $\mathbf{D}_1 \supset I \times \{O\}$. For each $t \in I$, let $\lambda_1(t)$ denote the Robin constant for $(D_1(t), O)$. Then $\lambda_1(t) = \Lambda(\xi + at)$ and $H(a, \xi) = (\log(-\lambda_1))''(0)$. Since D is a convex domain of R^m with smooth boundary, the domain $\mathbf{D}_1$ of $I \times R^m$ satisfies Condition 9.2. It follows from Lemma 9.2 that

$$(\log(-\lambda_1))''(0) \geq \frac{1}{m-2}\{(\log(-\lambda_1))'(0)\}^2$$

or, equivalently,

$$(9.14) \qquad \sum_{i,j=1}^{m} \frac{\partial^2 \log(-\Lambda(\xi))}{\partial \xi_i \partial \xi_j} a_i a_j \geq \frac{1}{m-2}\left(\sum_{i=1}^{m} \frac{\partial \log(-\Lambda(\xi))}{\partial \xi_i} a_i\right)^2.$$

In the case where $(\log(-\lambda_1))'(0) \neq 0$, we thus have $(\log(-\lambda_1))''(0) > 0$. In the case where $(\log(-\lambda_1))'(0) = 0$, we directly have

$$(\log(-\lambda_1))''(0) = \frac{(-\Lambda(\xi + at))''(0)}{(-\Lambda(\xi))}.$$

The right-hand side is positive, for $-\Lambda(\xi)$ has a positive Hessian in D. Therefore we always have $H(a, \xi) > 0$ and Theorem 9.2 is proved. □

We thus have the next corollary.

COROLLARY 9.2. *Under the same circumstances as in Theorem 9.2, the quadratic form*

$$ds^2 = \sum_{i,j=1}^{m} \frac{\partial^2 \log(-\Lambda(x))}{\partial x_i \partial x_j} dx_i \otimes dx_j$$

is a Riemannian metric in D.

In [19] we study the properties of this metric. Then inequality (9.14) will be useful.

We now apply Lemma 9.2 to study variations of the capacity $c(t)$ of a conductor $(S(t))$ when $(S(t))$ varies in R^3 with real parameter t. Using the sphere $\pi: \|x\|^2 = 1$, we use the notation $x^* = x/\|x\|^2$ $(x \neq 0, \infty)$, $O^* = \infty$, $\infty^* = O$, and $A^* = \{x^* \in R^3 \cup \{\infty\} \mid x \in A\}$ for $A \subset R^3 \cup \{\infty\}$. Let I be an open interval. Consider two transformations of $I \times (R^3 \cup \{\infty\})$ as follows:

$$L: (t, x) \to (t, L(x)) = (t, x^*),$$
$$M: (t, x) \to (t, M(t, x)) = (t, A(t)x + b(t)),$$

where $A(t)$ is a 3×3 matrix with $A(t)^t \cdot A(t) = E_3$ (the identity matrix), $b(t) \in R^3$, and $M(t, \infty) = \infty$. We call L a *symmetric* transformation with respect to π and M a *Euclidean motion*. For each $t \in I$, put $(\tilde{S}(t)) = M(t, (S(t)))$. If we denote by $c(t)$ (resp. $\tilde{c}(t)$) the capacity of the conductor $(S(t))$ (resp. $(\tilde{S}(t))$), then we easily have $c(t) = \tilde{c}(t)$. The next definition will be needed.

DEFINITION 9.1. A domain $\mathbf{E}$ of $I\times(R^3\cup\{\infty\})$ is said to be *symmetrically convex* if there exists a Euclidean motion M such that $L\circ M(\mathbf{E})$ is a convex domain in $I\times R^m$.

Now let $(\mathbf{S})\colon t\to(S(t))$ $(t\in I)$ be a variation of a conductor $(S(t))$ with total charge $+1$ which is bounded by the smooth surface $S(t)$ in R^3. We denote by $\mu(t,x)$ the equilibrium distribution on $S(t)$. Then $\mu(t,x)$ induces an electric field $e(t,x)$ spread over $R^3-(S(t))$. We have the total energy $\|e(t)\|^2$ of $e(t,x)$ and the capacity $c(t)$ of $(S(t))$. As usual, the variation $(\mathbf{S})$ is regarded as a subset $(\mathbf{S})$ of $I\times R^3$:

$$(\mathbf{S})=\{(t,x)\in I\times R^3\mid t\in I \text{ and } x\in(S(t))\}.$$

Consider the complement $\mathbf{E}$ of $(\mathbf{S})$ in $I\times(R^3\cup\{\infty\})$:

$$\mathbf{E}=I\times(R^3\cup\{\infty\})-(\mathbf{S}). \tag{9.15}$$

If, for $t\in I$, we denote by $E(t)$ the fiber at $\mathbf{E}$ at t, then

$$R^3\cup\{\infty\}=(S(t))\cup E(t) \quad\text{and}\quad S(t)\cap E(t)=\emptyset.$$

Since (9.2) implies that $\lim_{x\to\infty}e(t,x)=0$, we put $e(t,\infty)=0$. Hence the variation $\mathbf{E}\colon t\to E(t)$ $(t\in I)$ is the same as that of the electric field $\mathbf{e}\colon t\to e(t,x)$ $(t\in I)$. We now show the next theorem.

THEOREM 9.3. *Let $(\mathbf{S})\colon t\to(S(t))$ $(t\in I)$ be a variation of conductors such that the complement* $\mathbf{E}$ *of* $(\mathbf{S})$ *in $I\times(R^3\cup\{\infty\})$ is symmetrically convex. Then the capacity $c(t)$ of $(S(t))$ is a logarithmically convex function on I, and the total energy $\|e(t)\|^2$ is a logarithmically concave function on I.*

Proof. Suppose that $\mathbf{E}$ defined by (9.15) is symmetrically convex. Then there exists a Euclidean motion M of $I\times(R^3\cup\{\infty\})$ such that the domain $\mathbf{D}=L\circ M(\mathbf{E})$ of $I\times R^3$ satisfies Condition 9.2. Since each $(S(t))$ $(t\in I)$ is bounded, the fiber $D(t)$ of $\mathbf{D}$ at t contains the origin O. We thus have the Robin constant $\lambda(t)$ for $(D(t),O)$. By Lemma 9.2, $\log(-\lambda(t))$ is a convex function on I. For each $t\in I$, set $M(t,(S(t)))=(\tilde S(t))\subset\subset R^3$ and denote by $\tilde c(t)$ the capacity of $(\tilde S(t))$. Then $\tilde c(t)=c(t)$ and

$$D(t)=L(t,M(t,E(t)))=\{R^3\cup\{\infty\}-(\tilde S(t))\}^*.$$

Since $(\tilde S(t))-\tilde S(t)\ni O$, we see from Proposition 9.1 that $\tilde c(t)=-\lambda(t)=\omega_3/\|e(t)\|^2$. It follows that $\log c(t)$ and $-\log\|e(t)\|^2$ are convex on I. Theorem 9.3 is proved. □

References

1. L. V. Ahlfors ana L. Sario, *Riemann surfaces,* Princeton Univ. Press, Princeton, N.J., 1960.
2. H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderichen,* Springer, Berlin, 1934.

3. G. Faber, *Potentialtheorie und konforme Abbildung,* Sitzungsber. Math.-Physik Akad. Wiss. 49-64, (1920).
4. O. Fujita, *Sur les systèmes de fonctions holomorphes de plusieurs variables complexes. (III) Uniformité des fibres d'une application,* J. Math. Kyoto Univ. 27 (1987), 679-708.
5. J. Hadamard, *Mémoire sur les problème d'analyse relatif à l'équilibre des plaques élastiques éncastrées,* Mémoires présentés par divers savants à l'Académie des Science 33 (1907). (Ouvres II, C.R.N.S., Paris, 1968, pp. 515-641).
6. A. Kaneko, *On the analyticity of the locus of singularity of real analytic solutions with minimal dimension,* Nagoya Math. J. 104 (1986), 63-84.
7. T. Nishino, *On value distributions of analytic functions of two complex variables,* Sûgaku 32 (1980), 230-246 (Japanese).
8. K. Oka, *Théorie des fonctions analytiques de plusieurs variables, (IX) Domains finis sans point critique intérieur,* Japan. J. Math. 27 (1953), 97-155.
9. G. Robin, *Sur la distribution de l'éléctricité à la surface des conducteurs fermés et des conducteurs ouverts,* Ann. Sci. École Norm. Sup. (3) III (1886), 3-58.
10. M. Suzuki, *Sur les intégrales premières de certaines feuilletages analytiques complexes.* Fonctions de Plusieurs Variables Complexes, III (Sém. François Norguet, 1975-1977), 53-79, Lecture Notes in Math., 670, Springer, Berlin, 1978.
11. ———, *A function-theoretic proof to an analogue of Mordell's conjecture for algebraic curves over function fields,* preprint.
12. J. Wermer, *Potential theory and function algebras.* Visiting Scholars' Lectures—1980 (Lubbock, Tex., 1980), 113-125, Math. Ser., 14, Texas Tech. Univ., Lubbock, Tex., 1981.
13. H. Yamaguchi, *Sur une uniformité des surfaces constantes,* J. Math. Kyoto Univ. 13 (1973), 417-433.
14. ———, *Sur le mouvement des constants de Robin,* J. Math. Kyoto Univ. 15 (1975), 53-71.
15. ———, *Parabolicité d'une fonction entière,* J. Math. Kyoto Univ. 16 (1976), 71-92.
16. ———, *Calcul des variations analytiques,* Japan. J. Math. (N.S.) 7 (1981), 319-377.
17. ———, *Variations of pseudoconvex domains,* Proc. Japan Acad. Ser. A Math. Sci. 59A (1983), 289-292.
18. ———, *Variations of pseudoconvex domains in the complex manifold,* Proc. Japan Acad. Ser. A Math. Sci. 63A (1987), 17-20.
19. N. Levenberg and H. Yamaguchi, *Metric induced by Robin constants,* to appear.

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