On the Fourier Series of a Step Function

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Introduction

In recent years there has been a growth of interest in the interplay between Fourier series and harmonic mappings of the disc, with particular emphasis on connections with the topology of curves. Examples of this are:

(1) Hall's work [3] establishing the correct value of the Heinz constant concerning the Fourier coefficients of a circle mapping;
(2) the paper by Clunie and the author [2] taking the first steps towards establishing a general theory of univalent harmonic functions;
(3) several papers by Hengartner and Schober, including one containing a harmonic version of the Riemann mapping theorem [4] and a second studying a class of open harmonic mappings and their boundary behaviour [5]; and
(4) the author's paper [7] proving Shapiro's conjecture on the Fourier coefficients of an N-fold mapping of the circle and the discovery of an unexpected connection with certain classically defined multivalent analytic functions.

A result which continues to dominate this theory, and which has received a number of new proofs, is that due to Kneser [6], Rado, and (independently) Choquet [1] concerning the univalence of the harmonic extension into the disc of a homeomorphism of the circle onto a convex curve. Indeed, much of the current work either uses this result in a direct way or develops a variety of generalizations and analogous ideas. This function-theoretic development of harmonic mappings in the plane leads naturally to a variety of extremal problems. In a number of cases the extremal functions turn out to have boundary values which are step functions on the unit circle. For example, the extremal function giving the sharp value for the Heinz constant is

$$f(t)=\omega_k \quad (2\pi k/3<t<2\pi(k+1)/3,\ k=0,1,2),$$

where $\omega_k=\exp(2\pi ik/3)$ [3]. The harmonic extension of this function is a homeomorphism of the disc onto the interior of the triangle formed by the vertices ω_k. Furthermore, Hengartner and Schober [5] show that step-function solutions arise naturally as a class of solutions of their differential equation

Received October 26, 1988. Revision received March 2, 1989.
Michigan Math. J. 36 (1989).

and, moreover, that certain geometrical hypotheses force such solutions. All this makes it desirable to study in detail the connection between a step function and its harmonic extension.

In this paper I will show that there is a correspondence between step functions on the circle and rational functions which are ratios of finite Blaschke products. Furthermore, the degrees of these Blaschke products depend in general on the number of values attained by the step function and, more interestingly, on the geometry of the distribution of these values—at least in certain interesting cases. The general correspondence, proved in Section 1, is essentially algebraic. However, the geometric connection lies deeper and is proved in Section 2 as an application of the Kneser–Rado–Choquet theorem and more generally of the author's results on N-fold mappings [7]. However the density of suitable classes of step functions implies that the result we obtain is essentially equivalent to these earlier results. Therefore it would be of considerable interest to explore more deeply the interaction between the geometry of the step values and the zero-pole distribution of the associated Blaschke products. Also, since finite Blaschke products and their ratios occur in a number of function-theoretic situations, their association with step functions and harmonic mappings takes on an added significance.

1. The Harmonic Extension of a Step Function

Let T denote the unit circle and U the unit disc. We consider a step function $f: T \to \mathbf{C}$ defined by

$$f(e^{it}) = c_k \quad (t_{k-1} < t < t_k,\ 1 \le k \le n), \tag{1.1}$$

where $t_0 < t_1 < t_2 < \cdots < t_n = t_0 + 2\pi$. The harmonic extension of f to U is given by

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\operatorname{Re}\frac{1+ze^{-it}}{1-ze^{-it}}\right) f(e^{it})\,dt$$

and takes the form $f(z) = \bar{g}(z) + h(z)$, where g and h are analytic in U and $g(0) = 0$. We obtain

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{it})}{1-ze^{-it}}\,dt = \frac{1}{2\pi}\sum_{k=1}^{n} c_k \int_{t_{k-1}}^{t_k}\frac{dt}{1-ze^{-it}}.$$

Differentiating, we have

$$\begin{aligned} h'(z) &= \frac{1}{2\pi}\sum_{k=1}^{n} c_k \int_{t_{k-1}}^{t_k}\frac{e^{-it}}{(1-ze^{-it})^2}\,dt \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n} c_k\left(\frac{1}{z-\zeta_k} - \frac{1}{z-\zeta_{k-1}}\right), \end{aligned}$$

where $\zeta_k = e^{it_k}$ $(0 \le k \le n)$, so that $\zeta_0 = \zeta_n$. Writing $c_{n+1} = c_1$, we obtain

$$h'(z) = \sum_{k=1}^{n}\frac{\alpha_k}{z-\zeta_k}, \tag{1.2}$$

where

$$\alpha_k = \frac{1}{2\pi i}(c_k - c_{k+1}) \quad (1 \le k \le n). \tag{1.3}$$

We deduce that

$$\sum_{k=1}^{n} \alpha_k = 0. \tag{1.4}$$

Similarly, we have

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{ze^{-it}}{1-ze^{-it}} \bar{f}(e^{it})\, dt,$$

from which we obtain

$$g'(z) = -\sum_{k=1}^{n} \frac{\bar{\alpha}_k}{z - \zeta_k}. \tag{1.5}$$

In the case $n=1$, f is constant and $g'=h'=0$. For $n \ge 2$ we have, from (1.2) and (1.4),

$$\begin{aligned} h'(z) &= \sum_{k=1}^{n-1} \frac{\alpha_k}{z-\zeta_k} - \sum_{k=1}^{n-1} \frac{\alpha_k}{z-\zeta_n} \\ &= \sum_{k=1}^{n-1} \alpha_k \frac{\zeta_k - \zeta_n}{(z-\zeta_k)(z-\zeta_n)}. \end{aligned}$$

From this we obtain, since $|\zeta_k| = 1$ $(1 \le k \le n)$,

$$\bar{h}'(1/\bar{z}) = z^2 g'(z). \tag{1.6}$$

Let

$$S(z) = \prod_{k=1}^{n} (z - \zeta_k).$$

Then

$$z^n \bar{S}(1/\bar{z}) = S(z) \prod_{k=1}^{n} (-\bar{\zeta}_k).$$

Furthermore, we can write

$$h'(z) = \frac{P(z)}{S(z)} \quad \text{and} \quad g'(z) = \frac{Q(z)}{S(z)},$$

where P and Q are polynomials of degree at most $n-1$. We have

$$\frac{\bar{Q}(1/\bar{z})}{\bar{S}(1/\bar{z})} = \bar{g}'(1/\bar{z}) = z^2 h'(z) = z^2 \frac{P(z)}{S(z)}.$$

Hence

$$z^{n-2} \bar{Q}(1/\bar{z}) = P(z) \prod_{k=1}^{n} (-\bar{\zeta}_k). \tag{1.7}$$

Hence P and Q are polynomials of degree at most $n-2$. If Q has degree q with q_1 zeros at the origin and q_2 zeros away from the origin, then P has degree p, where $p = n-2-q_1$. Furthermore, P has $n-2-q$ zeros at the origin and q_2 zeros away from the origin. The q_2 zeros of P are the conjugates relative to the unit circle of the q_2 zeros of Q. This means that

(1.8) $$\frac{g'(z)}{h'(z)}=\frac{Q(z)}{P(z)}=R(z),$$

where $R(z)$ is a rational function of degree at most $n-2$ such that

(1.9) $$|R(z)|=1 \quad (|z|=1).$$

$R(z)$ takes the form of a Blaschke product formed with the q_2 zeros of Q away from the origin, together with a zero-pole term z^{q+q_1-n+2}: zero or pole depending on the sign of $q+q_1-n+2$.

Let us now show that we can reconstruct the step function $f(e^{it})$ up to an arbitrary additive constant from the following two pieces of information:

(1) an arbitrary polynomial Q of degree at most $n-2$;
(2) a polynomial $S(z)=\prod_{k=1}^{n}(z-\zeta_k)$ with n distinct zeros ζ_k on the unit circle.

We define

(1.10) $$P(z)=z^{n-2}\bar{Q}(1/\bar{z})\prod_{k=1}^{n}(-\zeta_k)$$

and set

(1.11) $$G(z)=\frac{Q(z)}{S(z)}, \quad H(z)=\frac{P(z)}{S(z)}.$$

Then we can write

(1.12) $$G(z)=-\sum_{k=1}^{n}\frac{\bar{\alpha}_k}{z-\zeta_k},$$

where $-\bar{\alpha}_k=Q(\zeta_k)/S'(\zeta_k)$ $(1\le k\le n)$. Now the coefficient of z^{n-1} in the polynomial $S(z)G(z)$ is $-\sum_1^n\bar{\alpha}_k$, and so we deduce, since $SG=Q$ has degree at most $n-2$, that

(1.13) $$\sum_{k=1}^{n}\alpha_k=0.$$

With the help of this and some simple algebra we easily obtain

(1.14) $$H(z)=\sum_{k=1}^{n}\frac{\alpha_k}{z-\zeta_k}.$$

However, even without this calculation we can now define c_k by solving the equations

(1.15) $$c_k-c_{k+1}=2\pi i\alpha_k \quad (1\le k\le n)$$

with $c_{n+1}=c_1$. The solution is $c_1=\mu$ and

(1.16) $$c_k=\mu-2\pi i\sum_{j=1}^{k-1}\alpha_j \quad (2\le k\le n+1),$$

where μ is an arbitrary constant. With the ζ_k assumed to be ordered on the unit circle, so that $\zeta_k=e^{it_k}$, where $t_0<t_1<\cdots<t_n=t_0+2\pi$, we find by the

earlier algebra that the step function

$$f(e^{it})=c_k \quad (t_{k-1}<t<t_k,\ 1\le k\le n) \tag{1.17}$$

extends harmonically into U as $\bar{g}+h$, where $g'=G$ and $h'=H$. Thus, up to an arbitrary additive constant, f is uniquely determined in U from (1) and (2).

Notice that the relations $g'=G$ and $h'=H$ imply that the functions G and H defined by (1.11) are automatically primitives in U. This is true in a larger domain than U. Let Γ_k denote the arc of the unit circle (ζ_{k-1},ζ_k) and D_k the domain

$$D_k=U\cup\Gamma_k\cup\{|z|>1\}, \tag{1.18}$$

so that D_k is the plane cut along the closed complementary circular arc to Γ_k. Let γ be a rectifiable loop in D_k. Then

$$\frac{1}{2\pi i}\int_\gamma G(z)\,dz=-\sum_{j=1}^{n}\bar{\alpha}_j n(\gamma,\zeta_j)=0,$$

since $n(\gamma,\zeta_j)$ has the same value for each j, as all the ζ_j lie on a connected set in the complement of γ. Hence G is a primitive in D_k and similarly H. Thus f has a harmonic extension across Γ_k into D_k. This is unique (up to the additive constant) in U, but may vary in $\{|z|>1\}$ depending across which arc Γ_k we continue f. In fact, if we denote by f_k the continuation of f into $\{|z|>1\}$ across Γ_k, it is easily shown that $f_k-f_j=2(c_k-c_j)$. Summarising, we have the following.

THEOREM 1. *Let $f(z)=\bar{g}(z)+h(z)$ $(z\in U)$ denote the harmonic extension into U of a step function on T given by*

$$f(e^{it})=c_k \quad (t_{k-1}<t<t_k,\ 1\le k\le n), \tag{1.19}$$

where $t_0<t_1<\cdots<t_n=t_0+2\pi$. Then, writing

$$S(z)=\prod_{k=1}^{n}(z-e^{it_k}), \tag{1.20}$$

we have for $n\ge 2$ that

$$g'(z)=\frac{Q(z)}{S(z)} \quad \textit{and} \quad h'(z)=\frac{P(z)}{S(z)}, \tag{1.21}$$

where P and Q are polynomials of degree at most $n-2$. Furthermore,

$$\frac{g'(z)}{h'(z)}=\frac{Q(z)}{P(z)}=R(z) \tag{1.22}$$

satisfies

$$|R(z)|=1 \quad (|z|=1), \tag{1.23}$$

and so takes the form of a Blaschke product of degree at most $n-2$. Conversely, if $Q(z)$ is an arbitrary polynomial of degree at most $n-2$ and if

$S(z)$ has the form (1.20), *then there exists a polynomial $P(z)$ of degree at most $n-2$ and a step function of the form* (1.19) *unique up to an additive constant such that* (1.21), (1.22), *and* (1.23) *are satisfied. The harmonic function f in U can be continued across any one of the arcs $(e^{it_{k-1}}, e^{it_k})$ into $\{|z|>1\}$ to give a function harmonic in the domain D_k of* (1.18).

2. Convex Polygons

Let $c_1, c_2, \ldots, c_n$ be $n \geq 2$ points in $\mathbf{C}$ lying on a convex curve J and assumed arranged in counter-clockwise order on J. Equivalently, the polygon $[c_1, c_2, \ldots, c_n, c_1]$ is convex. We assume that $c_1 \neq c_n$, but for greater generality we do not assume that all the points are distinct. We consider a step function of the form

$$f(e^{it}) = c_{r(k)} \quad (t_{k-1} \leq t < t_k) \tag{2.1}$$

for $1 \leq k \leq n$, where $t_0 < t_1 < \cdots < t_n = t_0 + 2\pi$ and where $k \to r(k)$ is a permutation of $\{1, 2, \ldots, n\}$. Thus f attains all the values c_k, but not necessarily in the order of the polygon.

LEMMA 1. *Let $F\colon T \to J$ be a positively oriented homeomorphism. Then there exist a nondecreasing step function $\varphi\colon \mathbf{R} \to \mathbf{R}$ and a nonincreasing step function $\psi\colon \mathbf{R} \to \mathbf{R}$ such that*

$$f(e^{it}) = F(e^{i\varphi(t)}) = F(e^{i\psi(t)}). \tag{2.2}$$

Proof. We extend $r(k)$ to the set of integers $\mathbf{Z}$ by the rule

$$r(k) = r(j) \quad \text{if } k \equiv j \pmod{n}. \tag{2.3}$$

Then $r(k)$ is periodic of period n and $r\colon \mathbf{Z} \to \{1, 2, \ldots, n\}$ is surjective. Notice that $r(k+1) \neq r(k)$ for any k. We define a function $\rho(k)$ inductively by the following recursion relationship. With $\rho(1) = r(1)$ set

$$\rho(k+1) = \begin{cases} \rho(k) + r(k+1) - r(k) & \text{if } r(k+1) > r(k), \\ \rho(k) + r(k+1) - r(k) + n & \text{if } r(k+1) < r(k). \end{cases} \tag{2.4}$$

Then $\rho(k)$ is strictly increasing and $\rho(k) \equiv r(k) \pmod{n}$. Also note that $\rho(n+1) = \rho(1) + \mu n$, where $1 \leq \mu \leq n-1$. We can write

$$c_k = F(e^{is_k}) \quad (1 \leq k \leq n+1), \tag{2.5}$$

where $s_1 \leq s_2 \leq \cdots \leq s_{n+1} = s_1 + 2\pi$. We define

$$\varphi(t) = s_{\rho(k)} \quad (t_{k-1} \leq t < t_k) \tag{2.6}$$

for $1 \leq k \leq n$, and $\varphi(t_n) = s_{\rho(n+1)}$, where, in general, if $j \equiv k \pmod{n}$ and $1 \leq k \leq n$, we have set

$$s_j = s_k + 2\pi \frac{j-k}{n}. \tag{2.7}$$

Then $\varphi(t)$ is increasing on $\mathbf{R}$ and $f(e^{it})=F(e^{i\varphi(t)})$. Furthermore, we note that

$$\varphi(t_0+2\pi)-\varphi(t_0)=s_{\rho(n+1)}-s_{\rho(1)}=2\pi\mu. \tag{2.8}$$

Next we define

$$\sigma(k)=\rho(k)-(k-1)n. \tag{2.9}$$

Then $\sigma(k)$ is strictly decreasing and $\sigma(k)\equiv r(k) \pmod n$. Also, $\sigma(n+1)=\sigma(1)+(\mu-n)n=\sigma(1)-\nu n$, where $1\le\nu\le n-1$. We set

$$\psi(t)=s_{\sigma(k)} \quad (t_{k-1}\le t<t_k) \tag{2.10}$$

for $1\le k\le n$ and $\psi(t_n)=s_{\sigma(n+1)}$. Then $\psi(t)$ is decreasing and

$$f(e^{it})=F(e^{i\psi(t)}).$$

Furthermore,

$$\psi(t_0+2\pi)-\psi(t_0)=-2\pi\nu. \tag{2.11}$$

□

REMARK 1. Lemma 1 is a detailed formulation of a simple but important fact concerning the jumps of a step function (where the step values are taken to lie on a closed curve): namely, a jump can be interpreted either as a forward jump along one arc of the curve or as a backward jump along the complementary arc. This observation was made to the author in conversations with R. R. Hall. The important numbers to arise out of the lemma are the periodic factors μ and ν related by $\mu+\nu=n$. These numbers control the zero-pole distribution of the Blaschke products g'/h' arising in Theorem 1. With the convexity hypothesis of this section we reformulate Theorem 1 to include this additional information.

THEOREM 2. *Let f be a step function on T given by* (2.1), *where the c_k satisfy the hypotheses of that paragraph. Then the harmonic extension to U of $f=\bar g+h$ satisfies, for a real constant λ,*

$$\frac{g'(z)}{h'(z)}=e^{i\lambda}\frac{B_2(z)}{B_1(z)}, \tag{2.12}$$

where B_1 and B_2 are Blaschke products with zeros in U. B_1 has degree at most $\mu-1$ and B_2 has degree at most $\nu-1$, where μ and ν are the periods satisfying (2.8) *and* (2.11) *respectively. In particular, g' has at most $\nu-1$ zeros in U and h' has at most $\mu-1$ zeros in U.*

Proof. Since the curve J is convex and $f(e^{it})=F(e^{i\varphi(t)})$, where φ is increasing and satisfies (2.8), we can apply Theorem A of [7] to deduce that h' has at most $\mu-1$ zeros in U and that

$$|g'(z)|\le\left|\frac{h'(z)}{B_1(z)}\right| \quad (z\in U),$$

where B_1 is the Blaschke product formed with these zeros. Since $f(e^{it}) = F(e^{i\psi(t)})$, where ψ is decreasing and satisfies (2.11), we can apply the same theorem to the function $z \to f(\bar{z})$ to deduce that g' has at most $\nu - 1$ zeros in U and that

$$|h'(z)| \le \left| \frac{g'(z)}{B_2(z)} \right| \quad (z \in U),$$

where B_2 is the Blaschke product formed with these zeros. An invariant form of Schwarz's lemma applied to these two inequalities gives

$$\left| \frac{g'(z)}{B_2(z)} \right| \le \left| \frac{h'(z)}{B_1(z)} \right| \le \left| \frac{g'(z)}{B_2(z)} \right| \quad (z \in U).$$

We immediately deduce (2.12). □

3. Topological Properties of the Harmonic Extension

In this section we shall assume that the points c_k ($1 \le k \le n$, $n \ge 2$) are distinct but otherwise arbitrary points of $\mathbf{C}$, and denote by Π the polygon $[c_1, c_2, \ldots, c_n, c_1]$. With f defined by (1.1) on T and denoting the harmonic extension in U, we shall be concerned with the range $f(U)$, its closure, and the valence of f in U.

THEOREM 3. *Π is the set of limit points of $f(z)$ as z approaches T from inside U.*

Proof. Since f is continuous on T except for jump discontinuities at the points ζ_k, it follows from a well-known result on angular limits [8, p. 131] that every point of Π is a limiting value of $f(z)$ as z approaches T. It is therefore sufficient to show that as $z \to \zeta_k$ from inside U, then $f(z)$ approaches only points on the segment $[c_k, c_{k+1}]$. This follows easily from the Poisson integral representation as follows: We have

$$f(z) = \sum_{j=1}^{n} c_j \frac{1}{2\pi} \int_{t_{j-1}}^{t_j} \frac{1-|z|^2}{|1-ze^{-it}|^2}\, dt \tag{3.1}$$

and hence, as $z \to e^{it_k}$, all terms in the sum tend to zero except the terms $j = k$ and $j = k+1$. Thus

$$f(z) \to c_k \alpha_k + c_{k+1} \alpha_{k+1} \quad \text{as} \quad z \to e^{it_k}, \tag{3.2}$$

where α_k, α_{k+1} are respectively the limiting values of

$$\frac{1}{2\pi} \int_{t_{k-1}}^{t_k} \frac{1-|z|^2}{|1-ze^{-it}|^2}\, dt, \quad \frac{1}{2\pi} \int_{t_k}^{t_{k+1}} \frac{1-|z|^2}{|1-ze^{-it}|^2}\, dt. \tag{3.3}$$

We see that $0 \le \alpha_k \le 1$, $0 \le \alpha_{k+1} \le 1$. Furthermore,

$$\alpha_k+\alpha_{k+1}=\lim_{z\to e^{it_k}}\frac{1}{2\pi}\int_{t_{k-1}}^{t_{k+1}}\frac{1-|z|^2}{|1-ze^{-it}|^2}\,dt$$

$$(3.4)\qquad =1-\lim_{z\to e^{it_k}}\frac{1}{2\pi}\int_{t_{k+1}}^{t_{k-1}+2\pi}\frac{1-|z|^2}{|1-ze^{-it}|^2}\,dt$$

$$=1.$$

So the limiting values of $f(z)$ as $z\to e^{it_k}$ are $c_k\alpha_k+c_{k+1}(1-\alpha_k)\in[c_k, c_{k+1}]$, as required. □

It is a well-known topological result that if Π has a nonzero winding number about a point w, and if F is a continuous parametrisation of Π, then $w\in F(U)$ for every continuous extension of F on T to $\bar{U}$. Even though f is not continuous on T, this result remains valid for the harmonic extension.

THEOREM 4. *$f(U)$ contains all points w about which Π has nonzero winding number.*

Proof. First, we consider the following parametrisation of Π given by

$$(3.5)\qquad F(e^{it})=\frac{(t_k-t)c_k+(t-t_{k-1})c_{k+1}}{t_k-t_{k-1}}\quad(t_{k-1}\le t\le t_k)$$

for $1\le k\le n$. Writing

$$(3.6)\qquad \varphi(t)=t_{k-1}\quad(t_{k-1}\le t<t_k,\ 1\le k\le n),$$

we see that $\varphi(t)$ is nondecreasing and that

$$(3.7)\qquad F(e^{i\varphi(t)})=c_k\quad(t_{k-1}\le t<t_k,\ 1\le k\le n),$$

so $F(e^{i\varphi(t)})=f(e^{it})$ except possibly at the t_k. We now choose a sequence $\{\varphi_j(t)\}$ of continuous approximations to φ by

$$(3.8)\qquad \varphi_j(t)=\begin{cases} t_{k-1} & (t_{k-1}\le t\le t_k-\delta_j),\\ (1/\delta_j)(t_k-t_{k-1})(t-t_k)+t_k & (t_k-\delta_j<t\le t_k)\end{cases}$$

for $1\le k\le n$, where $\{\delta_j\}$ is a sequence such that

$$\delta_j\to 0\ \text{as}\ j\to\infty\quad\text{and}\quad 0<\delta_j<\max\{t_k-t_{k-1}:1\le k\le n\}.$$

We define $\{f_j(e^{it})\}$ by

$$(3.9)\qquad \begin{aligned} f_j(e^{it})&=F(e^{i\varphi_j(t)})\\ &=\begin{cases} c_k & (t_{k-1}\le t\le t_k-\delta_j),\\ [(t-t_k)(c_{k+1}-c_k)]/\delta_j+c_{k+1} & (t_k-\delta_j<t<t_k).\end{cases}\end{aligned}$$

The harmonic extensions to U of f_j are then given by

$$(3.10)\qquad \begin{aligned} f_j(z)=\sum_{k=1}^{n}\Bigg[&c_k\frac{1}{2\pi}\int_{t_{k-1}}^{t_k-\delta_j}\operatorname{Re}\frac{1+ze^{-it}}{1-ze^{-it}}\,dt\\ &+\frac{1}{2\pi}\int_{t_k-\delta_j}^{t_k}\operatorname{Re}\frac{1+ze^{-it}}{1-ze^{-it}}\left(\frac{(t-t_k)(c_{k+1}-c_k)}{\delta_j}+c_{k+1}\right)dt\Bigg],\end{aligned}$$

and we can see directly that

(3.11) $\qquad f_j(z) \to f(z) \quad \text{as } j \to \infty$ locally uniformly in U.

Furthermore, the functions $f_j(z)$ are continuous in $\bar{U}$.

We require the following lemma.

LEMMA 2. *Suppose that* $w \in \mathbf{C} - \Pi$. *Then* $\exists R$ $(0 < R < 1)$ *and* j_0 *such that* $f_j(z) \neq w$ *for* $R \leq |z| < 1$ *and* $j \geq j_0$.

Proof. If not, then $\exists\{z_i\}$ with $|z_i| < 1$ and $|z_i| \to 1$ as $i \to \infty$, and also $\exists\{j(i)\}$ with $j(i) \to \infty$ as $i \to \infty$, such that $f_{j(i)}(z_i) = w$. Thus, without loss of generality we may assume that

$$f_j(z_j) = w \quad (j = 1, 2, \ldots),$$

where $z_j \to e^{is}$ as $j \to \infty$. Then $\exists k$ $(1 \leq k \leq n)$ such that $t_{k-1} < s \leq t_k$. We easily deduce that

$$\begin{aligned} w = \lim_{j\to\infty} \Bigg[& c_k \frac{1}{2\pi} \int_{t_{k-1}}^{t_k - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt \\ & + \frac{1}{2\pi} \int_{t_k - \delta_j}^{t_k} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \left(\frac{t - t_k}{\delta_j} (c_{k+1} - c_k) + c_{k+1} \right) dt \\ & + c_{k+1} \frac{1}{2\pi} \int_{t_k}^{t_{k+1} - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt \Bigg]. \end{aligned}$$

If $s < t_k$, then the second and third terms tend to zero. Hence

$$w = \lim_{j\to\infty} c_k \frac{1}{2\pi} \int_{t_{k-1}}^{t_k - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt = c_k.$$

Then $w \in \Pi$, contradicting the hypothesis. So we may assume that $z_j \to e^{it_k}$ as $j \to \infty$. We then see that

$$w = \lim_{j\to\infty} (A_j c_k + B_j c_{k+1}),$$

where

$$A_j = \frac{1}{2\pi} \int_{t_{k-1}}^{t_k - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt + \frac{1}{2\pi} \int_{t_k - \delta_j}^{t_k} \frac{t_k - t}{\delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt$$

and

$$B_j = \frac{1}{2\pi} \int_{t_k}^{t_{k+1} - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt + \frac{1}{2\pi} \int_{t_k - \delta_j}^{t_k} \left(1 + \frac{t - t_k}{\delta_j}\right) \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt.$$

Then clearly $A_j \geq 0$ and $B_j \geq 0$. Furthermore,

$$A_j + B_j = \frac{1}{2\pi} \int_{t_{k-1}}^{t_{k+1} - \delta_j} \operatorname{Re} \frac{1 + z_j e^{-it}}{1 - z_j e^{-it}} \, dt \to 1 \quad \text{as } j \to \infty.$$

Hence, choosing a suitable sequence of j so that $A_j \to A$, we have

$$w = Ac_k + (1-A)c_{k+1},$$

where $0 \le A \le 1$. This gives $w \in \Pi$, contradicting the hypothesis. □

Consider now a point $w \in \mathbf{C} - \Pi$ and suppose that $f(z) \ne w$ $(z \in U)$. Choose R $(0 < R < 1)$ and j_0 such that, for $j \ge j_0$, $f_j(z) \ne w$ for $R \le |z| < 1$. Since $f_j(z) \to f(z)$ uniformly on $|z| = R$ as $j \to \infty$, we have

$$\frac{f_j(z) - w}{f(z) - w} \to 1 \quad \text{as } j \to \infty$$

uniformly on $|z| = R$. Writing $T_r = \{z = re^{i\theta},\ 0 \le \theta \le 2\pi\}$, we deduce that $\exists j_1 \ge j_0$ such that, for $j \ge j_1$,

$$d(f_j - w, T_R) = d(f - w, T_R) = 0,$$

where $d(g, \Gamma)$ is the degree of the continuous mapping g on the curve Γ. The second equality follows from the degree principle, since $f - w \ne 0$ inside and on T_R. However, taking $j = j_1$, $f_j(z) - w$ is continuous and $\ne 0$ for $R \le |z| \le 1$. Hence, by the degree principle,

$$d(f_j - w, T) = d(f_j - w, T_R) = 0. \tag{3.12}$$

But from (3.9), f_j on T is a parametrisation of the polygon Π, so (3.12) gives

$$n(\Pi, w) = 0.$$

Hence $f(U)$ contains all points w about which Π has nonzero winding number. This completes the proof of Theorem 4. □

THEOREM 5. *Suppose that the polygon $\Pi = [c_1, \ldots, c_n, c_1]$ is a positively oriented Jordan curve bounding a domain D. Then $D \subset f(U)$. Furthermore, f is a homeomorphism of U onto D if and only if one of the following conditions holds:*

(i) *$h'(z) \ne 0$ for $z \in U$;*
(ii) *$Q(z)$ has degree $n-2$ and all its zeros in $\bar{U}$;*
(iii) *$P(z) \ne 0$ for $z \in U$;*
(iv) *$|R(z)| \le 1$ for $z \in U$;*
(v) *$|R(z)| < 1$ for $z \in U$.*

Proof. $D \subset f(U)$ is immediate from Theorem 4. Furthermore, Theorem 4 implies that f is a homeomorphism of U onto D if and only if (A) f is univalent in U. Now (A) ⇒ (i) (see, e.g., [2]) ⇒ (iii) ⇒ that $R(z)$ has no poles in U, which implies (iv). Now if (iv) holds and if $R(z)$ is a constant of modulus 1, then f takes only values on a line segment for $z \in U$ and therefore Π is not a Jordan curve. Hence (iv) ⇒ (v). Since the equivalence of (ii) and (iii) follows easily from (1.7) and (1.10), it remains to show that (v) ⇒ (A). Assume that (v) holds and let $a \in U$. Then either $|g'(a)| < |h'(a)|$, or g' and h' have a common zero at a, say

$$g'(z)=(z-a)^m G(z), \qquad h'(z)=(z-a)^m H(z),$$

where $|G(a)|<|H(a)|$. We then obtain

$$\frac{f(z)-f(a)}{(m+1)\rho^{m+1}H(a)e^{i(m+1)\theta}}=1+\frac{\bar{G}(a)}{H(a)}e^{-2i(m+1)\theta}+o(1)$$

for $z=a+\rho e^{i\theta}$ and $\rho>0$ small. Thus $d(f-f(a),C_\rho)=m+1$, where C_ρ is the positively oriented circle with centre a and radius ρ. Hence the function $f-f(a)$ has a zero of multiplicity $m+1$ at a.

Consider now the notations of Theorem 4 and choose $w\in\mathbf{C}-\Pi$. By Lemma 2, $\exists R$ $(0<R<1)$ and j_0 such that $f_j(z)\neq w$ for $R\leq|z|\leq 1$ and $j\geq j_0$. Arguing similarly as in the proof of Theorem 4, we obtain

$$n(\Pi,w)=d(f-w,T_R) \quad \text{for } R \text{ near to } 1.$$

Now from the above we see that $f-w$ has at most isolated zeros a in U with multiplicities $m(a)+1$, where $m(a)$ is the multiplicity of the zero of h' at a. By the general argument principle, if R is chosen near to 1 then

$$d(f-w,T_R)=\sum(m(a)+1),$$

the sum being taken over all zeros a in U of $f-w$. We obtain

$$\sum(m(a)+1)=n(\Pi,w),$$

and so there is exactly one simple zero a of $f-w$ in U for each $w\in D$, and no zeros in U of $f-w$ for those w exterior to Π. Finally, as f is an open mapping on U, there are no zeros in U of $f-w$ for $w\in\Pi$. Thus f is a homeomorphism of U onto D. This completes the proof. □

COROLLARY. *Suppose that the polygon Π is a convex polygon bounding the domain D. Then f is a homeomorphism from U onto D.*

Proof. Applying Lemma 1 we see that $\rho(k)=r(k)=k$ and so $\mu=1$. Therefore (by Theorem 2) $h'(z)\neq 0$ for $z\in U$, and so the result follows from Theorem 5. □

EXAMPLE 1. The following example shows that the corollary need not hold for a nonconvex Jordan polygon Π. Let $\Pi=[1,i,-1,\frac{1}{2}i,1]$. If we take $\zeta_k=\exp(2\pi i(k-1)/4)$ $(1\leq k\leq 4)$ then we obtain

$$f(0)=\frac{1}{2\pi}\sum_{k=1}^{n}c_k(t_k-t_{k-1})=\frac{1}{4}\sum_{k=1}^{n}c_k=\frac{3}{8}i. \tag{3.13}$$

Hence $f(0)$ lies outside the domain bounded by Π and so f is not one-to-one.

4. The Mapping Problem for Schlicht Polygons

In this section we shall assume that the polygon $\Pi=[c_1,\ldots,c_n,c_1]$ is a positively oriented Jordan curve with distinct vertices c_k bounding a domain D.

The mapping problem for Π is to find $\zeta_k = e^{it_k} \in T$ $(1 \le k \le n)$ distinct points appearing in positive cyclic order on T, such that the harmonic extension to U of the mapping f defined by (1.1) on T gives a homeomorphism of U onto D. By Theorem 5 the condition for this is that

$$h'(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z-\zeta_k} \neq 0 \quad (z \in U). \tag{4.1}$$

We shall discuss this problem in general terms and conjecture that for some polygons there may be no solutions at all. For a convex polygon there are many solutions, as any such set of $\{\zeta_k\}$ will do. Therefore we shall also consider the more special problem of whether we may pre-assign the zeros of Q—or, less stringently, choose in advance the zeros of the Blaschke product g'/h'. Surprisingly, this is not always possible even for convex polygons. On the other hand, we shall solve the problem in a particular case for the non-convex polygon of Example 1.

Given that we have found a set of $\{\zeta_k\}$ to solve the schlicht problem for the polygon Π, the following equation will hold identically:

$$\sum_{k=1}^{n} \frac{\bar{\alpha}_k P(z) + \alpha_k Q(z)}{z-\zeta_k} = 0, \tag{4.2}$$

where P and Q are related by (1.7) and where $P(z) \neq 0$ $(z \in U)$. On the other hand, the existence of such ζ_k, distinct and appearing in positive cyclic order on T, clearly provides a solution to the problem. We observe immediately that each point ζ_k must satisfy the equation

$$\bar{\alpha}_k P(z) + \alpha_k Q(z) = 0 \quad (1 \le k \le n). \tag{4.3}$$

Furthermore, no ζ_k can be one of the (say) q_4 common zeros on T of P and Q. For otherwise the pole of h' at ζ_k would be deleted, which implies that $\alpha_k = 0$, contradicting the distinctness of the vertices c_k of Π. Thus each ζ_k lies among the $n-2-q_4$ roots of the equation

$$B(z) = -\frac{\bar{\alpha}_k}{\alpha_k} \quad (1 \le k \le n), \tag{4.4}$$

where $B(z) = Q(z)/P(z)$ is a Blaschke product with its $n-2-q_4$ zeros in U. Writing

$$\alpha_k = |\alpha_k| e^{ia_k}, \quad \omega_k = -e^{-2ia_k} \quad (1 \le k \le n), \tag{4.5}$$

the general mapping problem for Π may be formulated as follows: We seek ζ_k occurring in positive cyclic order on T, and a Blaschke product $B(z)$ of degree at most $n-2$ with all its zeros in U, so that

$$\sum_{k=1}^{n} \alpha_k \frac{B(z) - \omega_k}{z - \zeta_k} = 0. \tag{4.6}$$

If this is possible then we obtain $h'(z)B(z) = g'(z)$, and so by Theorem 5 the mapping f is schlicht in U.

Let us consider the case $B(z)=cz$, where c is a constant satisfying $|c|=1$. Then we require ζ_k on the unit circle so that we have identically

$$\sum_{k=1}^{n} \frac{\bar{\alpha}_k + c\alpha_k z}{z-\zeta_k} = 0.$$

For distinct ζ_k this is possible only if

$$\zeta_k = -\bar{\varphi}\frac{\bar{\alpha}_k}{\alpha_k} \quad (1 \le k \le n).$$

On the other hand, since $\sum \alpha_k = 0$, the identity holds with these ζ_k. The $n-3$ common zeros of P and Q on T are now determined after choosing the value of c ($|c|=1$) and so cannot be pre-assigned, a fact first noted by Hengartner and Schober [5]. If this only possible solution is to work, the points ζ_k must appear in positive cyclic order on T. Thus the numbers ω_k must appear in positive cyclic order; that is, they must form a convex polygon. If they do, then we have by Theorem 5 a univalent solution. Summarising, we have the following theorem.

THEOREM 6. *Let* $\Pi = (c_1, c_2, \ldots, c_n, c_1)$ *be a positively oriented Jordan polygon with distinct vertices* c_k *bounding a domain D. We can find points* ζ_k $(1 \le k \le n)$ *occurring in positive cyclic order on T such that the mapping* $f = \bar{g} + h$ *given by* (1.1) *is a homeomorphism of U onto D with* $\gamma'/\eta' = \varphi\zeta$, *where c is a constant with* $|c|=1$, *if and only if the points* $\omega_k = -\bar{\alpha}_k/\alpha_k$ *occur in positive cyclic order on T. We then have*

$$\zeta_k = \bar{c}\omega_k \quad (1 \le k \le n).$$

This condition is quite restrictive, as the following considerations will show. Writing $\alpha_k = |\alpha_k|\exp(ia_k)$, we may assume (on the basis that Π is a Jordan polygon) that $|a_{k+1} - a_k| < \pi$ and $a_{n+1} = a_1 + 2\pi$. Then the condition can be written

$$2m_1\pi - 2a_1 < 2m_2\pi - 2a_2 < \cdots < 2m_{n+1}\pi - 2a_{n+1} \tag{4.7}$$

for suitable integers m_k. We easily deduce that

$$m_{k+1} \ge m_k \quad \text{and} \quad m_{n+1} = m_1 + 3, \tag{4.8}$$

the latter condition following from $2m_{n+1}\pi - 2a_{n+1} = 2m_1\pi - 2a_1 + 2\pi$. Therefore, if $k > j$ then we obtain

$$2m_k\pi - 2a_k < 2m_{n+1}\pi - 2a_{n+1} = 2m_1\pi - 2a_1 + 2\pi < 2m_j\pi - 2a_j + 2\pi, \tag{4.9}$$

and so

$$a_k - a_j > \pi(m_k - m_j) - \pi \ge -\pi. \tag{4.10}$$

We thus deduce the following.

THEOREM 7. *If* $g'/h' = cz$ ($|c|=1$) *for a Jordan polygon* Π, *then* Π *is close-to-convex.*

We give two examples which show that the condition of convexity of Π is neither necessary nor sufficient for the criterion to hold.

EXAMPLE 2. *Let Π be a Jordan polygon having two sides parallel to each other. Then there is no mapping f of the form* (1.1) *mapping U one-to-one onto the interior of Π for which $g'/h' = cz$. For example, there is no mapping of this form if Π is a rectangle.*

Proof. If Π has two parallel sides then, for some j and k $(j \neq k)$, α_j/α_k is real and so $\omega_j = \omega_k$ and the criterion of Theorem 6 does not hold. □

EXAMPLE 3. *Let Π be the nonconvex polygon* $(1, i, -1, \frac{1}{2}i, 1)$. *Then there exists a mapping f of the form* (1.1), *giving a homeomorphism of U onto the interior of Π, for which $g'/h' = -z$.*

Proof. We obtain $\omega_1 = -i$, $\omega_2 = i$, $\omega_3 = (-3+4i)/5$, and $\omega_4 = -(3+4i)/5$; these are in cyclic order on T. The result follows from Theorem 6. Explicit calculations give

$$\pi i g'(z) = \frac{6z(z+1)}{(z^2+1)(5z^2+6z+5)}, \quad \pi i h'(z) = \frac{-6(z+1)}{(z^2+1)(5z^2+6z+5)}.$$

REMARK 2. Hengartner and Schober [5] have shown that the mapping problem can be solved for a triangle with g'/h' having just one simple zero in U, a fact which is easily deduced from the above considerations. Furthermore, they have proved the remarkable fact that if f is any convex harmonic function with g'/h' equal to a product of N Blaschke factors and with $f(U)$ bounded, then $f(U)$ is a polygon with at most $N+2$ vertices and $f(e^{it})$ a step function with $N+2$ values. In particular, if g'/h' has one simple zero, then f takes three values on T and $f(U)$ is a triangle. On the assumption that f is a step function on T, the conclusion that $f(U)$ is a triangle can be seen by observing that, from Theorem 6, if Π is convex then $a_{k+1} - a_k \geq 0$ and we deduce that $m_{k+1} - m_k \geq 1$. Since $m_{n+1} = m_1 + 3$, we obtain $n = 3$. More generally, we have the following result.

THEOREM 8. *Let Π be a convex polygon with n vertices. Then f is a homeomorphism of U onto D such that g'/h' is a Blaschke product with its zeros in U and of exact degree $n-2$.*

Proof. Only the exactness of the degree requires proof, and this follows from Hengartner and Schober's result. In fact, essentially their method is to proceed similarly to the argument of (4.7) and (4.8), making use of the fact that $\arg B(e^{it_k})$ is increasing. Here is a second proof. We need to show that Q has no zeros on T, so that g' and h' have no common zeros. It is therefore sufficient to show that $|h'(z)|$ is bounded below as $|z| \to 1$ in U. For convex harmonic f, it is shown in [2] that for each ϵ $(|\epsilon| = 1)$, the function $h + \epsilon g$ is univalent in U. Hence

$$|h'(z)|-|g'(z)|\geq c(1-|z|) \quad (z\in U),$$

where c is a positive constant. We deduce that

$$|h'(z)|\geq c\frac{1-|z|}{1-|B(z)|}.$$

Now for a Blaschke product $B(z)$ of degree N we can write

$$\frac{1+B(z)}{1-B(z)}=\sum_{k=1}^{N} p_k \frac{x_k+z}{x_k-z}+w,$$

where $|x_k|=1$, $p_k>0$, and $\operatorname{Re} w=0$. Taking real parts we obtain

$$\frac{1-|B(z)|^2}{1-|z|^2}=\sum_{k=1}^{N} p_k \frac{|B(z)-1|^2}{|z-x_k|^2}.$$

As $B(x_k)=1$, the right-hand expression clearly remains continuous on $\bar{U}$, and therefore bounded. The results follows. □

EXAMPLE 4. The case when Π is a quadrilateral and $B(z)=z^2$. We are required to find ζ_k $(1\leq k\leq 4)$ appearing in positive cyclic order on T so that

$$\sum_{k=1}^{4} \alpha_k \frac{z^2-\omega_k}{z-\zeta_k}=0.$$

For this to hold, each ζ_k is one of the two square roots of ω_k, and we obtain

$$\zeta_k=\pm i|\alpha_k|/\alpha_k \quad (1\leq k\leq 4);$$

$$\sum_{k=1}^{4} \alpha_k\zeta_k=0.$$

Thus the problem can be solved if and only if these two conditions hold with the ζ_k appearing in positive cyclic order on T. The two conditions reduce to

$$\zeta_k=\epsilon_k\frac{|c_{k+1}-c_k|}{c_{k+1}-c_k} \quad \text{and} \quad \sum_{k=1}^{4}\epsilon_k|c_{k+1}-c_k|=0,$$

where $\epsilon_k=\pm 1$ $(1\leq k\leq 4)$. Note that, for the second condition to hold, $\epsilon_k=1$ for exactly two values of k. Thus the sum of the lengths of two sides of the quadrilateral must equal the sum of the lengths of the remaining two sides. To explore the first condition, we again write $\alpha_k=|\alpha_k|\exp(ia_k)$, so that $|a_{k+1}-a_k|<\pi$ and $a_5=a_1+2\pi$. We then have

$$\begin{aligned}(2m_1+\gamma_1)\pi-a_1<(2m_2+\gamma_2)\pi-a_2<(2m_3+\gamma_3)\pi-a_3\\ <(2m_4+\gamma_4)\pi-a_4<(2m_5+\gamma_5)\pi-a_5,\end{aligned}$$

where m_k are integers and $\gamma_k=0$ or 1 depending on whether $\epsilon_k=1$ or -1. We deduce that $m_{k+1}\geq m_k$. Also $\gamma_5=\gamma_1$ and

$$(2m_5+\gamma_5)\pi-a_5=(2m_1+\gamma_1)\pi-a_1+2\pi,$$

and we obtain $m_5=m_1+2$. We obtain a solution with

$$m_1 = m_2 = 0, \; m_3 = m_4 = 1, \; m_5 = 2; \quad \gamma_1 = \gamma_3 = \gamma_5 = 0, \; \gamma_2 = \gamma_4 = 1.$$

Thus, with $B(z) = z^2$, the mapping problem can be solved for Π provided that the sum of the lengths of two nonadjacent sides is equal to the sum of the lengths of the remaining two nonadjacent sides of Π. We leave it as an exercise to show that this condition is also necessary. Notice that by Theorem 8 the only solutions for any convex polygon Π are for convex quadrilaterals satisfying this property.

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