MAXIMAL SPACELIKE SUBMANIFOLDS OF A PSEUDORIEMANNIAN SPACE OF CONSTANT CURVATURE

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1. Introduction. Generalizing Lawson's result [4], Chern–DoCarmo–Kobayashi proved the following in [3]. Let M be an n-dimensional minimal submanifold of a unit sphere S^{n+p}. Let S be the square of the length of the second fundamental form of M. If M is compact, it follows from Simon's result that if $S \leq n/(2-1/p)$ everywhere on M then either $S=0$ or $S=n/(2-1/p)$. The Veronese surface in S^4 and $M_{m,n-m}$ in S^{n+1} are the only compact minimal submanifolds of dimension n in S^{n+p} satisfying $S=n/(2-1/p)$, where $M_{m,n-m}$ is the manifold $S^m(\sqrt{m/n}\,)\times S^{n-m}(\sqrt{(n-m)/n}\,)$ which is naturally imbedded in S^{n+1}.

On the other hand, in this paper we investigate maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature. Let $N_p^{n+p}(c)$ be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature c whose index is p. Let M be an n-dimensional complete Riemannian manifold isometrically immersed in $N_p^{n+p}(c)$. Note that the codimension is equal to the index.

The pseudohyperbolic space of radius r (>0) is the hyperquadric

$$H_p^{n+p}(r)=\{x\in \mathbf{R}_{p+1}^{n+p+1};\ \langle x,x\rangle = x_1^2+\cdots+x_n^2-x_{n+1}^2-\cdots-x_{n+p+1}^2=-r^2\}.$$

This is a space of constant curvature $-1/r^2$. Let $H^n(r)$ be the component of $H_0^n(r)$ through $(0,\dots,0,r)$. Here, we describe two examples of maximal spacelike immersions. We consider the mapping defined by

$$u_1=\frac{1}{\sqrt3}yz,\quad u_2=\frac{1}{\sqrt3}zx,\quad u_3=\frac{1}{\sqrt3}xy,\quad u_4=\frac{1}{2\sqrt3}(x^2-y^2),$$
$$u_5=\tfrac{1}{6}(x^2+y^2+2z^2),$$

where (x,y,z) is the natural coordinate system in $\mathbf{R}_1^3$ and (u_1,u_2,u_3,u_4,u_5) is the natural coordinate system in $\mathbf{R}_3^5$. This defines an isometric maximal immersion of $H^2(\sqrt3)$ into $H_2^4(1)$. We may call this the *hyperbolic Veronese surface.* Let $n_1,\dots,n_{p+1}$ be positive integers and $n=n_1+\cdots+n_{p+1}$. Let x_i be a point of $H^{n_i}(\sqrt{n_i/n}\,)$. Then $x=(x_1,\dots,x_{p+1})$ is a vector in $\mathbf{R}_{p+1}^{n+p+1}$ with $\langle x,x\rangle=-1$. This defines also an isometric immersion of

$$H_{n_1,\dots,n_{p+1}}=H^{n_1}(\sqrt{n_1/n}\,)\times\cdots\times H^{n_{p+1}}(\sqrt{n_{p+1}/n}\,)$$

into $H_p^{n+p}(1)$. Now, it has been proved by Cheng and Yau [2] that a complete maximal spacelike hypersurface in the Minkowski $(n+1)$-space is totally geodesic (see also [1]). First, we generalize this result slightly.

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THEOREM 1.1. *Let M be an n-dimensional complete Riemannian manifold isometrically immersed in $N_p^{n+p}(c)$, $c \geq 0$. If M is maximal, then the immersion is totally geodesic and M is a Riemannian space of constant curvature c.*

In [6], Nishikawa gives another extension of the Cheng-Yau result. Next, we consider a manifold immersed in a space of negative constant curvature. Denote by S the square of the length of the second fundamental form of an immersion.

THEOREM 1.2. *Let M be an n-dimensional complete Riemannian manifold isometrically immersed in a pseudo-Riemannian space $N_p^{n+p}(-c)$ of constant curvature $-c$ $(c>0)$. Assume that M is maximal. Then we have $0 \leq S \leq npc$.*

The hyperbolic Veronese surface is a maximal submanifold of $H_2^4(1)$ with $S = 4/3$. The submanifolds $H_{n_1, \ldots, n_{p+1}}$ of $H_p^{n+p}(1)$ satisfy $S = np$. Conversely, we can show the following.

THEOREM 1.3. *The submanifolds $H_{n_1, \ldots, n_{p+1}}$ in $H_p^{n+p}(1)$ are the only complete* connected *maximal spacelike submanifolds of dimension n in $H_p^{n+p}(1)$ satisfying $S = np$.*

2. Local formula. Let N be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature c, whose index is p. Let M be an n-dimensional Riemannian manifold isometrically immersed in N. As the pseudo-Riemannian metric of N induces the Riemannian metric of M, the immersion is called *spacelike*. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in N such that, at each point of M, $e_1, \ldots, e_n$ spans the tangent space of M and forms an orthonormal frame there. We make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, D \leq n+p; \quad 1 \leq i, j, k, l \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

We shall agree that repeated indices are summed over the respective ranges. Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the pseudo-Riemannian metric of N is given by $ds_N^2 = \sum \omega_i^2 - \sum \omega_\alpha^2 = \sum \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ for $1 \leq i \leq n$ and $\epsilon_\alpha = -1$ for $n+1 \leq \alpha \leq n+p$. Then the structure equations of N are given by

$$(2.1) \qquad \begin{cases} d\omega_A = \sum \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} = \sum \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum \epsilon_C \epsilon_D K_{ABCD} \omega_C \wedge \omega_D, \\ K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \end{cases}$$

We restrict these forms to M. Then

$$(2.2) \qquad \omega_\alpha = 0 \quad \text{for } n+1 \leq \alpha \leq n+p,$$

and the Riemannian metric of M is written as $ds_M^2 = \sum \omega_i^2$. We may put

$$(2.3) \qquad \omega_{i\alpha} = \sum h_{\alpha ij} \omega_j.$$

Then $h_{\alpha ij}$ are the components of the second fundamental form of M. From (2.1), we obtain the structure equations of M

$$(2.4) \qquad \begin{cases} d\omega_i = \sum \omega_{ij} \wedge \omega_j, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{cases}$$

and the Gauss formula

$$(2.5)\qquad \begin{cases} R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})-\sum (h_{\alpha ik}h_{\alpha jl}-h_{\alpha il}h_{\alpha jk}), \\ R_{\alpha\beta ij}=\sum (h_{\alpha ki}h_{\beta kj}-h_{\alpha kj}h_{\beta ki}). \end{cases}$$

We also have the structure equations of the normal bundle of M:

$$(2.6)\qquad \begin{cases} d\omega_\alpha=-\sum \omega_{\alpha\beta}\wedge\omega_\beta, \\ d\omega_{\alpha\beta}=-\sum \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j. \end{cases}$$

We call $H=\sum (1/n)(\sum_i h_{\alpha ii})e_\alpha$ the *mean curvature normal.* An immersion is said to be *maximal* if its mean curvature normal vanishes identically. From the Gauss formula (2.6) we obtain the following equalities about the Ricci curvature $R_{ij}=\sum R_{ikjk}$:

$$(2.7)\qquad R_{ij}=c(n-1)\delta_{ij}-\sum h_{\alpha ij}\sum_k h_{\alpha kk}+\sum_{\alpha,k} h_{\alpha ki}h_{\alpha kj}.$$

From this, the following is immediate.

PROPOSITION 2.1. *Let M be an n-dimensional Riemannian manifold immersed in $N_p^{n+p}(c)$ isometrically. If M is maximal, the Ricci curvature of M satisfies $((n-1)c\delta_{ij})\leq(R_{ij})$, and the equality holds everywhere if and only if M is totally geodesic in N.*

3. The Simons-Calabi type equation. Let $h_{\alpha ijk}$ denote the covariant derivative of $h_{\alpha ij}$ so that

$$(3.1)\qquad \sum h_{\alpha ijk}\omega_k=dh_{\alpha ij}+\sum h_{\alpha ik}\omega_{kj}+\sum h_{\alpha kj}\omega_{ki}-\sum h_{\beta ij}\omega_{\beta\alpha}.$$

Then we have $h_{\alpha ijk}=h_{\alpha ikj}$. Next, take the exterior derivative of (3.1) and define the second covariant derivative of $h_{\alpha ij}$ by

$$\sum h_{\alpha ijkl}\omega\ =dh_{\alpha ijk}+\sum h_{\alpha ijl}\omega_{lk}+\sum h_{\alpha ilk}\omega_{lj}+\sum h_{\alpha ljk}\omega_{li}-\sum h_{\alpha ijk}\omega_{\beta\alpha}.$$

Then we obtain the Ricci formula

$$(3.2)\qquad h_{\alpha ijkl}-h_{\alpha ijlk}=\sum h_{\alpha im}R_{mjkl}+\sum h_{\alpha mj}R_{mikl}+\sum h_{\beta ij}R_{\alpha\beta kl}.$$

The Laplacian $\Delta h_{\alpha ij}$ of the second fundamental form $h_{\alpha ij}$ is defined by $\Delta h_{\alpha ij}=\sum h_{\alpha ijkk}$. Using the same method as in [3] (see also [2] and [6]), we have

$$(3.3)\qquad \begin{aligned} \Delta h_{\alpha ij}=nch_{\alpha ij}-c\delta_{ij}\sum h_{\alpha kk}-\sum h_{\alpha mi}h_{\beta mj}h_{\beta kk}-2\sum h_{\alpha km}h_{\beta ki}h_{\beta mj} \\ +\sum h_{\alpha km}h_{\beta km}h_{\beta ij}+\sum h_{\alpha mi}h_{\beta mk}h_{\beta kj}+\sum h_{\alpha mj}h_{\beta ki}h_{\beta mk}. \end{aligned}$$

If we assume that M is maximal in N, and since we have

$$\tfrac{1}{2}\Delta(\sum (h_{\alpha ij})^2)=\sum (h_{\alpha ijk})^2+\sum h_{\alpha ij}\Delta h_{\alpha ij},$$

we obtain

$$(3.4)\qquad \begin{aligned} \tfrac{1}{2}\Delta(\sum h_{\alpha ij})^2=\sum (h_{\alpha ijk})^2+nc\sum (h_{\alpha ij})^2+\sum h_{\alpha ij}h_{\alpha lk}h_{\beta ij}h_{\beta kl} \\ +\sum (h_{\alpha ik}h_{\beta kj}-h_{\beta ik}h_{\alpha kj})(h_{\alpha il}h_{\beta lj}-h_{\beta il}h_{\alpha lj}). \end{aligned}$$

We will follow the argument in [3]. The square of the length of the second fundamental form h of M in N is given by $S=-\langle h,h\rangle=\sum_{i,j,\alpha}(h_{\alpha ij})^2$. For each α, let H_α be the symmetric matrix $(h_{\alpha ij})$ and put $S_{\alpha\beta}=\sum_{i,j} h_{\alpha ij}h_{\beta ij}$. Then, the $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1},\dots,e_{n+p}$. Set $S_\alpha=S_{\alpha\alpha}$ and we have $S=\sum_\alpha S_\alpha$. In general, for a matrix $A=(a_{ij})$, we put $N(A)=\operatorname{trace} A\cdot{}^tA$. Now, (3.4) can be rewritten as follows:

$$\tag{3.5} \tfrac{1}{2}\Delta S=\sum_{\alpha,i,j,k}(h_{\alpha ijk})^2+ncS+\sum_{\alpha,\beta}N(H_\alpha H_\beta-H_\beta H_\alpha)+\sum_\alpha S_\alpha^2.$$

It is clear that

$$\tag{3.6} 0\le N(H_\alpha H_\beta-H_\beta H_\alpha),$$

and the equality holds if and only if $H_\alpha H_\beta=H_\beta H_\alpha$. Put

$$p\sigma_1=\sum S_\alpha=S,\qquad \frac{p(p-1)}{2}\sigma_2=\sum_{\alpha<\beta}S_\alpha S_\beta.$$

Then we have

$$\sum_\alpha S_\alpha^2=\left(\sum_\alpha S_\alpha\right)^2-2\sum_{\alpha<\beta}S_\alpha S_\beta$$
$$=p\sigma_1^2+p(p-1)(\sigma_1^2-\sigma_2).$$

On the other hand, we know that

$$p^2(p-1)(\sigma_1^2-\sigma_2)=\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2.$$

Hence, we obtain

$$\tag{3.7} \sum S_\alpha^2=\frac{1}{p}S^2+\frac{1}{p}\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2.$$

Thus formula (3.5) is reduced to

$$\tag{3.8} \begin{aligned}\frac{1}{2}\Delta S=&\sum(h_{\alpha ijk})^2+ncS+\frac{1}{p}S^2+\frac{1}{p}\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2\\ &+\sum_{\alpha,\beta}N(H_\alpha H_\beta-H_\beta H_\alpha).\end{aligned}$$

4. Proofs of Theorems 1.1 and 1.2. We need the following theorem of [5] and [7] to prove our main results.

THEOREM 4.1 (Omori–Yau). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then for all $\epsilon>0$ there exists a point x in M such that, at x,*

$$\|\operatorname{grad} f\|<\epsilon,\quad \Delta f>-\epsilon,\quad \textit{and}\quad f(x)<\inf f+\epsilon.$$

LEMMA 4.2. *Let M be an n-dimensional complete Riemannian manifold isometrically immersed in $N_p^{n+p}(c)$. If M is maximal in $N_p^{n+p}(c)$, then $S=0$ or $S\le -cnp$.*

Proof. Using Proposition 2.1, we see that M satisfies the assumption of Theorem 4.1. We use the maximum principle argument as in [8]. Put $f=1/\sqrt{S+a}$ for any positive constant a. Then f is a bounded C^∞-function on M. By calculation, we have

$$\Delta f = -\frac{f^3}{2}\Delta S + 3f^5\|\text{grad } S\|^2. \tag{4.1}$$

Let ϵ be any positive number. Then Theorem 4.1 implies there is a point x on M such that, at x,

$$\frac{f^6}{4}\|\text{grad } S\|^2 < \epsilon, \quad \Delta f > -\epsilon, \quad f(x) < \inf f + \epsilon. \tag{4.2}$$

From (4.1) and (4.2) we get

$$\frac{f^4}{2}\Delta S < \epsilon(\inf f+\epsilon)+12\epsilon. \tag{4.3}$$

On the other hand, formula (3.8) states the following:

$$ncS + \frac{1}{p}S^2 \leq \frac{1}{2}\Delta S.$$

Substituting this into (4.3), we obtain

$$\frac{S}{(S+a)^2}\left(-nc - \frac{1}{p}S\right) \geq -\epsilon(\inf f+\epsilon) - 12\epsilon. \tag{4.4}$$

When $\epsilon \to 0$, $f(x)$ goes to the infimum and $S(x)$ goes to the supremum. Thus, we conclude from (4.4) that the function S is bounded on M, and that if $S\neq 0$ then $S \leq -npc$. □

Proofs of Theorems 1.1 *and* 1.2. First, assume that $c \geq 0$. Then Lemma 4.2 implies that $S=0$; that is, M is totally geodesic in $N_p^{n+p}(c)$. Next, if $c<0$ then (from Lemma 4.2) we obtain $S \leq -npc$. This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.3. Let M be an n-dimensional complete maximal space-like submanifold of $H_p^{n+p}(1)$ with $S=np$. Then from (3.8) it is clear

$$h_{\alpha ijk}=0, \quad S_\alpha = S_\beta, \quad H_\alpha H_\beta = H_\beta H_\alpha \quad \text{for any } i,j,k,\alpha,\beta. \tag{5.1}$$

Thus we have

$$S_\alpha = n. \tag{5.2}$$

The equalities $H_\alpha H_\beta = H_\beta H_\alpha$ imply that all of H_α are simultaneously diagonalizable, and that the normal connection in the normal bundle of M is flat. Hence, choosing a suitable base $e_1,\dots,e_n$, we have $h_{\alpha ij}=0$ for $i\neq j$, and put

$$h_{\alpha i} = h_{\alpha ii}. \tag{5.3}$$

As M is maximal,

$$\sum_i h_{\alpha i} = 0. \tag{5.4}$$

LEMMA 5.1. *All $h_{\alpha i}$ are constant on M. By changing the order of $e_1, \ldots, e_n$ we can put*

$$h'_{\alpha 1} = h_{\alpha 1} = \cdots = h_{\alpha m_1}, \quad h'_{\alpha 2} = h_{\alpha m_1+1} = \cdots = h_{\alpha m_2}, \quad \ldots,$$
$$h'_{\alpha s+1} = h_{\alpha m_s+1} = \cdots = h_{\alpha m_{s+1}} \quad \textit{for } n+1 \le \alpha \le n+p,$$

where, if $a \neq b$, $h'_{\alpha a} \neq h'_{\alpha b}$ for some α. Set

$$\mathbf{h}_\alpha = (h'_{\alpha 1}, \ldots, h'_{\alpha s+1}) \quad \textit{for } n+1 \le \alpha \le n+p,$$
$$\mathbf{h}_a = (h'_{n+1a}, \ldots, h'_{n+pa}) \quad \textit{for } 1 \le a \le s+1.$$

Then $\mathbf{h}_a \neq \mathbf{h}_b$ for $a \neq b$ and

$$\langle \mathbf{h}_a, \mathbf{h}_b \rangle = \sum_\alpha h'_{\alpha a} h'_{\alpha b} = -1. \tag{5.5}$$

Moreover, if we put $m_0 = 0$ we have

$$\omega_{ij} = 0 \quad \textit{for } m_{a-1}+1 \le i \le m_a, \ m_{b-1}+1 \le j \le m_b \ (a \neq b). \tag{5.6}$$

Proof. We modify slightly the argument in the proof of Lemma 3 in [3]. As $h_{\alpha ijk} = 0$, setting $i = j$ in (3.1) we get

$$0 = dh_{\alpha i} + \sum h_{\alpha ik}\omega_{ki} + \sum h_{\alpha ki}\omega_{ki} = dh_{\alpha i},$$

where we use $\omega_{\alpha\beta} = 0$ because the normal connection is flat. Hence $h_{\alpha i}$ are constant. If $h_{\alpha i} \neq h_{\alpha j}$, since (3.1) implies

$$0 = \sum h_{\alpha ik}\omega_{kj} + \sum h_{\alpha kj}\omega_{ki} = (h_{\alpha i} - h_{\alpha j})\omega_{ij},$$

we get $\omega_{ij} = 0$, and hence (5.6). If $h_{\alpha i} \neq h_{\alpha j}$ for some i, j, α, we also have

$$0 = d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} + \sum \omega_{i\beta} \wedge \omega_{\beta j} - \omega_i \wedge \omega_j.$$

As $\sum \omega_{ik} \wedge \omega_{kj} = 0$, we obtain

$$\left(\sum_\beta h_{\beta i} h_{\beta j} + 1\right)\omega_i \wedge \omega_j = 0.$$

Hence we obtain $\sum_\beta h_{\beta i} h_{\beta j} + 1 = 0$ if $h_{\alpha i} \neq h_{\alpha j}$ for some α. This shows (5.5).

Put $n_a = m_a - m_{a-1}$ and

$$H_{\alpha a} = \sqrt{n_a/n}\, h'_{\alpha a}, \quad H_{n+p+1a} = \sqrt{n_a/n} \quad \text{for } n+1 \le \alpha \le n+p, \ 1 \le a \le s+1.$$

Now, we put

$$\mathbf{H}_\alpha = (H_{\alpha 1}, \ldots, H_{\alpha s+1}), \qquad \mathbf{H}_a = {}^t(H_{n+1a}, \ldots, H_{n+p+1a}).$$

Since (5.4) is rewritten as $\sum n_a h'_{\alpha a} = 0$, we have

$$\langle \mathbf{H}_{n+p+1}, \mathbf{H}_\alpha \rangle = 0 \quad \text{for } n+1 \le \alpha \le n+p. \tag{5.7}$$

From (5.5), it follows that

$$\langle \mathbf{H}_a, \mathbf{H}_b \rangle = 0 \quad \text{for } 1 \le a < b \le s+1. \tag{5.8}$$

As (5.2) implies $\sum_a n_a h'^2_{\alpha a} = n$, we obtain

(5.9) $$\|\mathbf{H}_\alpha\|^2=1 \quad \text{for } n+1\le \alpha\le n+p+1. \qquad \square$$

We consider the matrix $\mathbf{H}=(H_{\alpha a})$ as the linear mapping $\mathbf{H}:\mathbf{R}^{s+1}\to\mathbf{R}^{p+1}$. Then we have the following.

LEMMA 5.2. *The matrix* $\mathbf{H}$ *is square and orthogonal.*

Proof. First we will show $s\le p$. As nonzero vectors $\mathbf{H}_1,\dots,\mathbf{H}_{s+1}$ of $\mathbf{R}^{p+1}$ satisfy (5.8), they are linearly independent. Hence $s+1\le p+1$. Put

$$T_{\alpha\beta}=\sum_a H_{\alpha a}H_{\beta a}, \qquad T=\sum_{\alpha,\beta}(T_{\alpha\beta})^2.$$

Taking a suitable base of $\mathbf{R}^{p+1}$, we diagonalize the symmetric matrix $(T_{\alpha\beta})$ and put $T_\alpha=T_{\alpha\alpha}=\|\mathbf{H}_\alpha\|^2$. Then we have

(5.10) $$T=\sum T_\alpha^2.$$

Put

$$U=\sum_\alpha T_\alpha=\sum_{\alpha,a}(H_{\alpha a})^2$$

and

$$(p+1)\sigma_1=\sum_\alpha T_\alpha=U, \qquad \frac{p(p+1)}{2}\sigma_2=\sum_{\alpha<\beta}T_\alpha T_\beta.$$

Then, as in (3.7), we have

$$T=\frac{1}{p+1}U^2+\frac{1}{p+1}\sum_{\alpha<\beta}(T_\alpha-T_\beta)^2.$$

Using (5.9), we obtain

(5.11) $$T=\frac{1}{p+1}U^2.$$

On the other hand, we set

$$T_a=\|\mathbf{H}_a\|^2=\sum_\alpha H_{\alpha a}H_{\alpha a} \quad \text{for } 1\le a\le s+1.$$

Then as $\langle\mathbf{H}_a,\mathbf{H}_b\rangle=0$ for $a\ne b$, we get

$$T=\sum_{\alpha,\beta,a}H_{\alpha a}H_{\alpha a}H_{\beta a}H_{\beta a}=\sum_a(T_a)^2.$$

As $U=\sum(H_{\alpha a})^2=\sum_a T_a$, we have

(5.12) $$T=\frac{1}{s+1}U^2+\frac{1}{s+1}\sum_{a<b}(T_a-T_b)^2.$$

From (5.11) and (5.12) we obtain

$$\frac{s-p}{(p+1)(s+1)}U^2=\frac{1}{s+1}\sum_{a<b}(T_a-T_b)^2.$$

This implies that $s=p$ and $\|\mathbf{H}_a\|=\|\mathbf{H}_b\|$. From (5.8) and (5.9), it follows that the matrix $\mathbf{H}$ is orthogonal. $\square$

Proof of Theorem 1.3. First, we assume that M is connected, simply connected and complete. For each $1\le a\le p+1$, define the distribution D_a by

$$\omega_1=0,\ldots,\omega_{m_1}=0,\ \omega_{m_1+1}=0,\ldots,\omega_{m_2}=0,\ldots,\omega_{m_a}=0,\ \omega_{m_{a+1}+1}=0,\ldots,\omega_n=0.$$

Then, D_a is globally defined on M. From (5.6), it is clear that D_a is integrable and parallel. Take a fixed point x of M. Let M_a be the maximal integrable submanifold of D_a through x; then it is n_a-dimensional, complete, connected and totally geodesic in M. Since M is simply connected and complete and each D_a is parallel, we conclude by a standard argument that M is a Riemannian product $M_1\times M_2\times\cdots\times M_{p+1}$. Since M is complete and simply connected, and since each M_a is totally geodesic in M, M_a is complete and simply connected. If $n_a=1$ then $M_a=R$. We may consider R as $H^1(\sqrt{1/n})$. From the Gauss formula (2.5) it follows that the Riemannian curvature of M_a is expressed as

$$R_{ijkl}=-(1+\sum (h'_{\alpha a})^2)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).$$

From Lemma 5.2 we have $\|\mathbf{H}_a\|^2=\sum_\alpha H_{\alpha a}^2=1$, that is, $(\sum (h'_{\alpha a})^2+1)(n_a/n)=1$. Hence, M_a is a space of constant curvature $-n/n_a$. Thus, in any case, we can put $M_a=H^{n_a}(\sqrt{n_a/n})$.

If M is complete and connected but not simply connected, let M be its simply connected Riemannian covering manifold. Then the composition mapping of M in $H_p^{n+p}(1)$ under the covering mapping and the immersion of M in $H_p^{n+p}(1)$ satisfies the assumption of the theorem and is imbedded in $H_p^{n+p}(1)$ as above. Thus, M is immersed as the product submanifold of the theorem. □

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