A BOUNDEDNESS THEOREM FOR L^1/H_0^1

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1. Introduction. The purpose of this note is to prove a version of the classical Vitali–Hahn–Saks theorem, which concerns the dual pair (L^1, L^∞), in the context of $(L^1/H_0^1, H^\infty)$. Namely:

THEOREM. *Let* $U_n \in L^1(\mathbf{T})$, $n=1,2,\ldots$, *and assume that for each inner function* φ *we have:*

$$\left|\int_0^{2\pi} U_n(e^{i\theta})\varphi(e^{i\theta})\,d\theta\right| \le C(\varphi) < \infty, \quad n=1,2,\ldots;$$

then

$$\sup_n \|U_n\|_{L^1/H_0^1} < \infty.$$

The Vitali–Hahn–Saks theorem for the circle (in the real valued case) claims:

THEOREM A. *Let* $U_n \in L^1_{\mathbf{R}}(\mathbf{T})$, $n=1,2,\ldots$, *and assume for each unimodular* $v \in L^\infty_{\mathbf{R}}(\mathbf{T})$ *that*

$$\left|\int_0^{2\pi} U_n(e^{i\theta})v(e^{i\theta})\,d\theta\right| \le C(v) < \infty, \quad n=1,2,\ldots;$$

then

$$\sup_n \|U_n\|_{L^1} < \infty.$$

If $\mathfrak{M}$ denotes the linear span of the inner functions then it is easy to see that $\mathfrak{M}$ is of first category in H^∞, which makes the theorem relevant as in the case of Theorem A. On the other hand, a theorem of Marshall (see [6], [7]) asserts that $\mathfrak{M}$ is dense in H^∞. But that does not imply the theorem.

An immediate consequence of the theorem is the elementary fact that $\mathfrak{M}$ is weak* dense in H^∞.

Now Theorem A has a stronger version (see [2, p. 80]).

THEOREM B. *Let* $\Lambda_n \in L^\infty_{\mathbf{R}}(\mathbf{T})^*$ *and assume that for any unimodular* $v \in L^\infty_{\mathbf{R}}(\mathbf{T})$ *we have that*

$$|\Lambda_n(v)| \le C(v) < \infty, \quad n=1,2,\ldots;$$

then

$$\sup_n \|\Lambda_n\|_{L^\infty(\mathbf{T})^*} < \infty.$$

The author does not know whether or not the corresponding analog of Theorem B holds in the case of H^∞, which would be a stronger result than Marshall's theorem. The ideas behind the proof of the theorem (which comes from [4]) are

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also akin to one of the proofs of the existence of inner functions in the unit ball of $\mathbf{C}^n$ [10] and work there also; because the theorem holds there too, these ideas alone cannot settle the above question in the affirmative since they would then imply that Marshall's theorem holds also in $\mathbf{C}^n$, where it does not ([9]). Thus an extra ingredient would be needed to prove (if it is true) this analytic Nikodym–Grothendieck theorem.

A consequence of the theorem is the following corollary, which settles a question of H. Shapiro (private communication).

COROLLARY 1. *There exists an inner function* $\varphi(z)=\sum_{n=0}^{\infty} a_n z^n$ *with* $\{\sum_{n=0}^{N} a_n\}_N$ *unbounded.*

This follows from the result of Landau that $\|\sum_{n=0}^{N} e^{-in\theta}\|_{L^1/H_0^1} \sim (1/\pi)\log N$ as $N\to\infty$. See for example [3, p. 144] and the references therein.

Another corollary is the following extension of the fact that L^1/H_0^1 is weakly sequentially complete (Mooney's theorem) ([8], [1]).

COROLLARY 2. *If* $U_n \in L$, $n=1,2,\ldots$, *and if for each inner function* φ *the sequence* $\{\int_0^{2\pi} U_n(e^{i\theta})\varphi(e^{i\theta})\,d\theta\}_{n=1}^{\infty}$ *converges, then there exists an* L^1*-function* U *such that*

$$\int_0^{2\pi} U_n(e^{i\theta})\varphi(e^{i\theta})\,d\theta \to \int_0^{2\pi} U(e^{i\theta})\varphi(e^{i\theta})\,d\theta.$$

To see this, use the theorem to obtain that $\{\|U_n\|_{L^1/H_0^1}\}_n$ is bounded. Since $\mathfrak{M}$ is dense one gets that $\int_0^{2\pi} U_n(e^{i\theta})f(e^{i\theta})\,d\theta$ converges for every $f\in H^\infty$, and thus Mooney's theorem gives the corollary.

By B we denote the unit ball of H^∞ and by $\mathcal{P}$ the vector space of all analytic polynomials; P_N and Q_N consist respectively of those elements of $\mathcal{P}$ of degree at most N and those whose lowest-order term is at least of degree N.

2. Proof of the Theorem.

If the result does not hold then there exists a sequence of L^1-functions U_n such that

(a) $\lim_n \int U_n\varphi = 0$ for each φ inner, and

(b) $\|U_n\|_{L^1/H_0^1} \uparrow \infty$.

We use the following two lemmas.

LEMMA 1. *If* $\{U_n\}_{n=1}^{\infty} \subset L^1$ *satisfies* (a) *and* (b), *then given* $\delta>0$ *and* $N\in\mathbf{N}$ *there exist a polynomial p and n > N such that*

$$p\in Q_N; \tag{1}$$

$$\|p\|_\infty < \delta; \tag{2}$$

$$\int U_n p > \frac{1}{\delta}. \tag{3}$$

LEMMA 2. *Let* $g\in\mathcal{P}$, $\|g\|<1$, *and* $\{v_j\}_{j=1}^{m}\subset L^1$. *Then, given* $r\in(0,1)$ *and* $\epsilon>0$, *there exist* $q\in\mathcal{P}$ *and* $s\in(r,1)$ *satisfying*

$$\|g+q\|<1; \tag{4}$$

$$|q(z)|\le\epsilon \quad \text{if } |z|<r; \tag{5}$$

$$|\{e^{i\theta}: |(g+q)(se^{i\theta})|>1-\epsilon\}|>(1-\epsilon)2\pi; \tag{6}$$

$$\left|\int v_j q\right|<\epsilon, \quad j=1,\dots,m. \tag{7}$$

Assuming the lemmas above, the proof of the theorem goes as follows.

Inductively we shall obtain sequences of polynomials $\{h_j\}_{j=1}^{\infty}$, integers $\{n_j\}_{j=1}^{\infty}$, and positive numbers $\{r_j\}_{j=0}^{\infty}$, $r_j\uparrow 1$, so that if we set $f_k=\sum_{j=1}^{k} h_j$ then for each k:

$$\|f_k\|_\infty<1; \tag{8}$$

$$U_{n_k}(h_k)\ge(k+1)+|U_{n_k}(f_{k-1})|; \tag{9}$$

$$|U_{n_j}(h_k)|\le 2^{-k}, \quad j<k; \tag{10}$$

$$|\{e^{i\theta}: 1-|f_k(r_k e^{i\theta})|<2^{-k}\}|\ge(1-k^{-1})2\pi; \tag{11}$$

$$|h_k(z)|<2^{-k} \quad \text{if } |z|\le r_{k-1}. \tag{12}$$

Because of (12) we have that $\{f_k\}$ converges locally uniformly to an analytic function f, which by (8) has $\|f\|_\infty\le 1$. Moreover, f_k converges to f in the weak* topology of H^∞ and so (9) and (10) combine to show that $|U_{n_k}(f)|>k$, but by (11) and (12) we have

$$|\{e^{i\theta}: 1-|f(r_k e^{i\theta})|>2^{-k+1}\}|>1-k^{-1}$$

which in particular shows that f is inner, providing us with a contradiction.

We begin the induction by using Lemma 1 to obtain $f_1=h_1$ and $h_1\in\mathbf{N}$ so that (8)–(12) hold ($f_0=h_0=0$, $U_0=0$).

Assume now that we have obtained $\{h_j\}_{j=1}^{k}$, $\{n_j\}_{j=1}^{k}$, $\{r_j\}_{j=1}^{k}$ so that (8)–(12) hold. Let $N=1+\deg f_k+n_k$ and $M=\sup_{m\in\mathbf{N}}|U_m(f_k)|+\max_{j\le k}\|U_{n_j}\|+k+2$.

We now use Lemma 1 with $\delta=2^{-k-2}M^{-1}(1-\|f_k\|)$ to obtain a polynomial p and $n>N$ such that (1), (2), and (3) hold. We set $n_{k+1}=n$. Now with $\epsilon=2^{-(k+2)}$, $g=f_k+p$, $v_j=U_{n_j}$ $(j=1,\dots,k+1)$, and $r=r_k$, we use Lemma 2 to produce a polynomial q and $s\in(r,1)$ such that (4)–(7) hold. We also set $h_{k+1}=p+q$ and $r_{k+1}=s$.

For $k+1$, (8) is clearly satisfied, and (11) follows from (6). Also, since $\delta+\epsilon<2^{-(k+1)}$, (12) holds. Now

$$|U_{n_{k+1}}(h_{k+1})|\ge|U_{n_{k+1}}(p)|-|U_{n_{k+1}}(q)|\ge\frac{1}{\delta}-\epsilon>M-1,$$

and so (9) holds.

Finally, if $j<k+1$ then

$$|U_{n_j}(h_{k+1})|\le|U_{n_j}(p)|+|U_{n_j}(q)|\le\delta\|U_{n_j}\|+\epsilon\le 2^{-k-1}$$

and (10) holds.

This finishes the proof of the theorem. □

3. Proof of Lemma 1. By Banach–Steinhaus there is $f \in H^\infty$, $\|f\|_\infty < \delta/2N$ such that $\sup_m |U_m(f)| = \infty$. Let $f_N(z) = \sum_{n=0}^N \hat{f}(n) z^n$; then $\|f_N\|_\infty \le \delta/2$ so that if $g = f - f_N$ we have $\|g\|_\infty < \delta$. Now $\sup_m |U_m(f_N)| < \infty$, because f_N is a linear combination of inner functions. Therefore $\sup_m |U_m(g)| = \infty$. Choose $n > N$ with $|U_n(g)| > 1/\delta$ and then take $p = \sum_{n=N+1}^M \hat{f}(n) r^k z^n$ with appropriate $r \in (0,1)$ and $M > N$ to get the desired polynomial. □

4. Proof of Lemma 2. We may assume $\|f\|_\infty + 3\epsilon < 1$.

We can partition $\mathbf{T}$ into intervals $\{I_j\}_{j=1}^L$ so that the oscillation of g in each of them is at most η. We can find $s \in (r,1)$ and a_j ($|a_j| < 1$) with $|g(te^{i\theta}) - a_j| < \eta$ if $e^{i\theta} \in I_j$, $j = 1, \dots, L$, $s \le t \le 1$. There $\eta \in (0, \epsilon)$ is to be specified later in terms of ϵ.

Let p_j be trigonometrical polynomials such that

$$|p_j(e^{i\theta})| < \frac{\eta}{2L} \quad \text{if } e^{i\theta} \notin I_j;$$

$$|\{e^{i\theta} \in I_j : |p_j(e^{i\theta}) - 1| < \eta\}| > (1-\epsilon)|I_j|;$$

and

$$|p_j(e^{i\theta})| + |1 - p_j(e^{i\theta})| < 1 + \eta \quad \text{for each } \theta \in [0, 2\pi].$$

Let $K_0 \in \mathbf{N}$ be such that $z^{K_0} p_j$ is, for each j, an analytic polynomial and such that $r^{K_0} < \epsilon/4L$.

Now if β_j are analytic polynomials, $\|\beta_j\|_\infty \le 2$, and $\beta_j(0) = 0$, then: if $k \ge K_0$ then $q(z) = \sum_{j=1}^L \beta_j(z^k) p_j(z)$ is an analytic function, and if $|z| \le r$ then $|q(z)| \le 4L|z|^k$ by Schwarz' lemma, and so (5) holds.

Moreover, if β_j satisfies

$$1 - 8\eta < |\beta_j(e^{i\theta}) + a_j| < 1 - 6\eta \quad \text{for each } \theta \in [0, 2\pi],$$

then if $e^{i\theta} \in I_j$ we have that

$$\begin{aligned} |g + q(e^{i\theta})| &\le 2\eta + |\beta_j(e^{ik\theta}) p_j(e^{i\theta}) + a_j| \\ &= 2\eta + |(\beta_j(e^{ik\theta}) + a_j) p_j(e^{i\theta}) + a_j(1 - p_j(e^{i\theta}))| \\ &\le 2\eta + (1 - 6\eta)[|p_j(e^{i\theta})| + |1 - p_j(e^{i\theta})|] \\ &= 2\eta + (1 - 6\eta)(1 + \eta) < 1 - 3\eta < 1. \end{aligned}$$

Furthermore, if $|p_j(e^{i\theta}) - 1| < \eta$ then, as above,

$$|(g+q)(e^{i\theta})| \ge |\beta_j(e^{ik\theta}) + a_j|(1 - \eta) - 3\eta \ge (1 - 8\eta)(1 - \eta) - 3\eta \ge 1 - \epsilon$$

if η is small enough. Thus,

$$|\{e^{i\theta} : |(g+q)(e^{i\theta})| \ge 1 - \epsilon\}| > (1 - \epsilon) 2\pi.$$

Therefore (4), (5), and (6) are satisfied as long as $k \ge K_0$ and if β_j are as above. Now to obtain (7) we need only observe that if $u \in L^1$ then $\int uq \to 0$ as $k \to \infty$. But this just follows from the Riemann–Lebesgue lemma.

Finally, for $\lambda = 1 - 7\eta$ consider

$$\tilde{\beta}_j(z) = \lambda \frac{\lambda z + a_j}{\lambda + z \bar{a}_j} - a_j$$

and let β_j be an analytic polynomial with $\|\beta_j - \tilde{\beta}_j\| < \eta$, $\beta_j(0) = 0$. □

5. Example. Let φ be an infinite Blaschke product and let

$$U_n(e^{i\theta}) = \overline{\varphi'\left(\left(1-\frac{1}{n}\right)e^{i\theta}\right)e^{i\theta}}.$$

Then $U_n \in L^1$. Consider U_n as linear functionals on A (the disk algebra). If b is a finite Blaschke product then

$$U_n(b) = \int_0^{2\pi} b(e^{i\theta})\overline{\varphi'\left(\left(1-\frac{1}{n}\right)e^{i\theta}\right)e^{i\theta}}\,d\theta = \int_0^{2\pi} e^{i\theta}b'\left(\left(1-\frac{1}{n}\right)e^{i\theta}\right)\overline{\varphi(e^{i\theta})}\,d\theta.$$

Thus

$$\lim_{n\to\infty} |U_n(b)| \leq 2\pi\,(\text{degree of } b).$$

But $\|U_n\|_{A^*} = \|U_n\|_{(H^\infty)^*}$ and $U_n(\varphi) \to \sum_{n=0}^{\infty} n|\hat{\varphi}(n)|^2$, and this series is infinite because otherwise $\varphi \in$ Dirichlet space, rendering φ a finite Blaschke product. Thus the theorem does not have an analog for the disk-algebra and finite Blaschke products—that is, the inner functions in the disk algebra—although the closure of their convex combination is the unit ball of A (see [5]).

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