ON FRACTIONAL DERIVATIVES AND STAR INVARIANT SUBSPACES

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1. Introduction and statement of main results. Let $\phi(z)$ be an inner function defined on the unit disk $D=\{|z|<1\}$. Factor ϕ canonically as

$$\phi(z)=\lambda B(z)s_\sigma(z),$$

where $|\lambda|=1$,

$$B(z)=\prod_{k=1}^{\infty}\frac{\bar{a}_k}{|a_k|}\frac{a_k-z}{1-\bar{a}_k z}$$

is a Blaschke product and

$$s_\sigma(z)=\exp\left(-\int_T \frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta)\right)$$

where σ is a positive singular measure on the unit circle T.

In [4] we proved the following result, extending earlier work of Frostman, Riesz, and Ahern and Clark (see [6] and [1]).

THEOREM A. *Let* $\zeta_0\in T$, $\phi=Bs_\sigma$, *and* $1<p<\infty$.

(1) *Necessary and sufficient that* $\lim_{r\to 1} f(r\zeta_0)$ *exist for all* $f\in K_*(\phi)$ *is that*

$$\sum_k \frac{1-|a_k|}{|\zeta_0-a_k|}+\int_T \frac{d\sigma(\zeta)}{|\zeta_0-\zeta|}<\infty.$$

(2) *Necessary and sufficient that* $\lim_{r\to 1} f(r\zeta_0)$ *exist for all* $f\in K_p(\phi)$ *is that*

$$\sum_k \frac{1-|a_k|}{|\zeta_0-a_k|^q}+\int_T \frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^q}<\infty.$$

(Here and in the sequel, by $\lim_{r\to 1} f(r\zeta_0)$ we mean $\lim_{r\to 1-} f(r\zeta_0)$.)

The spaces $K_p=K_p(\phi)$ and $K_*=K_*(\phi)$ are the "star-invariant" subspaces of H^p and BMOA determined by

$$K_p(\phi)=\phi\bar{H}_0^p\cap H^p \quad\text{and}\quad K_*(\phi)=K_2(\phi)\cap \mathrm{BMO},$$

where $\bar{H}_0^p=\{\bar{z}\bar{f}(z): f\in H^p\}$.

Although derivatives are not mentioned in [4] it is not difficult to conjecture (and prove) the correct results for the radial behavior of $f^{(1)}, f^{(2)}, \dots$ if $f\in K_p$ or K_*, and arrive at the following result.

THEOREM A′. *Let* $\zeta_0\in T$, $\phi=Bs_\sigma$, $1<p<\infty$ *and* $n=0,1,2,\dots$.

(1) *Necessary and sufficient that* $\lim_{r\to 1} f^{(n)}(r\zeta_0)$ *exist for all* $f\in K_*$ *is that*

$$\sum \frac{1-|a_k|}{|\zeta_0-a_k|^{n+1}}+\int_T \frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^{n+1}}<\infty.$$

Received September 17, 1986. Revision received February 5, 1987.
Research supported in part by NSF.
Michigan Math. J. 34 (1987).

(2) *Necessary and sufficient that* $\lim_{r\to 1} f^{(n)}(r\zeta_0)$ *exist for all* $f\in K_p$ *is that*

$$\sum \frac{1-|a_k|}{|\zeta_0-a_k|^{q(n+1)}}+\int_T \frac{d\sigma}{|\zeta_0-\zeta|^{q(n+1)}}<\infty,$$

where $1/p+1/q=1$.

In this paper, we will use the methods of [4] to study fractional derivatives and integrals of functions in K_p and K_*. It turns out that in this situation we can provide more information about the radial behavior of derivatives of K_p and K_* functions than one might first guess after glancing at Theorems A and A′.

If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is holomorphic on D we define the fractional derivative

$$(D^\alpha f)(z)=\sum_{n=0}^{\infty}(n+1)^\alpha a_n z^n,$$

where $\alpha\in\mathbf{R}$. We define the fractional integral

$$(I^\alpha f)(z)\sum_{n=0}^{\infty}(n+1)^{-\alpha} a_n z^n,$$

where $\alpha\in\mathbf{R}$. It is obvious that $D^\alpha f=I^{-\alpha}f$. If $\beta>0$ and $\alpha>0$ the following formulas are easily verified:

$$(1.1)\qquad (I^\beta f)(z)=\frac{1}{\Gamma(\beta)}\int_0^1\left(\log\frac{1}{t}\right)^{\beta-1} f(tz)\,dt$$

$$(1.2)\qquad f(z)=\frac{1}{\Gamma(\alpha)}\int_0^1\left(\log\frac{1}{t}\right)^{\alpha-1}(D^\alpha f)(tz)\,dt.$$

Notice also that if $n=1,2,3,\dots$ then

$$(1.3)\qquad (D^n f)(z)=\left(\frac{d}{dz}\cdot z\right)^n f(z)=z^n f^{(n)}(z)+\sum_{k=0}^{n-1} c(k,n)z^k f^{(k)}(z)$$

for nonzero constants $c(k,n)$.

Our main results are summarized below.

THEOREM 1. *Let* $\phi=Bs_\sigma$, $n=-1,0,1,\dots$, *and* $0<\alpha<1$. *Suppose* $1<p<\infty$, $1/p+1/q=1$, *and* $1\le q(n+1+\alpha)$. *If* $\zeta_0\in T$ *then the following conditions are equivalent:*

(1) $\lim_{r\to 1}(D^{n+\alpha}f)(r\zeta_0)$ *exists for all* $f\in K_p$.

$$(2)\qquad \sum\frac{1-|a_k|}{|\zeta_0-a_k|^{q(n+1+\alpha)}}+\int_T\frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^{q(n+1+\alpha)}}<\infty.$$

$$(3)\qquad \int_0^1 |D^{n+1}f(t\zeta_0)|\left(\log\frac{1}{t}\right)^{-\alpha}dt<\infty \quad \textit{for all } f\in K_p.$$

THEOREM 2. *Let* $\phi=Bs_\sigma$, $n=0,1,\dots$, $0<\alpha<1$, *and* $\zeta_0\in T$. *The following conditions are equivalent:*

(1) $\sup_{0\le r<1}|D^{n+\alpha}f)(r\zeta_0)|\le C\|f\|_*$ *for all* $f\in K_*$, *with* C *independent of* f.

(2) $$\sum \frac{1-|a_k|}{|\zeta_0-a_k|^{n+1+\alpha}}+\int_T \frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^{n+1+\alpha}}<\infty.$$

(3) $$\int_0^1 |D^{n+1}f(t\zeta_0)|\left(\log\frac{1}{t}\right)^{-\alpha} dt \le C\|f\|_* \quad \textit{for all } f\in K_*,$$

with C independent of f.

REMARK 1. Notice that in the fractional case condition (3) of Theorems 1 and 2 do not have analogues in Theorems A and A′.

REMARK 2. For $1<p<\infty$, if $n=-1$ and $q\alpha<1$, that $D^{n+\alpha}f=I^{1-\alpha}f$ is continuous on the closed disk for any $f\in H^p$ follows from formula (1.1) by Hölder's inequality and a standard approximation argument. This explains the restrictions on n and p in Theorems 1 and 2.

We also get analogues of Theorems 1 and 2 for Cauchy type integrals of functions $f\in K_p$ and K_*. These provide additional conditions equivalent to conditions of type (2) above. The following results are contained in Theorems 3 and 4.

THEOREM 3′. *Under the assumptions of Theorem* 1 *the following conditions are equivalent:*

(1) *Condition* (2) *of Theorem* 1 *holds.*

(2) *There exists a* $g\in H^q$ *such that* $(1-\phi g)(\zeta_0-z)^{-n-1-\alpha}\in K_q$.

THEOREM 4′. *Under the assumptions of Theorem* 2 *the following conditions are equivalent:*

(1) *Condition* (2) *of Theorem* 2 *holds.*

(2) *There exists a* $g\in H^1$ *such that* $(1-\phi g)(\zeta_0-z)^{-n-1-\alpha}\in H^1$.

As an application of Theorem 1 we prove the following characterization of Blaschke products whose zeros satisfy the Frostman condition.

THEOREM 5. *Let* $1<p<\infty$. *The following conditions are equivalent:*

(1) $(I^{1/p}f)(z)$ *extends to be continuous on the closed disk* $\bar{D}$ *for all* $f\in K_p(\phi)$.

(2) ϕ *is a Blaschke product whose zero sequence* $\{z_k\}$ *satisfies the Frostman condition*

$$\sup\left\{\sum\frac{1-|z_k|}{|\zeta-z_k|}:\zeta\in T\right\}<\infty.$$

(3) $(C_{(p-1)/p}f)(z)$ *extends to be continuous on the closed disk for all* $f\in K_p(\phi)$.

See Section 2 for the definition of $C_{n+\alpha}f$.

Theorem 5 may be regarded as a supplement to the following fact.

THEOREM (see Proposition 3.1 and its Corollary in [4]). *If* ϕ *is inner then* $K_*(\phi)=K_\infty(\phi)$ *if and only if* ϕ *is a Blaschke product whose zero sequence satisfies the Frostman condition.*

This theorem may be interpreted as the $p=\infty$ version of Theorem 5.

The proofs of Theorems 1–5 will be given in the next section. We will need the following result stated in [4, §2].

THEOREM B. *For an inner function* ϕ *and* $0<\delta<1$ *there is a region* $\mathcal{R}$ *and "Carleson curve"* Γ_δ *with the following properties:*

(i) $\Gamma=\partial\mathcal{R}\cap D$ *separates* $\{z:|\phi(z)|\geq\delta\}$ *from* $\{z:|\phi(z)|<\epsilon(\delta)\}$, *where* $\epsilon(\delta)<\delta$;

(ii) $\{z:|\phi(z)|<\epsilon(\delta)\}\subseteq\mathcal{R}$;

(iii) *arclength on* Γ_δ *is a Carleson measure*;

(iv) $|\phi(z)|\leq\delta$ *for* $z\in\Gamma_\delta$;

(v) $\Gamma=\bigcup_n\gamma_n$, *where* $\gamma_n=[a_n,b_n]$ *is a circular arc or radial segment and there are constants* σ_1 *and* σ_2 *such that*

$$0<\sigma_1\leq\left|\frac{a_n-b_n}{1-\bar{a}_nb_n}\right|\leq\sigma_2<1;$$

(vi) *if* ω_n *is the midpoint of* γ_n *then* $\{\omega_n\}$ *is a uniformly separated sequence, and if* $B_\phi=B_{\phi,\Gamma_\delta}$ *is the Blaschke product with zero sequence* $\{\omega_n\}$ *then there are constants* c_1 *and* c_2 *such that*

$$c_1(1-|\phi(z)|)\leq(1-|B_\phi(z)|)\leq c_2(1-|\phi(z)|);$$

(vii) *if* $\zeta_0\in T$ *then with* $\phi=Bs_\sigma$ *and* $\gamma\geq1$,

$$\sum\frac{1-|a_k|}{|\zeta_0-a_k|^\gamma}+\int_T\frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^\gamma}<\infty$$

if and only if

$$\sum\frac{1-|\omega_n|}{|\zeta_0-\omega_n|^\gamma}<\infty.$$

We assume familiarity with results in the literature concerning H^p, BMOA, Carleson measures, uniformly separated sequences, and the non-Euclidean metric. As references we cite [5] and [7]. For two analytic functions f and g defined on D we let $\langle f,g\rangle$ denote the usual pairing

$$\langle f,g\rangle=\lim_{r\to1}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(re^{i\theta})\overline{g(re^{i\theta})}\,d\theta$$

in the cases where the limit exists. By $\|f\|_*$ we mean the BMOA norm of f; $\|f\|_p$ will denote the H^p norm of f. The notation $A\doteq B$ will be used if there are positive constants m and M such that $mA\leq B\leq MA$. Finally, the letters $c,c_1,c_2,\ldots$ etc. will denote various constants which certainly differ in value in different inequalities.

2. Proofs of main results. We will need the following computational results which are well known.

LEMMA 1. *Let* $0\leq\alpha<1$, $|z|\leq1$, *and* $\gamma>1-\alpha$. *Then there are constants* $c_1(\alpha,\gamma)$ *and* $c_2(\alpha,\gamma)$ *independent of* z *such that*

$$\frac{c_1(\alpha,\gamma)}{|1-z|^{\gamma-1+\alpha}}\leq\int_0^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{dt}{|1-tz|^\gamma}\leq\frac{c_2(\alpha,\gamma)}{|1-z|^{\gamma-1+\alpha}}.\tag{2.1}$$

Now let $|w|<1$, $0<\alpha<1$, and choose a branch of the logarithm so $(1-w)^\alpha$ is positive if $w=0$.

LEMMA 2. *If* $n=0,1,2,\dots$ *then*

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{1}{(1-z)^{n+\alpha}}=(n!)\frac{\sin\pi\alpha}{\pi}\int_0^1\frac{t^{\alpha-1+n}(1-t)^{-\alpha}}{(1-tz)^{n+1}}\,dt.$$

Proof. By the binomial theorem

$$\begin{aligned}\frac{1}{(1-z)^\alpha}&=\sum_{n=0}^\infty\frac{\Gamma(\alpha+n)}{\Gamma(n+1)\Gamma(\alpha)}z^n\\&=\frac{\sin\pi\alpha}{\pi}\sum_{n=0}^\infty\frac{\Gamma(\alpha+n)\Gamma(1-\alpha)}{\Gamma(n+1)}z^n\\&=\frac{\sin\pi\alpha}{\pi}\sum_{n=0}^\infty\int_0^1 t^{n+\alpha-1}(1-t)^{-\alpha}\,dt\,z^n\\&=\frac{\sin\pi\alpha}{\pi}\int_0^1\frac{t^{\alpha-1}(1-t)^{-\alpha}}{(1-tz)}\,dt,\end{aligned}$$

where we have used the facts that $\sin\pi\alpha\Gamma(\alpha)\Gamma(1-\alpha)=\pi$ and

$$\int_0^1 t^{n+\alpha-1}(1-t)^{-\alpha}\,dt=\frac{\Gamma(n+\alpha)\Gamma(1-\alpha)}{\Gamma(n+1)};$$

see [10, pp. 239 and 254]. The result follows by differentiation with respect to z.

As a consequence of Lemma 2, suppose $0<t_0<1$. Since

$$\int_0^{t_0}\frac{t^{n+\alpha+1}(1-t)^{-\alpha}}{(1-tz)^{n+1}}\,dt$$

remains uniformly bounded as z varies (and n and α are held fixed), Lemma 2 shows that there is a constant $c(n,\alpha)>0$ independent of t_0 and a $\delta>0$ such that $|z-1|<\delta$ implies

$$\left|\int_{t_0}^1\frac{t^{n+\alpha-1}(1-t)^{-\alpha}}{(1-tz)^{n+1}}\,dt\right|\geq\frac{c(n,\alpha)}{|1-z|^{n+\alpha}}.$$

This observation gives the next fact.

COROLLARY 1. *If* t_1 *is sufficiently close to* 1 *then there exist constants* $\delta_1>0$ *and* $c(n,\alpha)>0$ *with* $c(n,\alpha)$ *independent of* t_1, *such that* $|z-1|<\delta_1$ *implies that*

$$\left|\int_{t_1}^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{1}{(1-tz)^{n+1}}\,dt\right|\geq\frac{c(n,\alpha)}{|1-z|^{n+\alpha}}.$$

Proof. Since

$$\left|(1-t)^{-\alpha}-\left(\log\frac{1}{t}\right)^{-\alpha}\right|=O(|1-t|^{1-\alpha})\quad\text{as } t\to1$$

and

$$|t^{\alpha-1+n}-1|=O(|1-t|)\quad\text{as } t\to1,$$

using Lemma 1 we see that

$$\left|\int_{t_1}^{1} \frac{1}{(1-tz)^{n+1}}\left(\left(\log\frac{1}{t}\right)^{-\alpha} - t^{n+\alpha-1}(1-t)^{-\alpha}\right)dt\right|$$
$$\le \int_{t_1}^{1} \frac{|t|^{n+\alpha-1}|(1-t)^{-\alpha}-(\log(1/t))^{-\alpha}| + |t^{\alpha-1+n}-1|(\log(1/t))^{-\alpha}}{|1-tz|^{n+1}}\,dt$$
$$\le c|1-t_1|\frac{1}{|1-z|^{n+\alpha}}.$$

Choosing t_1 close enough to 1 so $c|1-t_1|$ is sufficiently small, the desired result follows from the observation made after Lemma 2. □

The next corollary is the last preliminary fact we will need. For related results, see [2]. First we need two formulas.

If B is a Blaschke product with zero sequence $\{a_k\}$, then

$$B'(z) = B(z)\sum \frac{1-|a_k|^2}{(z-a_k)(1-\bar{a}_k z)}$$
$$= -\sum B_{(k)}(z)\frac{\bar{a}_k}{|a_k|}\frac{(1-|a_k|^2)}{(1-\bar{a}_k z)^2},$$

where $B_{(k)}$ is the Blaschke product with zeros $\{a_j\}_{j\neq k}$. Leibnitz's rule gives that

$$B^{(N)}(z) = -\sum_{j=0}^{N-1}\binom{N-1}{j}\sum_k B_{(k)}^{(N-1-j)}(z)\frac{\bar{a}_k^{j+1}}{|a_k|}\frac{(j+1)!\,(1-|a_k|^2)}{(1-\bar{a}_k z)^{j+2}}, \tag{2.2}$$

for $N=1,2,3,\ldots$. Also, if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is holomorphic on the disk, then for $0<\alpha<1$ and $n=-1,0,1,\ldots$

$$\begin{aligned}(D^{n+\alpha}f)(z) &= (I^{1-\alpha}(D^{n+1}f))(z)\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^1\left(\log\frac{1}{t}\right)^{-\alpha}(D^{n+1}f)(tz)\,dt.\end{aligned} \tag{2.3}$$

This follows from (1.1) and (1.2).

COROLLARY 2. *Let $\phi = Bs_\sigma$, $n=0,1,2,\ldots$, $0<\alpha<1$. Suppose in addition that whenever ψ divides ϕ, $\lim_{r\to 1}\psi(r)$ exists and has modulus 1. Then the following conditions are equivalent:*

(1) *There is a constant $M>0$ such that*

$$\sup_{0\le r<1}|(D^{n+\alpha}\psi)(r)|\le M$$

for all divisors ψ of ϕ.

(2) $$\sum_k \frac{1-|a_k|}{|1-a_k|^{n+1+\alpha}} + \int\frac{d\sigma}{|1-\zeta|^{n+1+\alpha}} < \infty.$$

(3) *There is a constant $M>0$ such that*

$$\int_0^1 |(D^{n+1}\psi)(t)|\left(\log \frac{1}{t}\right)^{-\alpha} dt \le M$$

for all divisors ψ *of* ϕ.

Proof. Suppose $\psi = B_1 s_\tau$ divides ϕ, that is, $0 \le \tau \le \sigma$ and the zero sequence of B_1, $Z(B_1)$, is a subsequence of $\{a_k\} = Z(B)$. It follows from [1, Lemma 3, p. 197] that the hypothesis that $\lim_{r\to 1} \psi(r) = L$ exists and $|L| = 1$ for all divisors ψ of ϕ is equivalent to the condition that

$$\sum_k \frac{1-|a_k|}{|1-a_k|} + \int \frac{d\sigma(\zeta)}{|1-\zeta|} < \infty.$$

Use (2.2) to get the estimate

$$|\psi'(t)| \le c\left(\sum \frac{1-|a|^2}{|1-\bar{a}t|^2} + \int_T \frac{d\sigma(\zeta)}{|1-\bar{\zeta}t|^2}\right).$$

It follows from the last two inequalities and Lemma 1 that

$$\lim_{r\to 1} \int_r^1 |\psi'(t)|\, dt = 0,$$

where the convergence is actually uniform over the entire collection $\{\psi : \psi$ is a divisor of $\phi\}$. Thus, given $\epsilon > 0$ we may find δ, independent of ψ, such that if $\delta < t < 1$ and ψ divides ϕ then $|\psi(t) - \psi(1)| < \epsilon$. Choose $\epsilon = \frac{1}{4}(c(1,\alpha)/c_2(\alpha, 2))$ where the constants on the right-hand side refer back to Lemma 1 and Corollary 1, and find the corresponding δ. By adjusting the δ_1 of Corollary 1 we may assume that the t_1 of Corollary 1 is bigger than δ.

With this taken care of we are ready to use induction to show that (1) implies (2).

Assume that $n = 0$ and (1) holds. Suppose B_1 is a finite Blaschke product whose zeros satisfy the conditions

(i) $|a_k - 1| < \delta_1$ and

(ii) $$\frac{1}{|a_k|}\frac{1-|a_k|}{|1-\bar{a}_k t|} < \frac{\epsilon}{2} \quad \text{for } t_1 < t < 1,$$

where ϵ is as above. Set $\tau \equiv 0$. We will show that the sum in condition (2), for the zeros of B_1, is finite. A similar argument will deal with the integral.

Letting $r \to 1$, use (2.2) and (2.3) to conclude from (1) that

$$\left|\int_{t_1}^1 \left(\log \frac{1}{t}\right)^{-\alpha} \sum \frac{\bar{a}_k}{|a_k|}[B_{(k)}(t) - B_1(t) + B_1(t) - B_1(1) + B_1(1)]\frac{(1-|a_k|^2)}{(1-\bar{a}_k t)^2}\, dt\right| \le M',$$

where

$$B_{(k)}(z)\frac{\bar{a}_k}{|a_k|}\frac{a_k - z}{1-\bar{a}_k z} = B_1(z)$$

and M' is an absolute constant.

Observe that for $t_1 < t < 1$,

$$|B_{(k)}(t)-B_1(t)|=\frac{|B_{(k)}(t)(|a_k|+\bar{a}_k t)(1-|a_k|)|}{|a_k||1-\bar{a}_k t|}<\epsilon$$

and $|B_1(t)-B_1(1)|<\epsilon$. Since $|B_1(1)|=1$, it follows that

$$\left|\sum\int_{t_1}^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{\bar{a}_k}{|a_k|}\frac{(1-|a_k|^2)}{(1-\bar{a}_k t)^2}\,dt\right|\le M'+2\epsilon\sum\int_{t_1}^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{1-|a_k|^2}{|1-\bar{a}_k t|^2}\,dt$$
$$\le M'+\frac{c(1,\alpha)}{2}\sum\frac{1-|a_k|^2}{|1-a_k|^{1+\alpha}}.$$

Corollary 1 shows that we may partition the zero sequence $Z(B)$ of B into finitely many sets such that if the zeros of B_1 belong to one of those sets, then the left-hand side of the last inequality is bounded below by

$$\frac{3c(1,\alpha)}{4}\sum\frac{1-|a_k|^2}{|1-a_k|^{1+\alpha}},$$

where the sum is over the zeros of B_1. From this it follows that

$$\sum\frac{1-|a_k|^2}{|1-a_k|^{1+\alpha}}<\infty,$$

where the sum is taken over the entire zero sequence of B.

A similar argument considering $s_\tau(z)$ where $0\le\tau\le\sigma$ and τ assigns zero mass to a small neighborhood of 1 shows that

$$\int_T\frac{d\sigma(\zeta)}{|1-\zeta|^{1+\alpha}}<\infty.$$

Thus (1) implies (2) in case $n=0$.

Assume now that (1) implies (2) is true for $0,1,2,\ldots,n-1$. We will show that this means that (1) implies (2) is true for n as well. Use (1.2) to obtain that

$$(D^n f)(r)=\frac{1}{\Gamma(\alpha)}\int_0^1\left(\log\frac{1}{t}\right)^{-\alpha}(D^{n+\alpha}f)(rt)\,dt \tag{2.4}$$

and argue recursively from this (using (1.3)) that (1) implies that all the limits $\lim_{r\to 1}(D^k\psi)(r)$ and $\lim_{r\to 1}\psi^{(k)}(r)$ exist and are uniformly bounded (independent of ψ) if $k=0,1,\ldots,n$ and ψ divides ϕ. Now use formula (2.2) to get that

$$B_1^{(n+1)}(t)=-\sum_k\frac{B_{(k)}(t)(n+1)!\,\bar{a}_k^{n+1}(1-|a_k|^2)}{|a_k|(1-\bar{a}_k t)^{n+2}}+R(t),$$

where the last remark and the induction hypothesis allow us to apply condition (2) of Corollary 2 (with $n-1$ in place of n) to deduce that $\int_{t_1}^1(\log(1/t))^{-\alpha}R(t)\,dt$ remains uniformly bounded as $t\to 1$, where the bound is independent of B_1. By (2.3), it follows then that if B_1 is a finite Blaschke product, condition (1) implies

$$\left|\int_{t_1}^1\left(\log\frac{1}{t}\right)^{-\alpha}\sum_{a_k\in Z(B_1)}B_{(k)}(t)\frac{(\bar{a}_k)^{n+1}(1-|a_k|^2)}{|a_k|(1-\bar{a}_k t)^{n+2}}\,dt\right|\le M',$$

where M' is independent of B_1. Argue as before to conclude that

$$\sum_{a\in Z(B)} \frac{1-|a|^2}{|1-a|^{n+1+\alpha}} < \infty.$$

A similar proof based on an analogue of (2.2) shows that

$$\int_T \frac{d\sigma(\zeta)}{|1-\zeta|^{n+1+\alpha}} < \infty.$$

Thus (1) implies (2) for all $n=0,1,2,\ldots$.

An argument by induction patterned after the one given above also shows that (3) implies (2); note that since

$$r(D^m\psi)(r) = \int_0^r (D^{m+1}\psi)(t)\,dt$$

for $m=0,1,2,\ldots$, condition (3) easily yields the facts that the limits

$$\lim_{r\to 1}(D^k\psi)(r) \quad \text{and} \quad \lim_{r\to 1}\psi^{(k)}(r)$$

exist and are uniformly bounded for $k=0,1,\ldots,n$, a key step in the induction.

Finally, that (2) implies both (1) and (3) is an easy consequence of (2.2), (2.3), Lemma 1, and the estimate

$$|\psi^{(n+1)}(t)| \le c\left[\sum \frac{1-|a_k|^2}{|1-ta_k|^{n+2}} + \int \frac{d\sigma(\zeta)}{|1-t\zeta|^{n+2}}\right],$$

which follows in a straightforward but tedious fashion by calculating $\psi^{(n+1)}(z)$ and applying condition (2) repeatedly in making elementary estimates. This concludes the proof. □

We are now ready to prove the main results. In the proofs of Theorems 1–4 we assume without loss of generality that $\zeta_0 = 1$.

Proof of Theorem 1. From (2.3), it follows that if $f\in K_p$ then

$$(D^{n+\alpha}f)(r) = \langle f, K(\cdot\,;r)\rangle$$

where $K(\cdot\,;r)\in K_q$ and

$$\Gamma(1-\alpha)\overline{K(z;r)} = \int_0^1 \left(\log\frac{1}{t}\right)^{-\alpha} D_\zeta^{n+1}\left(\frac{1-\phi(\zeta)\overline{\phi(z)}}{1-\zeta\bar{z}}\right)dt.$$

Under the pairing $\langle f,g\rangle$, Lemma 4.2 of [4] states that K_p^* is isomorphic to K_q. A standard duality argument shows that if (1) holds then $K(\cdot\,;r)$ converges weak-* in $K_p^* = K_q$ to $K(\cdot\,;1)$, where $K(\cdot\,;1)$ is defined by the formula above. In other words, (1) implies that $K(\cdot\,;1)$ belongs to K_q. By Lemma 4.1 of [4],

$$\|K(\cdot\,;1)\|_q^q \doteq \int_{\Gamma_\epsilon} |K(z;1)|^q\,|dz|$$

where Γ_ϵ is the Carleson curve associated with ϕ as described in Section 1. We claim that if ϵ is chosen sufficiently small then

$$|K(z;1)| \ge \frac{c}{|1-z|^{n+1+\alpha}}$$

for $z \in \Gamma_\epsilon$ and z close to 1. Accepting this claim as true, it follows that if (1) holds then

$$\begin{aligned}\|K(\cdot;1)\|_q^q &\geq c\int_{\Gamma_\epsilon} \frac{|dz|}{|1-z|^{q(n+1+\alpha)}} \\ &\doteq \sum_n \int_{\gamma_n} \frac{|dz|}{|1-z|^{q(n+1+\alpha)}} \\ &\doteq \sum_n \frac{1-|\omega_n|}{|1-\omega_n|^{q(n+1+\alpha)}},\end{aligned}$$

where $\{\omega_n\}$ and γ_n are as described in Section 1. Thus (1) implies that the sum above is finite and Theorem B of Section 1 shows therefore that (1) implies (2).

To prove the claim, (2.4) shows that (1) implies $\lim_{r\to 1}(D^n f)(r)$ exists if $f \in K_p$ and $n = 0, 1, \ldots$. So therefore do the limits $\lim_{r\to 1}(D^k f)(r)$ and $\lim_{r\to 1} f^{(k)}(r)$ if $f \in K_p$ and $k = 0, 1, \ldots, n$. Notice also that if $n = 0, 1, \ldots$ we also conclude from Theorem A that the hypothesis of Corollary 2 is satisfied.

Calculate now that

$$\begin{aligned}&D_t^{n+1}\left(\frac{1-\phi(t)\overline{\phi(z)}}{1+t\bar{z}}\right) \\ &= t^{n+1}\frac{d^{n+1}}{dt^{n+1}}\left(\frac{1-\phi(t)\overline{\phi(z)}}{1-t\bar{z}}\right) + \sum_{k=0}^{n} c(k,n)t^k \frac{d^k}{dt^k}\left(\frac{1-\phi(t)\overline{\phi(z)}}{1-t\bar{z}}\right) \\ &= (n+1)!\, t^{n+1}\bar{z}^{n+1}\frac{1-\phi(t)\overline{\phi(z)}}{(1-t\bar{z})^{n+2}} - t^{n+1}\frac{\phi^{(n+1)}(t)\overline{\phi(z)}}{(1-t\bar{z})} + \frac{R}{(1-t\bar{z})^{n+1}}\end{aligned}$$

where $c(k,-1) = 0$ and $c(k,n)$ is as in (1.3), and R is a sum of terms which remain uniformly bounded as $t \to 1$ by the remark prior to the calculation above. (We are using here that $1-\overline{\phi(0)}\phi(z) \in K_p(\phi)$, so "essentially" we may treat $\phi(z)$ as a function in $K_p(\phi)$. No generality is lost in assuming that $\phi(0) \neq 0$.)

By choosing ϵ so small that $z \in \Gamma_\epsilon$ implies that $|\phi(z)|$ is sufficiently small, we see that if $z \in \Gamma_\epsilon$ and is close enough to 1 then, if $n = 0, 1, \ldots$, Corollary 1 yields

$$\begin{aligned}|K(z;1)| \geq c_1 &\left|\int_0^1 \left(\log\frac{1}{t}\right)^{-\alpha}\frac{1}{(1-t\bar{z})^{n+2}}\,dt\right| - c_2\int_0^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{|\phi^{n+1}(t)|}{|1-t\bar{z}|}\,dt \\ &- c_3\int_0^1\left(\log\frac{1}{t}\right)^{-\alpha}\frac{dt}{|1+tz|^{n+1}}.\end{aligned}$$

To bound the integral involving $\phi^{(n+1)}$, since $1-\overline{\psi(0)}\psi(z) \in K_p(\phi)$ whenever ψ divides ϕ, condition (1) enables us to apply Corollary 2 and conclude that condition (3) of Corollary 2 holds. It follows that

$$|K(z;1)| \geq \frac{c_1}{|1-z|^{n+1+\alpha}} - \frac{c_2}{|1-z|} - \frac{c_3}{|1-z|^{n+\alpha}}$$

if z is close to 1 and on Γ_ϵ. This proves the claim for $n = 0, 1, 2, \ldots$. If $n = -1$ the argument is even simpler and depends only on Corollary 1 and correct choice of ϵ.

Assume next that (2) is true. We will show that (3) must follow. Use the proof of Lemma 3.1 in [4] to express $f \in K_p$ as

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{\overline{h(\zeta)}}{\phi(\zeta)}\right] \frac{d\bar{\zeta}}{1 - z\bar{\zeta}},$$

where $h \in K_p$ and $\|h\|_p = \|f\|_p$. Estimate that

$$\begin{aligned}
|(D^{n+1}f)(z)| &\le c \int_\Gamma |h(\zeta)| \frac{|d\zeta|}{|1 - z\bar{\zeta}|^{n+2}} \\
&= c \sum_n \int_{\gamma_n} |h(\zeta)| \frac{|d\zeta|}{|1 - z\bar{\zeta}|^{n+2}} \\
&\le c \sum_n \frac{|h(\zeta_n)|}{|1 - z\bar{\omega}_n|^{n+2}} (1 - |\omega_n|),
\end{aligned}$$

where $\zeta_n \in \gamma_n$ and $|h(\zeta_n)| = \sup\{|h(\zeta)| : \zeta \in \gamma_n\}$. It follows that

$$\begin{aligned}
\int_0^1 |D^{n+1}f(t)| \left(\log \frac{1}{t}\right)^{-\alpha} dt &\le c \sum_n |h(\zeta_n)| (1 - |\omega_n|) \int_0^1 \left(\log \frac{1}{t}\right)^{-\alpha} \frac{dt}{|1 - t\omega_n|^{n+2}} \\
&\doteq \sum_n |h(\zeta_n)| \frac{1 - |\omega_n|}{|1 - \omega_n|^{n+1+\alpha}} \\
&\le \left(\sum |h(\zeta_n)|^p (1 - |\zeta_n|)\right)^{1/p} \left(\sum \frac{1 - |\omega_n|}{|1 - \omega_n|^{q(n+1+\alpha)}}\right)^{1/q},
\end{aligned}$$

since $(1 - |\zeta_n|) \doteq (1 - |\omega_n|)$. Since $\zeta_n \in \gamma_n$ the sequence $\{\zeta_n\}$ is a Carleson sequence and it follows that this last product is finite. Thus (2) implies (3).

Next, assume that (3) holds. We will prove that (2) must follow. If $f \in K_p$, the dominated convergence theorem implies that

$$\lim_{r \to 1} \int_0^r |D^{n+1}f(t)| \left(\log \frac{1}{t}\right)^{-\alpha} dt < \infty.$$

The uniform boundedness principle shows therefore that with

$$\Lambda_r(f) = \int_0^r (D^{n+1}f)(t) \left(\log \frac{1}{t}\right)^{-\alpha} dt$$

then, as $r \to 1$, Λ_r converges weak-* on K_p^* to a bounded linear functional. It is easy to verify that $\Lambda_r(f) = \langle f, g_r \rangle$, where

$$\overline{g_r(z)} = \int_0^r \left(\log \frac{1}{t}\right)^{-\alpha} D_\zeta^{n+1} \left(\frac{1 - \phi(\zeta)\overline{\phi(z)}}{1 - \zeta\bar{z}}\right)\Bigg|_{\zeta = t} dt$$

belongs to K_q. Thus (3) implies that g_r converges weak-* to a function $g \in K_q$. It is clear that $g(z) = K(z; 1)$, where $K(\cdot; 1)$ is the kernel defined in the first part of this proof. We show now that (3) implies (2) by an argument similar to the one used to show (1) implies (2).

Because, if $n = 0, 1, 2, \ldots,$

$$|r(D^n f)(r)| \le \int_0^r |D^{n+1} f(t)|\, dt,$$

(3) implies that

$$\begin{aligned}\int_0^1 |rD^n f(r)| \left(\log \frac{1}{r}\right)^{-\alpha} dr &\le \int_0^1 \int_0^r |D^{n+1} f(t)|\, dt \left(\log \frac{1}{r}\right)^{-\alpha} dr \\ &= \int_0^1 \int_t^1 \left(\log \frac{1}{r}\right)^{-\alpha} dr\, |D^{n+1} f(t)|\, dt \\ &\le c \int_0^1 |D^{n+1} f(t)|\, dt < \infty.\end{aligned}$$

Arguing recursively, we see that (3) holds with k in place of n for $k = 0, 1, \dots, n$ (if $n \ne -1$). This means that for $n \ne -1$ we have that $\lim_{t\to 1} f^{(k)}(t)$ exists for $k = 0, 1, \dots, n$ and $f \in K_p$, and therefore Theorem A implies that the hypotheses of Corollary 2 are satisfied. (This is all unnecessary if $n = -1$.)

Since (3) implies also that

$$\int_0^1 |\phi^{n+1}(t)| \left(\log \frac{1}{t}\right)^{-\alpha} dt < \infty,$$

an analysis of $D_t^{n+1}((1-\phi(t)\overline{\phi(z)})/(1-t\bar{z}))$ similar to the previous one shows that

$$|K(z;1)| \ge \frac{c}{|1-z|^{n+1+\alpha}}$$

if $z \in \Gamma_\epsilon$ and is close to 1. Since (3) implies that $K(\cdot\,;1) \in K_q$, it follows exactly as before that (3) implies (2), as desired.

To complete the proof, we show that (2) implies (1). Suppose (2) is true. We claim that

$$\lim_{r\to 1} \int_0^1 f^{(k+1)}(rt) \left(\log \frac{1}{t}\right)^{-\alpha} dt$$

exists whenever $f \in K_p$ and $k = -1, 0, \dots, n$. From (2.3) it follows that (1) must hold.

To prove the claim, use the integral representation of $f \in K_p$ and write

$$\begin{aligned}\int_0^1 f^{(k+1)}(rt) \left(\log \frac{1}{t}\right)^{-\alpha} dt &= \sum_m \int_0^1 \left[\int_{\gamma_m} \left(\frac{\overline{h(\zeta)}}{\overline{\phi(\zeta)}}\right) \frac{\bar{\zeta}^{k+1}(k+1)!}{(1-rt\bar{\zeta})^{k+2}}\, d\bar{\zeta}\right] \left(\log \frac{1}{t}\right)^{-\alpha} dt \\ &= \sum_m F_{m,k}(r)\end{aligned}$$

where

$$[(k+1)!]^{-1} F_{m,k}(r) = \int_0^1 \left[\int_{\gamma_m} \left[\frac{\overline{h(\zeta)}}{\overline{\phi(\zeta)}}\right] \frac{\bar{\zeta}^{k+1}\, d\bar{\zeta}}{(1-rt\bar{\zeta})^{k+1}}\right] \left(\log \frac{1}{t}\right)^{-\alpha} dt$$

and $F_{m,k}$ is continuous on the interval $[0,1]$. Estimate now that

$$\begin{aligned}\|F_{m,k}\|_\infty &= \sup\{|F_{m,k}(r)| : 0 \le r \le 1\} \\ &\le c \int_0^1 |h(\zeta_m)| \frac{1-|\omega_m|}{|1-t\omega_m|^{k+2}} \left(\log \frac{1}{t}\right)^{-\alpha} dt\end{aligned}$$

$$\leq c|h(\zeta_m)| \frac{1-|\omega_m|}{|1-\omega_m|^{k+1+\alpha}},$$

where $\zeta_m \in \gamma_m$ and $|h(\zeta_m)| = \sup\{|h(\zeta)|; \zeta \in \gamma_m\}$. Thus

$$\sum_m \|F_{m,k}\|_\infty \leq c\left(\sum |h(\zeta_m)|^p(1-|\zeta_m|)\right)^{1/p} \left(\sum \frac{1-|\omega_m|}{|1-\omega_m|^{q(k+1+\alpha)}}\right)^{1/q}$$
$$\leq C\|f\|_p < \infty$$

for $k=-1, 0, \ldots, n$. The Weierstrass-M test implies now that $\sum_m F_{m,k}$ extends to be continuous on $[0,1]$ for $k=0,1,\ldots,$ and the proof is complete. □

Proof of Theorem 2. Notice that (2.4) shows that if either (1) or (3) holds, then $\lim_{r\to 1} f(r)$ exists whenever $f \in K_*(\phi)$, so by Theorem A the hypothesis of Corollary 2 must hold. That (1) implies (2) then follows from Corollary 2 since whenever ψ divides ϕ, $1-\overline{\psi(0)}\psi(z) \in K_*(\phi)$ and $\|1-\overline{\psi(0)}\psi\|_* \leq c$, where c does not depend on ψ. That (2) implies (3) follows from the integral representation (see [4, Lemma 3.1]):

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \left[\frac{\overline{h(\zeta)}}{\phi(\zeta)}\right] \frac{d\bar{\zeta}}{1-z\bar{\zeta}},$$

where $h \in H^\infty$ and $\|h\|_\infty = \|f\|_*$, for $f \in K_*$, by an argument similar to the corresponding one in the proof of Theorem 1. To see that (3) implies (2), an argument similar to the parallel point in the proof of the implication (3) implies (2) in Theorem 1 shows that if $k=0,1,\ldots,n+1$, ψ divides ϕ, and (3) holds, then, for an M' independent of ψ,

$$\int_0^1 |\psi^{(k)}(t)| \left(\log\frac{1}{t}\right)^{-\alpha} dt < M'.$$

Corollary 2 shows now that (3) implies (2).

Finally, if (2) holds use the integral representation for $f \in K_*$ and the Weierstrass-M test to deduce (1). This completes the proof. □

We consider now integrals of K_p functions with respect to fractional Cauchy kernels. As before, if $|w|<1$ and $0<\alpha<1$ choose a branch of the logarithm so $(1-w)^\alpha$ is positive if $w=0$. Let

$$(C_{n+\alpha}f)(z) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{1}{2\pi i}\int_T (\bar{\zeta})^n f(\zeta)(1-\bar{\zeta}z)^{-n-\alpha}\,|d\zeta|$$

where $n=0,1,\ldots$ and $|z|<1$. It follows from Lemma 2 that

$$(C_{n+\alpha}f)(z) = \frac{\sin\pi\alpha}{\pi}\int_0^1 t^{\alpha-1+n}(1-t)^{-\alpha} f^{(n)}(tz)\,dt,$$

and we expect therefore that $C_{n+\alpha}f$ will behave like $D^{n-1+\alpha}f$.

If $f \in K_p(\phi)$ and $1<p<\infty$ then it is easy to verify that

$$(C_{n+\alpha}f)(r) = \langle f, K_r\rangle,$$

where $K_r \in K_q(\phi)$ and

$$\overline{K_r(z)} = \frac{\sin \pi\alpha}{\pi} \int_0^1 t^{\alpha-1+n}(1-t)^{-\alpha} \frac{d^n}{d\zeta^n}\left(\frac{1-\phi(\zeta)\overline{\phi(z)}}{1-\zeta\bar{z}}\right)\Big|_{\zeta=tr} dt.$$

Because $\log(1/t) \doteq 1-t$ for t close to 1, it is almost obvious that the following properties are all equivalent:

(1) $\lim_{r\to 1}(C_{n+\alpha}f)(r)$ exists for all $f \in K_p$, if $1 \leq (n+\alpha)q$.

(2) $$\sum \frac{1-|a|}{|1-a|^{q(n+\alpha)}} + \int_T \frac{d\sigma}{|1-\zeta|^{q(n+\alpha)}} < \infty \quad \text{if } 1 \leq (n+\alpha)q.$$

(3) $\lim_{r\to 1}(D^{n-1+\alpha}f)(r)$ exists for all $f \in K_p$, if $1 \leq (n+\alpha)q$.

We omit the proofs; however, we state the following result since it provides yet another condition equivalent to (2) above.

THEOREM 3. *Let $1 < p < \infty$, $n = 0, 1, \ldots,$ and $0 < \alpha < 1$. Suppose $1 \leq q(n+\alpha)$ and $\phi = Bs_\sigma$. Then the following conditions are equivalent:*

(1) *Condition* (1) *above holds.*

(2) *Condition* (2) *above holds.*

(3) $\overline{\lim}_{r\to 1} \|P_{K_2}(1-rz)^{-n-\alpha}\|_q < \infty$, *where P_{K_2} denotes orthogonal projection of H^2 onto K_2.*

(4) $(1-\phi g)(1-z)^{-n-\alpha} \in K_q$ *for some $g \in H^q$.*

Proof. The equivalence of (1), (3), and (4) follows from duality considerations; see also [4, §4], especially the remark preceding Theorem 4.2. That (1) and (2) are equivalent is essentially the same as the equivalence of (1) and (2) in Theorem 1. □

For $K_*(\phi)$ we state the following result.

THEOREM 4. *Let $\phi = Bs_\sigma$, $n = 1, 2, \ldots,$ and $0 < \alpha < 1$. The following conditions are equivalent:*

(1) $\lim_{r\to 1}(C_{n+\alpha}f)(r)$ *exists for all $f \in K_*(\phi)$.*

(2) *For some $g \in H^1$, $(1-\phi g)(1-z)^{-n-\alpha} \in H^1$.*

(3) $$\sum_{a\in Z(B)} \frac{1-|a|}{|1-a|^{n+\alpha}} + \int \frac{d\sigma}{|1-\zeta|^{n+\alpha}} < \infty.$$

Proof. The equivalence of (1) and (2) is again based on duality considerations; see [4, Thm. 3.3]. That (1) and (3) are equivalent is essentially contained in Theorem 2. □

We conclude this paper with the proof of Theorem 5.

Proof of Theorem 5. We will show only the equivalence of (1) and (2). Suppose first that ϕ satisfies (2). By Corollary 3.3 and Lemma 3.4 of [9] we may factor ϕ as $\phi = B_1 B_2 \cdots B_n$, where each B_i is a Blaschke product whose zeros are uniformly separated and (of course) still satisfy the Frostman condition. If $p = 2$ then by [1, Lemma 3.1, p. 196]

$$K_2(\phi) = K_2(B_1) \oplus B_1 K_2(B_2) \oplus \cdots \oplus B_1 B_2 \cdots B_{n-1} K_2(B_n).$$

An argument based on the M. Riesz theorem shows that $H^p=\phi H^p+K_p(\phi)$ if $1<p<\infty$. Using this fact and the argument used in [1] for the case $p=2$, it is easy to show that

$$K_p(\phi)=K_p(B_1)+B_1K_p(B_2)+\cdots+B_1B_2\cdots B_{n-1}K_p(B_n).$$

If $f\in K_p(B_i)$ for some i, then by the remarks on [8, p. 274]

$$f(z)=\sum \frac{c_k d_k^{1-1/p}}{1-\bar{z}_k z},$$

where $d_k=1-|z_k|$, $\{z_k\}$ is the zero sequence of B_i, and $\{c_k\}\in l^p$. Set

$$G=B_1B_2\cdots B_{i-1} \quad \text{and} \quad g_k(z)=\frac{G(z)c_k d_k^{1-1/p}}{1-\bar{z}_k z}.$$

It is simple to check that $(I^{1/p}g_k)(z)$ is continuous for $|z|\leq 1$.

Estimate next that if N is fixed then

$$\begin{aligned}
\left\|\sum_{k\geq N} I^{1/p}g_k\right\|_\infty &= \sup\left\{\left|\sum_{k\geq N}(I^{1/p}g_k)(z)\right| : |z|\leq 1\right\}\\
&\leq C\sup\left\{\sum_{k\geq N}|c_k|d_k^{1-1/p}\int_0^1\left(\log\frac{1}{t}\right)^{1/p-1}\frac{dt}{|1-t\bar{z}_k z|} : |z|\leq 1\right\}\\
&\leq C\sup\left\{\sum_{k\geq N}|c_k|d_k^{1-1/p}\frac{1}{|1-\bar{z}_k\zeta|^{1-1/p}} : \zeta\in T\right\}\\
&\leq C\left(\sum_{k\leq N}|c_k|^p\right)^{1/p}\sup\left(\sum_{k\geq N}\frac{d_k}{|1-\bar{z}_k\zeta|} : \zeta\in T\right)^{1/q}\\
&\leq C\left(\sum_{k\geq N}|c_k|^p\right)^{1/p}.
\end{aligned}$$

It follows that the series $\sum_{k=1}^\infty I^{1/p}(g_k)$ converges in the disk algebra of holomorphic functions on D with continuous extensions to the closed disk. Since $I^{1/p}(f)$ is a finite sum of such series, $I^{1/p}(f)$ must be continuous on the closed disk, as claimed.

Conversely, assume that (2) holds. Let $\zeta\in T$ and $K_\zeta(z)\in K_q$ be such that

$$(I^{1/p}f)(\zeta)=\langle f, K_\zeta\rangle;$$

such a K_ζ exists since the map $f\to(I^{1/p}f)(\zeta)$ is a bounded linear functional under assumption (2). In fact, since (2) implies that $\|I^{1/p}f\|_\infty\leq C\|f\|_p$, it is clear that $\|K_\zeta\|_q\leq C$ for all $\zeta\in T$. The proof of Theorem 1 ((1) implies (2)) makes it clear that with $\{\omega_n\}$, the points associated with the Carleson curve Γ_ϵ, we must have

$$\sup_{\zeta\in T}\sum\frac{1-|\omega_n|}{|1-\bar{\zeta}\omega_n|}\doteq\sup_{\zeta\in T}\|K_\zeta\|_q<C,$$

and it follows that $\{\omega_n\}$ is a Frostman sequence. It follows from this by the remark after the proof of Theorem 3.1 in [4] that $K_*(\phi)=K_\infty(\phi)$. By [4, Prop. 3.1] (and its Corollary), ϕ must also be a Blaschke product satisfying condition (F). This completes the proof. □

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