ON THE AUTOMORPHIC FORMS OF A NONCONGRUENCE SUBGROUP

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Our aim in this paper is to give the first example of a noncongruence subgroup which is "essentially cuspidal", that is, for which cuspidal eigenfunctions exist abundantly (see Section 1 for a definition). It is a discrete subgroup of $SL_2(\mathbf{C})$ obtained as the kernel of a Kubota symbol. The proof consists of the explicit evaluation of the Eisenstein matrix associated to the subgroup, as well as its determinant. From these follows a Weyl law which gives the precise asymptotics of the cusp forms. We use some computations of Kubota [4] and Patterson [5], as well as analogous work for congruence subgroups ([1; 2]).

1. Let $\mathcal{H}$ be the hyperbolic 3-space $\{w=(y,z)=(y,x_1,x_2) \mid y>0\}$. If we identify w with the quaternion x_1+ix_2+jy then the group $G=SL_2(\mathbf{C})$ acts on $\mathcal{H}$ via linear fractional transformations; namely, if

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G \text{ then } g(w)=(aw+b)(cw+d)^{-1}.$$

Let Γ be a discrete cofinite subgroup of G whose parabolic fixed points form h Γ-equivalence classes represented by the cusps $\kappa_1, \dots, \kappa_h$ and let $\Gamma_{\kappa_1}, \dots, \Gamma_{\kappa_h}$ be their stabilizers in Γ. If we choose maps $\rho_i\colon \kappa_i \to \infty$ and let $w^{(i)}=\rho_i w=(y^{(i)}, z^{(i)})$, then the Eisenstein series at κ_i is defined for $w\in\mathcal{H}$ and $s\in\mathbf{C}$ with $\operatorname{Re}(s)>2$ by

$$E_i(w,s)=\sum_{\gamma\in\Gamma_{\kappa_i}\backslash\Gamma} y^{(i)}(\gamma w)^s.$$

$E_i(w,s)$ admits a Fourier expansion at each cusp κ_j, whose zero coefficient is of the form

$$\delta_{ij}y^{(j)s}+\varphi_{ij}(s)y^{(j)2-s},$$

for some meromorphic function $\varphi_{ij}(s)$. We let $\Phi(s)=(\varphi_{ij}(s))_{i,j=1,\dots,h}$ and $\varphi(s)=\det\Phi(s)$. The dependency of these functions of s on the choice of the ρ_i's is not essential.

The function $\varphi(s)$ is closely tied up with the cusp forms of Γ. Let Δ be the Laplace operator on $\mathcal{H}$ and let λ be the eigenvalue of a square integrable automorphic eigenfunction of Δ, that is, $u\in L^2(\Gamma\backslash\mathcal{H})$ and $\Delta u+\lambda u=0$. If we count

$$N_\Gamma(T)=\#\{\lambda \mid \sqrt{\lambda}\in[0,T]\}$$

and also let

$$M_\Gamma(T)=\frac{1}{2\pi}\int_{-T}^{T}-\varphi'/\varphi(1+it)\,dt,$$

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then the Selberg trace formula for Γ gives the asymptotics

$$N_\Gamma(T)+M_\Gamma(T)\sim c_\Gamma T^3 \quad \text{as } T\to\infty,$$

where $c_\Gamma=\operatorname{vol}(\Gamma\backslash\mathcal{H})/6\pi^2$. Therefore a good estimation of $M_\Gamma(T)$ implies a Weyl law for Γ:

$$N_\Gamma(T)\sim c_\Gamma T^3 \quad \text{as } T\to\infty.$$

Such an estimate can be derived for the *congruence subgroups* that act on $\mathcal{H}$, that is, those that contain the principal congruence subgroups

$$\Gamma(\mathfrak{N})=\{\gamma\in \mathrm{SL}_2(\mathcal{O}_K)\mid \gamma\equiv I(\operatorname{mod}\mathfrak{N})\},$$

where K is an imaginary quadratic number field, $\mathcal{O}_K$ is its ring of integers, and $\mathfrak{N}$ is an ideal in $\mathcal{O}_K$. In these cases $\varphi(s)$ can be expressed in terms of certain L-functions associated to K so that $M_\Gamma(T)=O(T\log T)$ (see [7; 1; 2].

In [6] Sarnak calls a discrete subgroup for which the above Weyl law holds *essentially cuspidal,* and asks whether or not arithmetic groups other than these congruence subgroups are essentially cuspidal. Our objective here is to establish the first Weyl law for a noncongruence subgroup. This subgroup is the kernel of a Kubota symbol, which we now describe.

Let $\omega=(-1+\sqrt{-3})/2$ be a cubic root of unity and let K be the number field $\mathbf{Q}(\omega)$ with the ring of integers $\mathcal{O}_K=\mathbf{Z}[\omega]$. Let $\Gamma(3)\subset \mathrm{SL}_2(\mathcal{O}_K)$ be the principal congruence subgroup of level 3. Denote by $(-)_3$ the cubic residue symbol, taking values in $\{1,\omega,\omega^2\}$. Then a special case of a theorem of Kubota is the following.

THEOREM (Kubota [3]). *For $\gamma\in\Gamma(3)$ define*

$$\chi(\gamma)=\chi\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\begin{cases}(c/a)_3 & \text{if } c\neq 0,\\ 1 & \text{if } c=0.\end{cases}$$

Then χ is a character on $\Gamma(3)$, whose kernel $\Gamma(3)^1$ contains no congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$.

Although one can proceed to analyze $\Gamma(3)^1$ we prefer to work with a larger group, introduced by Patterson in [5], in order to have a smaller number of cusps. Let

$$\Gamma=\{\gamma\in \mathrm{SL}_2(\mathcal{O}_K)\mid \text{there is a } g\in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma\equiv g(\operatorname{mod}3)\}.$$

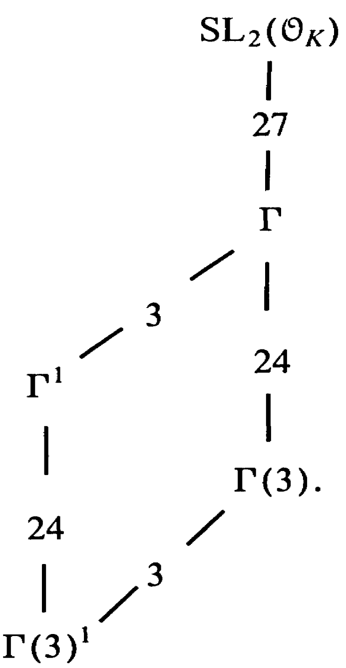
Thus every $\gamma\in\Gamma$ can be written as $\gamma=g\gamma_1$, with $g\in \mathrm{SL}_2(\mathbf{Z})$ and $\gamma_1\in\Gamma(3)$. Define

$$\chi(\gamma)=\chi(\gamma_1).$$

By [5, p. 127] this gives a character on Γ which extends χ.

COROLLARY. *The kernel Γ^1 of χ in Γ is a noncongruence subgroup.*

We summarize the groups we have considered with their indices:



To state our main theorems, let $v = 9\sqrt{3}/2$ be the volume of the lattice $3\mathcal{O}_K\backslash\mathbf{C}$ and let $\zeta_K(s)$ be the Dedekind zeta function of K.

THEOREM 1. *The Eisenstein matrix* $\Phi(s)$ *of the noncongruence subgroup* Γ^1 *is given, for a natural choice of* ρ_i*'s, by*

$$\Phi(s) = v^{-1}\pi(s-1)^{-1}\begin{bmatrix} A & B & B & C & C \\ B & B & A & C & C \\ B & A & B & C & C \\ C & C & C & D & E \\ C & C & C & E & D \end{bmatrix},$$

where

$$A = r_1(s)\frac{\zeta_K(s-1)}{\zeta_K(s)} + 2\frac{\zeta_K(3s-3)}{\zeta_K(3s-2)}, \qquad B = r_1(s)\frac{\zeta_K(s-1)}{\zeta_K(s)} - \frac{\zeta_K(3s-3)}{\zeta_K(3s-2)},$$

$$C = r_2(s)\frac{\zeta_K(s-1)}{\zeta_K(s)}, \qquad D = r_3(s)\frac{\zeta_K(s-1)}{\zeta_K(s)}, \qquad E = r_4(s)\frac{\zeta_K(s-1)}{\zeta_K(s)},$$

and $r_i(s)$, $1 \le i \le 4$, *are rational functions of* 3^s.

THEOREM 2. *The Eisenstein determinant* $\varphi(s)$ *of the noncongruence subgroup* Γ^1 *is given by*

$$\varphi(s) = r(s)(v^{-1}\pi(s-1)^{-1})^5\frac{\zeta_K(s-1)^3}{\zeta_K(s)^3}\frac{\zeta_K(3s-3)^2}{\zeta_K(3s-2)^2},$$

where $r(s)$ *is a rational function of* 3^s.

We remark that $\pi(s-1)^{-1}$ is the gamma factor of $\zeta_K(s-1)/\zeta_K(s)$ as well as $\zeta_K(3s-3)/\zeta_K(3s-2)$. We also note that the poles of $r(s)$ form an arithmetic progression on the imaginary axis, so that asymptotically most of the poles of $\varphi(s)$

come from the zeros of $\zeta_K(s)$ and $\zeta_K(3s-2)$ in the critical strip. By the method mentioned above one has the following.

COROLLARY. *As* $T\to\infty$,

$$N_{\Gamma^1}(T)\sim c_{\Gamma^1}T^3.$$

2. We begin the proofs of these theorems by identifying the cusps of Γ^1. By the method of primitive pairs, the group $\Gamma(3)$ has 12 cusps which can be represented by

$$\infty, 0, 1, -1, \omega, -\omega^2, \omega-1, (1-\omega)^{-1}, -\omega, \omega^2, 1-\omega, (\omega-1)^{-1}.$$

The first four are clearly Γ-equivalent, and it is easy to find elements of $SL_2(\mathbf{Z})$ that map ω to $-\omega^2$, $\omega-1$ and $(1-\omega)^{-1}$, so that the next four (and similarly the last four) are also Γ-equivalent. Furthermore, no additional equivalence of cusps occurs, since the Γ-equivalence of ω and 0, for example, would imply that ω is congruent to a rational number modulo 3. Thus Γ has three cusps, which we represent by (say) ∞, ω, and $-\omega$.

A parabolic fixed point x of Γ is said to be *essential* if the character χ is trivial on the stabilizer Γ_x. It is clear that this notion depends only on the equivalence classes of such points, that is, on the cusps.

LEMMA 1. ∞ *is essential, while* ω, $-\omega$ *are inessential.*

Proof. Let $\gamma=g\gamma_1\in\Gamma_\infty$. Then $\gamma_1(\infty)=g^{-1}(\infty)$, which is a parabolic fixed point of $SL_2(\mathbf{Z})$ and is therefore $SL_2(\mathbf{Z},3)$-equivalent to ∞, 0, 1, or -1. However, since $\gamma_1\in\Gamma(3)$, it is in fact equivalent to ∞. Take $\tau\in SL_2(\mathbf{Z},3)$ with $\tau(g^{-1}(\infty))=\infty$. Since χ is trivial on $SL_2(\mathbf{Z},3)$ ([5, p. 127]), we have $\chi(\gamma_1)=\chi(\tau\gamma_1)$. But $\tau\gamma_1\in\Gamma(3)_\infty$, so that $\chi(\tau\gamma_1)=1$.

Turning to Γ_ω, we give a $\tau\in\Gamma_\omega$ (in fact, $\tau\in\Gamma(3)_\omega$) such that $\chi(\tau)=\omega$:

$$\tau=\begin{pmatrix}1+3\omega & -3\omega^2\\ 3 & 1-3\omega\end{pmatrix}.$$

To compute $(3/(1+3\omega))_3$ we note that the norm $|1+3\omega|^2=7$ and so $1+3\omega$ is a prime. We thus need to express $3^{(7-1)/3}=9$ modulo $1+3\omega$. But

$$9=(3\omega^2-\omega)(1+3\omega)+\omega,$$

so $9\equiv\omega(\text{mod}(1+3\omega))$ and $\chi(\tau)=\omega$. A similar example can be found in $\Gamma_{-\omega}$. □

LEMMA 2. *The cusp* ∞ *of* Γ *splits into three cusps in* Γ^1.

Proof. Choose $\tau\in\Gamma(3)$ with $\chi(\tau)=\omega$, and define $\kappa=\tau(\infty)$. If $\gamma\in\Gamma^1$ satisfies $\gamma(\infty)=\kappa$ then $\gamma^{-1}\tau\in\Gamma_\infty$, so that by Lemma 1 $\chi(\gamma^{-1}\tau)=1$, and $\chi(\gamma)=\chi(\tau)=\omega\neq1$. Thus κ cannot be equivalent to ∞ in Γ^1. A similar argument shows that $\kappa'=\tau^2(\infty)$ is a third inequivalent cusp. This is a complete set of cusps because $[\Gamma:\Gamma^1]=3$. □

LEMMA 3. *Let* x *and* y *be inessential parabolic fixed points which are equivalent in* Γ. *Then* x *and* y *are also equivalent in* Γ^1.

Proof. Let $\gamma \in \Gamma$ with $y=\gamma(x)$. Since x is inessential, there is a $\gamma' \in \Gamma_x$ such that $\chi(\gamma')=\chi(\gamma)^{-1}$. Then $y=\gamma\gamma'(x)$ and $\chi(\gamma\gamma')=1$. □

To summarize, we have

$$\begin{array}{lccccccc} \text{cusps of } \Gamma: & & \infty & & & \omega & & -\omega \\ & / & | & \backslash & & | & & | \\ \text{cusps of } \Gamma^1: & \infty & \kappa & \kappa' & & \omega & & -\omega. \end{array}$$

3. In this section we relate the Eisenstein series $E_i^1(w,s)$ $(1 \le i \le 5)$ of Γ^1 to the Eisenstein series $E_i(w,s)$ $(1 \le i \le 3)$ of Γ.

Since ∞ is an essential cusp we can define

$$E_\infty(w,s,\chi)=\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi(\gamma) y(\gamma w)^s$$

and similarly define $E_\infty(w,s,\chi^2)$. Then

$$\begin{aligned} E_\infty(w,s)+E_\infty(w,s,\chi)+E_\infty(w,s,\chi^2) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (1+\chi(\gamma)+\chi^2(\gamma)) y(\gamma w)^s \\ &= 3 \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \chi(\gamma)=1}} y(\gamma w)^s = 3E_\infty^1(w,s). \end{aligned}$$

Also,

$$3E_\kappa^1(w,s)=E_\kappa(w,s)+E_\kappa(w,s,\chi)+E_\kappa(w,s,\chi^2).$$

But since κ and ∞ are Γ-equivalent,

$$\begin{aligned} E_\kappa(w,s)&=E_\infty(w,s), \\ E_\kappa(w,s,\chi)&=\chi(\tau)E_\infty(w,s,\chi), \\ E_\kappa(w,s,\chi^2)&=\chi^2(\tau)E_\infty(w,s,\chi^2). \end{aligned}$$

Thus

$$3E_\kappa^1(w,s)=E_\infty(w,s)+\omega E_\infty(w,s,\chi)+\omega^2 E_\infty(w,s,\chi^2).$$

Working the same way with κ', we obtain

$$\begin{bmatrix} E_\infty^1(w,s) \\ E_\kappa^1(w,s) \\ E_{\kappa'}^1(w,s) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} E_\infty(w,s) \\ E_\infty(w,s,\chi) \\ E_\infty(w,s,\chi^2) \end{bmatrix}.$$

Turning next to $E_\omega^1(w,s)$, we see that there is a bijection

$$\Gamma_\omega^1 \backslash \Gamma^1 \leftrightarrow \Gamma_\omega \backslash \Gamma$$

given by

$$\Gamma_\omega^1 \gamma^1 \to \Gamma_\omega \gamma^1.$$

This map is clearly one-to-one. To see that it is onto, take $\Gamma_\omega \gamma \in \Gamma_\omega \backslash \Gamma$. Since ω is inessential, there is a $\gamma' \in \Gamma_\omega$ with $\chi(\gamma'\gamma)=1$, so that $\Gamma_\omega \gamma = \Gamma_\omega \gamma' \gamma$ is the image of $\Gamma_\omega^1 \gamma' \gamma$. We can therefore conclude that

$$E_\omega^1(w,s)=E_\omega(w,s), \qquad E_{-\omega}^1(w,s)=E_{-\omega}(w,s).$$

It follows from this discussion that to calculate $\Phi(s)$ it is enough to look for the zero Fourier coefficients of $E_\infty(w,s)$, $E_\infty(w,s,\chi)$, $E_\infty(w,s,\chi^2)$, $E_\omega(w,s)$, and $E_{-\omega}(w,s)$ at the five cusps. Now

$$E_\infty(w,s)=E_\infty(\tau w,s)=E_\infty(\tau^2 w,s),$$
$$E_\infty(w,s,\chi)=\chi(\tau)^{-1}E_\infty(\tau w,s,\chi)=\chi(\tau^2)^{-1}E_\infty(\tau^2 w,s,\chi),$$
$$E_\infty(w,s,\chi^2)=\chi(\tau)^{-2}E_\infty(\tau w,s,\chi^2)=\chi(\tau^2)^{-2}E_\infty(\tau^2 w,s,\chi^2).$$

Let the zero coefficients at ∞ of these three functions be

$$y^s+\psi(s)y^{2-s},$$
$$y^s+\psi(s,\chi)y^{2-s},$$
$$y^s+\psi(s,\chi^2)y^{2-s}.$$

Then the upper 3×3 block in $\Phi(s)$ is given by

$$\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}\begin{bmatrix}\psi(s) & \psi(s) & \psi(s)\\ \psi(s,\chi) & \omega\psi(s,\chi) & \omega^2\psi(s,\chi)\\ \psi(s,\chi^2) & \omega^2\psi(s,\chi^2) & \omega\psi(s,\chi^2)\end{bmatrix}.$$

PROPOSITION. *Let* $\zeta(s,1)=\sum_{\substack{c\in \mathcal{O}_K\\ c\equiv 1(3)}}|c|^{-2s}$. *Then*

(1) $\quad \psi(s)=v^{-1}\pi(s-1)^{-1}\dfrac{3^s-3+2\cdot 3^{2-s}}{3^{s-1}-1}\dfrac{\zeta(s-1,1)}{\zeta(s,1)}.$

(2) $\quad \psi(s,\chi)=\psi(s,\chi^2)=v^{-1}\pi(s-1)^{-1}\dfrac{3^{3s-2}-1}{3^{3s-3}-1}\dfrac{\zeta(3s-3,1)}{\zeta(3s-2,1)}.$

Proof. See Patterson [5, pp. 137–139] (and cf. Kubota [4, pp. 50–52]). □

Using $\zeta(s,1)=(1-3^{-s})\zeta_K(s)$ (see §4) we obtain from this proposition and the preceding discussion the upper 3×3 block as stated in Theorem 1, with

$$\begin{aligned}3A &= \psi(s)+\psi(s,\chi)+\psi(s,\chi^2)\\ &= v^{-1}\pi(s-1)^{-1}\frac{1-3^{1-s}}{1-3^{-s}}\frac{3^s-3+2\cdot 3^{2-s}}{3^{s-1}-1}\frac{\zeta_K(s-1)}{\zeta_K(s)}\\ &\quad +2\frac{1-3^{3-3s}}{1-3^{2-3s}}\frac{3^{3s-2}-1}{3^{3s-3}-1}\frac{\zeta_K(3s-3)}{\zeta_K(3s-2)}\\ &= v^{-1}\pi(s-1)^{-1}3\frac{3^s-3+2\cdot 3^{2-s}}{3^s-1}\frac{\zeta_K(s-1)}{\zeta_K(s)}+6\frac{\zeta_K(3s-3)}{\zeta_K(3s-2)}.\end{aligned}$$

An identical calculation gives the expression for B.

4. Consider next the Eisenstein series of $\Gamma(3)$, which we denote by $\tilde{E}_i(w,s)$ or $\tilde{E}_{\kappa_i}(w,s)$, $1\le i\le 12$. We saw earlier that the cusp ω of Γ splits into ω, $-\omega^2$, $\omega-1$, and $(1-\omega)^{-1}$ in $\Gamma(3)$, so that if we choose the same map sending ω to ∞ for both groups we obtain

$$E_\omega(w,s)=\tilde{E}_\omega(w,s)+\tilde{E}_{-\omega^2}(w,s)+\tilde{E}_{\omega-1}(w,s)+\tilde{E}_{(1-\omega)^{-1}}(w,s).$$

This reduces the study of $E_\omega(w,s)$ to that of the corresponding $\tilde{E}$'s, to which we can apply the methods of [1] and [2].

If the cusp κ_i of $\Gamma(3)$ is written as $\kappa_i=-\delta_i/\gamma_i$, with $\gamma_i,\delta_i\in\mathcal{O}_K$, then for the right choice of coordinates at κ_i we have

$$\tilde{E}_i(w,s)=\sum_{\substack{(c,d)=(1)\\ c\equiv\gamma_i(3)\\ d\equiv\delta_i(3)}}\frac{y^s}{N(cw+d)^s}.$$

Here for a quaternion $w=x_1+ix_2+jy$ we let $N(w)=x_1^2+x_2^2+y^2$. Define

$$F_i(w,s)=\sum_{\substack{c\equiv\gamma_i(3)\\ d\equiv\delta_i(3)}}\frac{y^s}{N(cw+d)^s}.$$

To relate these two functions we make the simple (but key) observation that the group $(\mathcal{O}_K/(3))^\times$ of invertible elements modulo 3 can be represented by the six roots of unity ±1, $\pm\omega$, $\pm\omega^2$. Therefore, if we decompose the sum in $F_i(w,s)$ according to the greatest common divisor of c and d, we obtain the equality

$$F_i(w,s)=\Bigg(\sum_{\substack{(k)<\mathcal{O}_K\\ (k,3)=1}}|k|^{-2s}\Bigg)\tilde{E}_i(w,s).$$

We note that

$$\sum_{\substack{(k)\\ (k,3)=1}}|k|^{-2s}=\sum_{(k)}|k|^{-2s}-\sum_{\substack{(k)\\ \sqrt{-3}\,|\,k}}|k|^{-2s}=(1-3^{-s})\zeta_K(s).$$

PROPOSITION. *The zero coefficient of $F_i(w,s)$ at κ_j is given by*

$$\delta_{ij}\zeta(s,-\gamma_i\beta_j+\delta_i\alpha_j)y^{(j)s}+v^{-1}\pi(s-1)^{-1}\zeta(s-1,\gamma_i\delta_j-\delta_i\gamma_j)y^{(j)2-s}.$$

Here α_j,β_j are related to γ_j,δ_j via

$$\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}\in \mathrm{SL}_2(\mathcal{O}_K),$$

and, for $\lambda\in\mathcal{O}_K$,

$$\zeta(s,\lambda)=\sum_{\substack{c\in\mathcal{O}_K\\ c\equiv\lambda(3)}}|c|^{-2s}.$$

Proof. One expresses $F_i(w,s)$ as a sum over a lattice, and uses the Poisson summation formula to write it as a sum of exponentials over the dual lattice. The zero coefficient can then be read off. See [1] and [2] for details. □

To compute the coefficients of $E_\omega(w,s)$ and $E_{-\omega}(w,s)$ at the five cusps, we thus need some of the multiplication table of $\gamma_i\delta_j-\gamma_j\delta_i$ modulo 3:

	∞	0	1	-1	ω	$-\omega^2$	$\omega-1$	$(1-\omega)^{-1}$	$-\omega$	ω^2	$1-\omega$	$-(1-\omega)^{-1}$
ω	ϵ	ϵ	$1-\omega$	ϵ	0	ϵ	ϵ	ϵ	ϵ	$\omega-1$	ϵ	ϵ
$-\omega$	ϵ	ϵ	ϵ	$1-\omega$	ϵ	$\omega-1$	ϵ	ϵ	0	ϵ	ϵ	ϵ
∞	0	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	$1-\omega$	ϵ	ϵ	ϵ	$\omega-1$

Here ϵ denotes one of the six units, and we use the fact that 0, $\omega-1$, and $1-\omega$ represent the other three classes in $\mathcal{O}/(3)$. We can now express $\zeta(s,\gamma_i\delta_j-\gamma_j\delta_i)$ for these values in terms of $\zeta_K(s)$. Firstly, for a unit ϵ,

$$\zeta(s,\epsilon)=\frac{1}{6}\sum_{(c,3)=1}|c|^{-2s}=\frac{1}{6}\Big(\sum_{c}|c|^{-2s}-\sum_{\sqrt{-3}\,|\,c}|c|^{-2s}\Big)=(1-3^{-s})\zeta_K(s).$$

Writing $1-\omega=\sqrt{-3}\,\omega^2$, we also have

$$\begin{aligned}\zeta(s,1-\omega)=\zeta(s,\omega-1)&=\sum_{c\equiv\sqrt{-3}\,\omega^2(3)}|c|^{-2s}=|\sqrt{-3}|^{-2s}\sum_{c\equiv\omega^2(\sqrt{-3})}|c|^{-2s}\\&=3\cdot 3^s(1-3^{-s})\zeta_K(s).\end{aligned}$$

Finally, $\zeta(s,0)=6\cdot 3^{-2s}\zeta_K(s)$.

Combining these results with the table above, we can now find the remaining entries $\varphi_{ij}(s)$ in the matrix $\Phi(s)$. The zero coefficients of $E_\omega(w,s)$ and $E_{-\omega}(w,s)$ at ∞, κ, and κ' are all the same, and are given by the sum of the coefficients of the four $\tilde{E}_{\kappa_i}(w,s)$'s at ∞. This sum is

$$v^{-1}\pi(s-1)^{-1}\frac{3\zeta(s-1,1)+\zeta(s-1,1-\omega)}{(1-3^{-s})\zeta_K(s)}=v^{-1}\pi(s-1)^{-1}\frac{3^{s+1}-3^{3-s}}{3^s-1}\frac{\zeta_K(s-1)}{\zeta_K(s)}.$$

By our table, this calculation also gives the zero coefficient at $-\omega$. Similarly, the coefficient at ω is

$$\begin{aligned}&v^{-1}\pi(s-1)^{-1}\frac{3\zeta(s-1,1)+\zeta(s-1,0)}{(1-3^{-s})\zeta_K(s)}\\&=v^{-1}\pi(s-1)^{-1}\frac{3^{s+1}-9+2\cdot 3^{3-s}}{3^s-1}\frac{\zeta_K(s-1)}{\zeta_K(s)}.\end{aligned}$$

Working in the same way with $E_{-\omega}(w,s)$, and using the symmetry $\varphi_{ij}(s)=\varphi_{ji}(s)$, completes the proof of Theorem 1. □

5. To compute the determinant $\varphi(s)$, it is convenient to interchange the second and third rows of $\Phi(s)$ and look at

$$\begin{bmatrix}A&B&B&C&C\\B&A&B&C&C\\B&B&A&C&C\\C&C&C&D&E\\C&C&C&E&D\end{bmatrix}.$$

Consider the basis of $\mathbf{C}^5$ given by

$$\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}\begin{bmatrix}1\\\omega\\\omega^2\\0\\0\end{bmatrix}\begin{bmatrix}1\\\omega^2\\\omega\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}$$

Among these vectors, the second, third, and fifth are eigenvectors of our matrix, with eigenvalues $A-B$, $A-B$, and $D-E$ (respectively). To find the (product of the) other two, the action of the matrix on the subspace spanned by the first and fourth vectors is

$$\begin{bmatrix}\alpha\\\alpha\\\alpha\\\beta\\\beta\end{bmatrix}\to\begin{bmatrix}\alpha A+2\alpha B+2\beta C\\ \text{"}\\ \text{"}\\ 3\alpha C+\beta D+\beta E\\ \text{"}\end{bmatrix},$$

so that for this to be an eigenvector with eigenvalue λ we must have

$$\begin{bmatrix}A+2B & 2C\\ 3C & D+E\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}=\lambda\begin{bmatrix}\alpha\\\beta\end{bmatrix}.$$

Therefore the product of the remaining eigenvalues is $(A+2B)(D+E)-6C^2$, and the determinant of our matrix above is

$$(A-B)^2(D-E)((A+2B)(D+E)-6C^2)=-r(s)\frac{\zeta_K(s-1)^3}{\zeta_K(s)^3}\frac{\zeta_K(3s-3)^2}{\zeta_K(3s-2)^2},$$

where

$$r=27(r_4-r_3)(r_1(r_3+r_4)-2r_2^2).$$

Multiplying by -1 and by the gamma factor completes the proof of Theorem 2. □

REFERENCES

1. I. Efrat, *Cusp forms and higher rank,* preprint, 1984.
2. ———, *Eisenstein series and Cartan groups,* Illinois J. Math. 31 (1987), no. 3.
3. T. Kubota, *Ein arithmetischer Satz über eine Matrizengruppe,* J. Reine Angew. Math. 222 (1965), 55–57.
4. ———, *On automorphic functions and the reciprocity law in a number field,* Kinokuniya Bookstore, Tokyo, 1969.
5. S. Patterson, *A cubic analogue of the theta series,* J. Reine Angew. Math. 296 (1977), 125–161.

6. P. Sarnak, *On cusp forms. The Selberg trace formula and related topics* (Brunswick, Maine, 1984), 393–397, Contemp. Math., 53, Amer. Math. Soc., Providence, R.I., 1986.
7. A. Selberg, *Harmonic analysis,* Teil 2, Göttingen, 1954.

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