HILBERT FUNCTIONS AND SYMBOLIC POWERS

Craig Huneke

1. Introduction. The following question of Cowsik [4] inspired much of this paper: if R is a regular local ring and $p \subset R$ is a prime ideal, then is $\bigoplus_{n\geq 0} p^{(n)}$ Noetherian? Here $p^{(n)} = p^n R_p \cap R$ is the nth symbolic power of p. Cowsik proved if $\dim R/p = 1$ then $\bigoplus p^{(n)}$ Noetherian implies that p is a set-theoretic complete intersection. One of the main cases is when R is 3-dimensional and $\operatorname{ht} p = 2$. Positive results were obtained in [5] and [8], but recently Roberts [29] gave a counterexample to the general question. One of the main results of this paper is to give a necessary and sufficient criterion (Theorem 3.1) for $\bigoplus_{n\geq 0} p^{(n)}$ to be Noetherian which is relatively simple to apply. Namely, we are able to show that, if R is a 3-dimensional regular local ring and p is a height-2 prime of R, then $\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian if and only if there exist k, ℓ, elements $f \in p^{(k)}$, $g \in p^{(\ell)}$, and $x \notin p$ such that $\lambda(R/(f,g,x)) = ek\ell$, with $e = \lambda(R/(p,x))$. Here $\lambda(\;)$ denotes length.

The proof of this result requires an understanding of Hilbert functions of m-primary ideals in 2-dimensional Cohen–Macaulay (C–M for short) local rings. In general, if R, m is a d-dimensional local C–M ring and I is an m-primary ideal, then there is a polynomial $P_I(n)$ of the form

$$e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}\binom{n}{1} + (-1)^d e_d$$

such that, for $n \gg 0$, $P_I(n) = \lambda(R/I^n)$. We define $H_I(n) = \lambda(R/I^n)$ for every $n \geq 0$, and call H_I the Hilbert function of I and P_I the Hilbert polynomial of I. Not a great deal is known about the coefficients $e_0, \ldots, e_d$ of $P_I(n)$. However, see [10], [21], [28], and [30].

Of course, e_0 is called the multiplicity of I and can be computed as follows: if R/m is infinite and $x_1, \ldots, x_d$ is a minimal reduction of I, then $e_0 = \lambda(R/(x_1, \ldots, x_d))$. Northcott [22] showed that $\lambda(R/I) \geq e_0 - e_1$ always holds while Narita [21] showed that $e_2 \geq 0$. Recently Kubota [11] proved that if $\lambda(R/I) = e_0 - e_1$ and $\lambda(R/I^2) = e_0(d+1) - e_1 d$, then necessarily $e_2 = \cdots = e_d = 0$. For our theorem on symbolic powers we need to improve this theorem. We are able to show (Theorem 2.7) that if $\lambda(R/I) = e_0 - e_1$ then necessarily $e_2 = \cdots = e_d = 0$. (From this it follows that $P_I(n) = H_I(n)$ for all $n \geq 0$ and also that $I^2 = (x_1, \ldots, x_d)I$.)

The basic method of proof is essentially the same as in papers of Rees [26] and Kubota [11], but pushed slightly differently. The methods also apply to considering the difference $P_I(n) - H_I(n)$. Following the lead of Morales [20], we consider when $P_I(n) = H_I(n)$ for all $n \geq 1$ (Theorem 2.11) and when $P_I(n) \geq H_I(n)$ for all $n \geq 1$ (Proposition 2.12), and later apply these to the case where $\bigoplus_{n\geq 0} I^n/I^{n+1}$ is

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C–M (Theorem 4.6). We also discuss the independence of the reduction number. We refer the reader to the text for precise statements.

The work on $H_I(n)$ and $P_I(n)$ is contained in Section 2, while in Section 3 we give the main application to symbolic powers along with several examples. In particular, recovering a result of [9], we show that if $e(R/p)=3$ then $\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian; we also show that if p is the defining ideal of a curve $k[[t^4, t^n, t^m]]$ then $\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian. We prove a theorem which shows the difference between $\bigoplus_{n\geq 0} p^{(n)}$ being Noetherian and p being a set-theoretic complete intersection in the case where p is height 2 in a 3-dimensional regular local ring.

In Section 4 we carry over essentially the same proofs as in Section 2 to study $\bar{H}_I(n)$ and $\bar{P}_I(n)$, where I is an m-primary ideal in a 2-dimensional local C–M analytically unramified ring R. Here $\bar{H}_I(n)=\lambda(R/\overline{I^n})$, where $\overline{I^n}$ is the integral closure of I^n and $\bar{P}_I(n)$ is the polynomial of degree 2 such that $\bar{P}_I(n)=\bar{H}_I(n)$ for $n \gg 0$. Rees [25] proved such a polynomial exists. We prove exactly similar theorems regarding the coefficients of $\bar{P}_I(n)$ in this section as in Section 2, and as in the work of Northcott [22], Narita [21], and Kubota [11] for $P_I(n)$. We also discuss an interesting theorem of Morales [20] which states that if $\bar{P}_I(n)=\bar{H}_I(n)$ for $n\geq 1$ then $\bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$ is C–M. Here R is a germ of a normal surface over **C**. We prove $\bar{P}_I(n)=\bar{H}_I(n)$ for all $n\geq 1$ if and only if $\overline{I^n}=(x,y)\overline{I^{n-1}}$ for $n\geq 3$, where (x,y) is a minimal reduction of I. Under this condition, $\bigoplus \overline{I^n}/\overline{I^{n+1}}$ is C–M if and only if $\overline{I^2}\cap(x,y)=(x,y)\bar{I}$. In Theorem 4.7, if char $R=p>0$ we prove this latter condition for any d-dimensional C–M local ring; that is, if I is any ideal generated by a regular sequence $x_1,\dots,x_g$ then we are able to show $\overline{I^n}\cap I^{n-1}=I^{n-1}\bar{I}$.

As a corollary we obtain that if $\bar{P}_I(n)=\bar{H}_I(n)$ for $n\geq 1$ and char $R=p>0$, then $\bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$ is C–M—that is, we obtain the characteristic p analogue of the theorem of Morales. We continue with a related result showing if R is F-pure (see [6]) of dimension d and $x_1,\dots,x_d$ is a system of parameters for R then

$$\overline{(x_1,\dots,x_d)^{d+1}}\subseteq(x_1,\dots,x_d).$$

We also present (in an appendix) a proof shown to me by M. Hochster, which uses the technique of reduction to characteristic p to prove that if R is a C–M local ring containing a field and $x_1,\dots,x_g$ is a regular sequence generating an ideal I, then $\overline{I^n}\cap I^{n-1}=I^{n-1}\bar{I}$.

Hence we obtain a generalization of Morales' theorem to arbitrary 2-dimensional C–M analytically unramified local rings which contain a field.

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2. Hilbert functions. We begin this section by proving, in a d-dimensional C–M local ring R, that if $e_0-e_1=\lambda(R/I)$ then $e_2=\cdots=e_d=0$, $P_I(n)=H_I(n)$ for all $n\geq 1$, and I has reduction number at most one; that is, for any minimal reduction $(x_1,\dots,x_d)$ of I, $I^2=(x_1,\dots,x_d)I$. This is a stronger version of a theorem of Kubota [11] and Rhodes [28, Proposition 6.1(ii)], and is what we use for the criterion for the symbolic power algebra to be Noetherian (Theorem 3.1).

However we are also able to prove several other results dealing with the difference between $P_I(n)$ and $H_I(n)$. For instance, we give a necessary and sufficient condition in Theorem 2.11 for them to be the same for all $n \geq 1$. We also show under certain conditions (see Proposition 2.12) that $P_I(n) \geq H_I(n)$ for $n \geq 1$ and further that if $P_I(n) = H_I(n)$ then $P_I(j) = H_I(j)$ for $j \geq n$. We also briefly discuss the independence of the reduction number.

THEOREM 2.1. *Let (R, m) be a d-dimensional C–M local ring $(d > 0)$ with infinite residue field. Let I be an m-primary ideal. Write*

$$\lambda(R/I^n) = e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d$$

for $n \gg 0$. Assume $e_0 - e_1 = \lambda(R/I)$. Then, for any minimal reduction $x_1, \ldots, x_d$ of I, $I^2 = I(x_1, \ldots, x_d)$, $e_2 = \cdots = e_d = 0$, and $H_I(n) = P_I(n)$ for $n \geq 1$. Conversely, if there are $x_1, \ldots, x_d \in I$ such that $I^2 = I(x_1, \ldots, x_d)$ then $e_0 - e_1 = \lambda(R/I)$.

Proof. We induct on d. If $d = 1$, then this is the content of [14, Theorem 1.5 and Lemma 1.11].

Let $(x_1, \ldots, x_d)$ be any minimal reduction of I. Then denote by $x_1^*, \ldots, x_d^*$ their images in $I/I^2 \subseteq \mathrm{gr}_I(R) = G$, and by J denote $\bigoplus_{n \geq 1} I^n/I^{n+1}$. Then for some n, $(x_1^*, \ldots, x_d^*)J^{n-1} = J^n$ so that $(x_1^*, \ldots, x_d^*)$ is not contained in any associated prime of G which does not contain J. As R/m is infinite it follows that we may, after elementary transformations of $x_1, \ldots, x_d$, assume that each of them is superficial for I. Let $\bar{R} = R/(x_d)$, $\bar{I} = I/(x_d)$. Then Northcott [22] showed $e_k(I) = e_k(\bar{I})$ for $0 \leq k \leq d-1$. In particular $\lambda(\bar{R}/\bar{I}) = \lambda(R/I) = e_0(\bar{I}) - e_1(\bar{I})$, so that by induction $\bar{I}^2 = \bar{I}(\bar{x}_1, \ldots, \bar{x}_{d-1})$, $e_2(\bar{I}) = \cdots = e_{d-1}(\bar{I}) = 0$, and $P_{\bar{I}}(n) = H_{\bar{I}}(n)$ for all $n \geq 1$. Lifting to R we see that

$$I^2 \subseteq (x_1, \ldots, x_{d-1})I + (x_d). \tag{2.2}$$

We claim that

$$(I^2 : x_1) = I. \tag{2.3}$$

Let $a \in (I^2 : x_1)$. Then from (2.2) it follows that

$$x_1 a = \sum_{i=1}^{d-1} x_i b_i + x_d c,$$

where $b_1, \ldots, b_{d-1}$ are in I. Hence $x_1(a - b_1) \in (x_2, \ldots, x_d)$, and since $x_1, \ldots, x_d$ are a regular sequence it follows that $a - b_1 \in (x_2, \ldots, x_d)$ so that $a \in I$. Thus $(I^2 : x_1) = I$.

By our choice of $x_1, \ldots, x_d$, x_1 is superficial for I. Hence the same induction applied to $R/(x_1)$ and $I/(x_1)$ shows that

$$\begin{aligned} I^2 &= (I^2 : x_1)x_1 + (x_2, \ldots, x_d)I \\ &= Ix_1 + (x_2, \ldots, x_d)I \quad \text{by (2.3)} \\ &= (x_1, \ldots, x_d)I. \end{aligned}$$

Now Theorem (iv) of [11] shows that $e_2 = \cdots = e_d = 0$ and $H_I(n) = P_I(n)$ for all $n \geq 1$.

The converse is easy and can be found in [11]. □

FUNDAMENTAL LEMMA 2.4. *Let R, m be a 2-dimensional local C–M ring and let $x, y \in m$ be any system of parameters of R. Let I be any ideal integral over (x, y). Then*

$$\lambda(I^{n+1}/(x,y)I^n) - \lambda(I^n:(x,y)/I^{n-1})$$
$$= [P_I(n+1) - H_I(n+1)] + [P_I(n-1) - H_I(n-1)] - 2[P_I(n) - H_I(n)]$$

for all $n \geq 1$.

Proof. Consider the exact sequence

$$0 \to K \to (R/I^n)^2 \xrightarrow{\alpha} (x,y)/I^n(x,y) \to 0,$$

where $K = \ker(\alpha)$ and $\alpha((\bar{r}, \bar{s})) = rx + sy$. We calculate K. If $\alpha(\bar{r}, \bar{s}) = 0$ then $rx + sy \in I^n(x, y)$ so that there are $u, v \in I^n$ with $rx + sy = ux + vy$. Hence $x(r-u) = y(v-s)$ and there is a $t \in R$ with $r = u + yt$ and $s = v - xt$. Thus $(\bar{r}, \bar{s}) = t(\bar{y}, -\bar{x})$ which shows that $(\bar{y}, -\bar{x})$ generates K. Thus $K \simeq R/(I^n:(x,y))$. Hence

$$\lambda((x,y)/I^n(x,y)) = 2\lambda(R/I^n) - \lambda(R/I^n:(x,y)).$$

The length $\lambda(R/(x,y))$ is equal to the multiplicity of I. Hence

$$\lambda(I^{n+1}/I^n(x,y)) + \lambda(R/I^n:(x,y)) = 2\lambda(R/I^n) + e - \lambda(R/I^{n+1}),$$

so that

$$\text{(2.5)} \quad \lambda(I^{n+1}/I^n(x,y)) - \lambda(I^n:(x,y)/I^{n-1}) = e + 2H_I(n) - H_I(n+1) - H_I(n-1).$$

Now consider $P_I(n)$. We have that

$$P_I(n) = e_0(I)\binom{n+1}{2} - e_1(I)n + e_2(I),$$

where $e_0(I) = e$ is the multiplicity of I. Consider $P_I(n+1) + P_I(n-1) - 2P_I(n)$ for $n \geq 1$: this is equal to

$$\left[e_0\binom{n+2}{2} - e_1(n+1) + e_2\right] + \left[e_0\binom{n}{2} - e_1(n-1) + e_2\right] - 2\left[e_0\binom{n+1}{2} - e_1 n + e_2\right]$$
$$= e_0\left[\frac{(n+2)(n+1) + n(n-1) - 2(n+1)n}{2}\right]$$
$$= e_0 = e.$$

Hence, substituting this expression of e into (2.5), we obtain the lemma. □

Now we set $v_n = \lambda(I^{n+1}/(x,y)I^n) - \lambda(I^n:(x,y)/I^{n-1})$ for $n \geq 1$, with the assumptions as in the fundamental lemma.

COROLLARY 2.6. *Let R, I, x, y, and v_n be as above. Let a_1, a_2 be arbitrary real numbers and define $a_n = (n-1)a_2 - (n-2)a_1$. Then*

$$\sum_{n=1}^{\infty} a_n v_n = (a_2 - 2a_1)(e_0 - e_1 - \lambda(R/I)) + (a_2 - a_1)(e_2),$$

where

$$P_I(n) = e_0\binom{n+1}{2} - e_1 n + e_2.$$

Proof. Set $u_n = P_I(n) - H_I(n)$ for $n \geq 0$. By the fundamental lemma, $v_n = u_{n+1} + u_{n-1} - 2u_n$. Hence,

$$\sum_{n=1}^{\infty} a_n v_n = \sum_{n=1}^{\infty} a_n(u_{n+1} + u_{n-1} - 2u_n).$$

This sum is finite since $u_N = 0$ for $N \gg 0$, and we obtain

$$\sum_{n=1}^{\infty} a_n v_n = a_1 u_o + (a_2 - 2a_1)u_1 + \sum_{n=2}^{\infty} (a_{n+1} + a_{n-1} - 2a_n)u_n.$$

But $a_{n+1} + a_{n-1} - 2a_n = 0$ for $n \geq 2$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} a_n v_n &= a_1 u_0 + (a_2 - 2a_1)u_1 \\ &= a_1 e_2 + (a_2 - 2a_1)[e_0 - e_1 + e_2 - \lambda(R/I)] \\ &= (a_2 - a_1)e_2 + (a_2 - 2a_1)[e_0 - e_1 - \lambda(R/I)].\end{aligned}$$

□

As immediate consequences of Corollary 2.6 we obtain the following formulas for some of the coefficients of $P_I(n)$.

REMARK 2.7.

$$\lambda(R/I) - [e_0 - e_1] = \sum_{n=1}^{\infty} v_n.$$

Proof. Let $a_1 = a_2 = 1$ in the corollary. Then $a_n = 1$ for all n. □

REMARK 2.8.

$$e_2 = \sum_{n=1}^{\infty} n v_n.$$

Proof. Let $a_1 = 1$ and $a_2 = 2$. Then $a_n = n$ for all n. □

REMARK 2.9. *Let j be chosen least such that $P_I(n) = H_I(n)$ for all $n \geq j$. Then*

$$j(\lambda(R/I) - (e_0 - e_1)) - e_2 = \sum_{k=1}^{j-1} (j-k)v_k.$$

Proof. Since $P_I(n) = H_I(n)$ for $n \geq j$ it follows that $v_n = 0$ for $n \geq j+1$ by the fundamental lemma. Let $a_1 = j-1$, $a_2 = j-2$. Then $a_i = j-i$ for all $i \geq 1$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} a_n v_n &= \sum_{k=1}^{j-1} (j-k)v_k + 0 \cdot v_j - v_{j+1} - 2v_{j+2} \cdots \\ &= \sum_{k=1}^{j-1} (j-k)v_k \quad \text{since } v_n = 0 \text{ for } n \geq j+1.\end{aligned}$$

Thus by Corollary 2.6 we obtain Remark 2.9. □

REMARK. Recall (Theorem 2.1) that if $\lambda(R/I) = e_0 - e_1$ then necessarily $e_2 = 0$. This would immediately follow from Remark 2.9 if we knew that $v_k \geq 0$ for all k. Also, Northcott's theorem [22] that $\lambda(R/I) - (e_0 - e_1) \geq 0$ would follow immediately from Remark 2.7, while Narita's theorem [21] that $e_2 \geq 0$ would follow immediately from Remark 2.8. Clearly if grade $\mathrm{gr}_I(R)^+ \geq 1$ then $v_k \geq 0$ for all k; however, an example due to Huckaba shows that in general v_k need not be nonnegative (although "philosophically" they behave as if they were!).

We recall some material from [24]. Let I be an ideal of grade at least 1 in a Noetherian ring R. Then $\{(I^{n+1} : I^n)\}$ form an ascending chain of ideals and the stable value is denoted $\tilde{I}$. Then we have the following.

REMARK 2.10. $(\tilde{I})^n = I^n$ for $n \gg 0$. Furthermore $\tilde{\tilde{I}} = \tilde{I}$, so that $(\tilde{I}^n : \tilde{I}^{n-1}) = \tilde{I}$ for all $n \geq 1$. $\tilde{I}$ is the largest ideal with the same Hilbert polynomial as I.

THEOREM 2.11. *Let (R, m) be a 2-dimensional local C–M ring with infinite residue field and I an m-primary ideal such that $I = \tilde{I}$. Then the following are equivalent.*

(1) *$H_I(n) = P_I(n)$ for all $n \geq 1$.*
(2) *There exist x, y in I such that $I^3 = (x, y)I^2$ and further grade $\mathrm{gr}_I(R)^+ \geq 1$, where $\mathrm{gr}_I(R)^+ = \bigoplus_{n>0} I^n/I^{n+1}$.*

Proof. Assume (1). Set $u_n = P_I(n) - H_I(n)$. By the fundamental lemma, if x, y is a reduction of I then $\lambda(I^{n+1}/(x,y)I^n) - \lambda(I^n : (x,y)/I^{n-1}) = u_{n+1} + u_{n-1} - 2u_n$ for all $n \geq 1$. Assuming (1) we obtain (for all $n \geq 2$) that $\lambda(I^{n+1}/(x, y)I^n) = \lambda(I^n : (x,y)/I^{n-1})$ and (for $n = 1$) that $\lambda(I^2/(x,y)I) - \lambda(I : (x,y)/I^0) = u_0$. Thus $\lambda(I^2/(x,y)I) = P_I(0) - H_I(0) = e_2(I)$. For $n = 2$,

$$\lambda(I^3/(x,y)I^2) = \lambda(I^2 : (x,y)/I).$$

We claim that $I^2 : (x,y) = I$. For if $a \in I^2 : (x,y)$ then $a(x,y) \subseteq I^2$; so for $j \gg 0$, $aI^j \subseteq I^{1+j}$ since (x,y) is a reduction of I. Hence $a \in (I^{j+1} : I^j) \subseteq \tilde{I} = I$. It follows that $I^3 = (x,y)I^2$. Then $I^{n+1} = (x,y)I^n$ for all $n \geq 2$, so that (since $u_n = 0$ for $n \geq 1$) it follows that $I^n : (x,y)/I^{n-1} = 0$ for all $n \geq 2$. Hence grade $\mathrm{gr}_I(R)^+ > 0$.

Next assume (2). Since grade $\mathrm{gr}_I(R)^+ > 0$, $I^n : (x,y)/I^{n-1} = 0$ for all $n \geq 1$, where (x,y) is any reduction of I. Hence $u_{n+1} + u_{n-1} - 2u_n = \lambda(I^{n+1}/(x,y)I^n)$ for $n \geq 1$. If $n \geq 2$, $I^{n+1} = (x,y)I^n$ by assumption, so that $u_{n+1} + u_{n-1} - 2u_n = 0$ for all $n \geq 2$. For large n, $u_n = 0$. Let i be chosen maximal so that $u_i \neq 0$. Assume $i > 0$. Then $u_i + u_{i+2} - 2u_{i+1} = 0$, which contradicts the choice of i. □

PROPOSITION 2.12. *Fix the notation as in the fundamental lemma. If $v_n \geq 0$ for all n (e.g., if grade $\mathrm{gr}_I(R)^+ > 0$) then $P_I(n) - H_I(n) \geq P_I(m) - H_I(m) \geq 0$ if $n < m$ unless $P_I(n) = H_I(n)$. If $P_I(n) = H_I(n)$ for some n then $P_I(j) = H_I(j)$ for all $j \geq n$.*

Proof. Let $u_n = P_I(n) - H_I(n)$. For $n \gg 0$, $u_n = 0$. We will first show that necessarily $u_j \geq u_i$ if $j < i$. In particular $u_j \geq 0$. Clearly it suffices to show $u_i \geq u_{i+1}$ if $1 \leq i$. Suppose not, so that $u_i < u_{i+1}$. We obtain from the fundamental lemma that $v_{i+2} = u_{i+2} + u_i - 2u_{i+1}$, and (by assumption) $v_{i+1} \geq 0$ so that $u_{i+1} \leq (u_{i+2} + u_i)/2$

and so $u_{i+1} < (u_{i+2}+u_{i+1})/2$, or $u_{i+1} < u_{i+2}$. Similarly $u_j < u_{j+1}$ for all $j \geq i$. For large n, however, $u_n = 0$, contradicting this inequality. Thus $u_i \geq u_{i+1}$ if $1 \leq i$.

Now suppose $u_i = 0$ for some i. Then we have $0 \geq u_{i+1} \geq u_{i+2} \geq \cdots \geq 0$ so that $u_{i+1} = u_{i+2} = \cdots = u_n = \cdots = 0$ as required.

Finally, suppose $u_i = u_{i+1}$ for some i. Then, as above, $u_{i+1} \leq (u_{i+2}+u_i)/2$ so that

$$u_{i+1} \leq (u_{i+2}+u_{i+1})/2 \quad \text{or} \quad u_{i+1} \leq u_{i+2}.$$

As we have shown that $u_{i+1} \geq u_{i+2}$, it follows that $u_{i+1} = u_{i+2}$. Similarly $u_i = u_j$ if $j \geq i$ and hence $u_i = 0$. Thus we see that the u_i are strictly decreasing until they hit 0. □

EXAMPLE 2.13. The following example was shown to me by S. Huckaba. Let $R = k[[x,y]]$, k a field, and I be the ideal generated by (x^7, x^6y, x^2y^5, y^7). Then

$$P_I(n) = 49\binom{n+1}{2} - 21n + 3$$

and $P_I(n) = H_I(n)$ for $n \geq 4$. However $H_I(1) = 32$, $H_I(2) = 110$, and $H_I(3) = 235$ while $P_I(1) = 31$, $P_I(2) = 108$, and $P_I(3) = 234$. Thus $u_1 = -1$, $u_2 = -2$, and $u_3 = -1$. Thus

$$v_1 = u_2 + u_0 - 2u_1 = -2 + (3) + 2 = 3,$$
$$v_2 = u_3 + u_1 - 2u_2 = -2 + 4 = 2,$$
$$v_3 = u_4 + u_2 - 2u_3 = -2 + 2 = 0,$$
$$v_4 = u_5 + u_3 - 2u_4 = -1,$$

and $v_i = 0$ for $i \geq 5$. Hence the v_n do not have to be positive.

Another application of the fundamental lemma has to do with the independence of the reduction number. If J is a reduction of I, set $r_J(I) =$ least n such that $JI^n = I^{n+1}$. Sally raised the question if $r_J(m)$ is independent of J when J is a minimal reduction of m.

In his thesis [7] Huckaba showed that for the ideal of Example 2.13, the reduction number is *not* independent of the reduction chosen, but he shows independence if grade $\mathrm{gr}_I(R)^+ \geq 1$. Also see [31] for other work on the reduction number.

We can show the following.

PROPOSITION 2.14. *Let N be the least integer such that $(I^j : I) = I^{j-1}$ for all $j \geq N$. (Such an N exists; see* [19, Lemma 8.1].) *Let N' be the least integer such that $P_I(j) = H_I(j)$ for $j \geq N'$. If $N \leq N'$ then $r_{J_1}(I) = r_{J_2}(I)$ for any two minimal reductions J_1, J_2 of I.*

Proof. We claim that N is also the least integer such that $I^j : (x,y) = I^{j-1}$ for $j \geq N$, where (x,y) is any minimal reduction of I. If $I^j : (x,y) = I^{j-1}$ then $(I^j : I) \subseteq (I^j : (x,y))$, so that $j \geq N$. Let N_1 be chosen least so that $I^j : (x,y) = I^{j-1}$ for $j \geq N_1$. The above argument shows $N_1 \geq N$. If $N < N_1$ then $I^{N_1-1} : I = I^{N_1-2}$, but $I^{N_1-1} : (x,y) \neq I^{N_1-2}$. However, if $t \in I^{N_1-1} : (x,y)$ then $t(x,y) \subseteq I^{N_1-1}$ implies $It(x,y) \subseteq I^{N_1}$, so $It \subseteq (I^{N_1} : (x,y)) = I^{N_1-1}$ or $t \in I^{N_1-1} : I = I^{N_1-2}$. Thus $N_1 = N$.

Now we have that $P_I(N'-1) \neq H_I(N'-1)$ and hence if (x,y) is *any* minimal reduction of I, we have that $\lambda(I^{N'+1}/(x,y)I^{N'}) - \lambda(I^{N'}:(x,y)/I^{N'-1}) \neq 0$, while (by the fundamental lemma) $\lambda(I^{N'+2}/(x,y)I^{N'+1}) - \lambda(I^{N'+1}:(x,y)/I^{N'}) = 0$.

By assumption $N \leq N'$, so that $I^{N'}:(x,y)/I^{N'-1} = 0$ and $I^{N'+1}:(x,y)/I^{N'} = 0$ by the above calculation. Hence we obtain that $I^{N'+1} \neq (x,y)I^{N'}$ while $I^{N'+2} = (x,y)I^{N'+1}$. Thus $r_{(x,y)}(I) = N'+1$ is independent of (x,y). □

REMARK 2.15. Of course $N \leq N'$ if grade $\mathrm{gr}_I(R)^+ \geq 1$, since in this case $I^j : I = I^{j-1}$ for every j. But in this case the fundamental lemma shows that not only is the reduction number independent, but also (for every n) that $\lambda(I^{n+1}/(x,y)I^n)$ is independent of the minimal reduction (x,y)!

3. Symbolic powers. We will be able to use Theorem 2.1 to prove our main theorem, which gives a criterion for $\bigoplus_{n\geq 0} p^{(n)}$ to be Noetherian where p is a height-2 prime in a 3-dimensional regular local ring (R,m). Cowsik [4] proved that if this is the case then p is necessarily a set-theoretic complete intersection. Specific cases of when $\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian have been done in [5] and [8], and Roberts [29] recently gave an example to show that this is not always the case (although it is still possible under the above assumptions that if R is complete or has positive characteristic then $\bigoplus_{n\geq 0} p^{(n)}$ is always Noetherian).

We are able to apply our result to show, for curves of multiplicity three, that $\bigoplus_{n\geq 0} p^{(n)}$ is always Noetherian (see [9]), and to treat an example Moh mentions as one which was not even known to be a set-theoretic complete intersection.

Before beginning the proof we recall the definition of Serre's intersection multiplicity. Let R be a regular local ring and let M, N be two finitely generated R-modules with $\lambda(M\otimes N) < \infty$. Then

$$\chi(M,N) = \sum_{i=0}^{\infty} (-1)^i \lambda(\mathrm{Tor}_i^R(M,N)).$$

THEOREM 3.1. *Let (R,m) be a 3-dimensional regular local ring with infinite residue field; let $p \subseteq R$ be a height-2 prime ideal. Then the following are equivalent.*

(i) *$\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian.*

(ii) *There exist k, ℓ, two elements $f \in p^{(k)}$, $g \in p^{(\ell)}$, and an $x \notin p$ such that $\lambda(R/(f,g,x)) = \lambda(R/(p,x))\ell k$.*

Proof. It is known (see [4]) that (i) is equivalent to the following: (i′) There exists a k such that $(p^{(k)})^n = p^{(kn)}$ for all $n \geq 1$.

Now assume (i′). Then $\inf_n(\mathrm{depth}\, R/(p^{(k)})^n) = 1$ so that, by Burch's theorem [3], $\ell(p^{(k)}) = 2$. Hence there are $f, g \in p^{(k)}$ such that $p^{(k)} = \overline{(f,g)}$. (Note that $p^{(k)}$ is integrally closed.) Choose any $x \notin p$ and let $S = R/xR$, $I = (p^{(k)}, x)/(x)$. We compute $P_I(n)$. Since $I^n = ((p^{(k)})^n, x)/(x) = (p^{(kn)}, x)/(x)$ it follows that $\lambda(S/I^n) = \lambda(R/p^{(kn)}, x)$.

Since x is a nonzero divisor on $R/p^{(m)}$ for all m, it follows that

$$\lambda(R/(p^{(m)},x)) = \chi(R/p^{(m)}, R/(x))$$
$$= \chi(R/(p,x))\ell(R_p/p_p^m) = \lambda(R/(p,x))\ell(R_p/p_p^m)) =$$

$$=e\binom{m+1}{2},$$

where we have set $e=\lambda(R/(p,x))$. Hence

$$\lambda(S/I^n)=\lambda(R/p^{(kn)},x)=e\binom{kn+1}{2}=ek^2\binom{n+1}{2}-e\binom{k}{2}n$$

for all $n\geq 1$. Thus the multiplicity of J is ek^2. Since J is integral over $\bar{f},\bar{g}$ in S, it follows that $\lambda(S/(\bar{f},\bar{g}))=ek^2$. Hence $\lambda(R/(f,g,x))=ek^2$, which is (ii).

Next we show that (ii) is equivalent to the following: (ii′) There exists a k and elements f, $g\in p^{(k)}$ and an element $x\notin p$ such that $\lambda(R/(f,g,x))=ek^2$. ($e=\lambda(R/(p,x))$.)

Obviously, (ii′) implies (ii). Conversely, suppose $f\in p^{(k)}$, $g\in p^{(\ell)}$, $x\notin p$, and $\lambda(R/(f,g,x))=ek\ell$. Then $\lambda(R/(f^\ell,g^k,x))=k\ell(ek\ell)=e(k\ell)^2$ and $f^\ell,g^k\in p^{(k\ell)}$, which gives (ii′).

Now assume (ii′). Set $I=(p^{(k)},x)/(x)$, where k and x are chosen as in the statement of (ii′). Write

$$P_I(n)=e_0\binom{n+1}{2}-e_1n+e_2$$

so that, for $n\gg 0$, $P_I(n)=\lambda(S/I^n)$.

Our calculations above show that if we set $J_n=(p^{(kn)},x)/(x)$ then

$$\lambda(S/J_n)=ek^2\binom{n+1}{2}-e\binom{k}{2}n, \tag{3.2}$$

where $e=\lambda(R/(p,x))$.

By assumption there are $f,g\in p^{(k)}$ such that $\lambda(S/(\bar{f},\bar{g}))=ek^2$. As $(\bar{f},\bar{g})\subset I$, $e(I)\leq e((\bar{f},\bar{g}))=\lambda(S/(\bar{f},\bar{g}))=ek^2$. Hence,

$$e_0\leq ek^2. \tag{3.3}$$

On the other hand, since $p^{(k)n}\subseteq p^{(kn)}$ for all $n\geq 1$ it follows that $\lambda(S/I^n)\geq\lambda(S/J_n)$ for all $n\geq 1$, so that

$$e_0\binom{n+1}{2}-e_1n+e_2\geq ek^2\binom{n+1}{2}-e\binom{k}{2}n$$

for all $n\gg 0$. Hence,

$$e_0\geq ek^2 \quad\text{and}\quad e\binom{k}{2}\geq e_1. \tag{3.4}$$

From (3.4) and (3.3) we obtain that $e_0=ek^2$. Hence

$$e_0-e_1=ek^2-e_1\geq ek^2-e\binom{k}{2}$$

(from (3.4)), and

$$ek^2-e\binom{k}{2}=e\binom{k+1}{2}=\lambda(S/I)$$

by (3.2). Thus $e_0-e_1\geq\lambda(R/I)$.

Now Northcott's theorem [22] implies that $e_0 - e_1 = \lambda(S/I)$. Hence from Theorem 2.1 we conclude that $e_2 = 0$ and $P_I(n) = H_I(n)$ for all $n \geq 1$ so that, for all $n \geq 1$,

$$\begin{aligned}\lambda(S/I^n) &= e_0\binom{n+1}{2} - e_1 n \\ &= ek^2\binom{n+1}{2} - (e_0 - \lambda(S/I))n \\ &= ek^2\binom{n+1}{2} - \left(ek^2 - e\binom{k+1}{2}\right)n \\ &= ek^2\binom{n+1}{2} - e\binom{k}{2}n \\ &= \lambda(S/J_n).\end{aligned}$$

As $I^n \subset J_n$, we obtain $I^n = J_n$ for all $n \geq 1$. Hence

$$(p^{(k)n}, x) = (p^{(kn)}, x) \quad \text{for } n \geq 1.$$

This equation shows that $p^{(kn)} \subseteq (p^{(k)n}, x)$, so

$$\begin{aligned}p^{(kn)} &\subseteq p^{(k)n} + (x) \cap (p^{(kn)}) \\ &= p^{(k)n} + x(p^{(kn)} : x) \\ &= p^{(k)n} + xp^{(kn)}.\end{aligned}$$

Applying Nakayama's lemma gives that

$$p^{(kn)} = p^{(k)n} \quad \text{for all } n \geq 1,$$

which shows (i′). □

REMARK 3.5. Suppose, in Theorem 3.1, that $x \notin m^2$ and the leading forms $\bar{f}^*$ and $\bar{g}^*$ of $\bar{f}$ and $\bar{g}$ in $\operatorname{gr}_n(S)$ $(n = m/(x))$ are relatively prime. Then $\lambda(R/(f, g, x)) = \deg((\bar{f})^*)\deg((\bar{g})^*)$.

COROLLARY 3.6 ([9]). *Suppose* R, m, p *are as in Theorem* 3.1 *and* $e(R/p) = 3$. *Then* $\bigoplus_{n \geq 0} p^{(n)}$ *is Noetherian.*

Proof. Choose $x \notin p$ such that $x \notin m^2$ and $\lambda(R/(p, x)) = 3$. By length arguments (as was done in [9]) one sees that $(p^{(2)}, x)/(x) = (m^4, x, f)$, where $f \in m^3 \cap p^{(2)}$ and $\deg((\bar{f})^*) = 3$. Choose some $\bar{g} \in \bar{m}^4$ such that $(\bar{g})^*$ and $(\bar{f})^*$ are relatively prime, and lift $\bar{g}$ to $g \in p^{(2)}$. Then

$$\begin{aligned}\lambda(R/(f, g, x)) &= \lambda(S/(\bar{f}, \bar{g})) = \deg((\bar{f})^*)\deg((\bar{g})^*) \quad \text{(by Remark 3.5)} \\ &= 3 \cdot 4 = 12 = 3 \cdot 2^2.\end{aligned}$$

As $f, g \in p^{(2)}$ and $\lambda(R/(p, x)) = 3$, the conclusion follows from Theorem 3.1. □

EXAMPLE 3.7. Moh mentions the following example as not being known even though it is a set-theoretic complete intersection. Let $R = \mathbf{C}[[X, Y, Z]]$; let p be the kernel of the homomorphism of R onto $\mathbf{C}[[t^6, t^7+t^{10}, t^8]]$ sending X to t^6, Y to t^7+t^{10}, and Z to t^8. We claim that $\bigoplus_{n\geq 0} p^{(n)}$ is Noetherian and hence in particular that p is a set-theoretic complete intersection. In fact we will find $f, g \in p^{(10)}$ and $x \notin p$, so that $\lambda(R/(f, g, x)) = 600 = 6\times 10^2$ while $\lambda(R/(p, x)) = 6 = e(R/p)$.

First we calculate the generators of p. A straightforward check shows that

$$\begin{aligned} a &= 2xz^3 - 3x^2yz - 2x^4 + y^3 - xyz, \\ b &= x^3z - 2yz^2 + xy^2 - x^2z, \\ c &= x^2z^2 - 2x^3y + y^2z - xz^2, \quad \text{and} \\ d &= x^4 - z^3 \end{aligned}$$

are in p. We claim $p = (a, b, c, d)$.

To prove this claim we first observe that, up to unit, $(a, b, c, d) = I$ are the 3×3 minors of

$$\begin{pmatrix} x & -y & -2z & 2x \\ z & -2x^2 & -y & 2xz \\ 0 & -z & x & 2y \end{pmatrix}.$$

It follows that R/I is C–M.

Consider $(I, x) = (a, b, c, d, x) = (y^3, 2yz^2, y^2z, z^3, x)$ so that $\lambda(R/(I, x)) = 6$. Thus

$$\begin{aligned} e(x; R/I) &= \lambda(R/(I, x)) \quad (\text{as } R/I \text{ is C–M}) \\ &= 6. \end{aligned}$$

By the associativity formula,

$$e(x; R/I) = \sum_{\substack{I\subseteq q \\ \operatorname{ht} q = 2}} e(x; R/q)\,\ell(R_q/I_q).$$

Since $e(x; R/p) = \lambda(R/(p, x)) = 6$, it follows that $\ell(R_p/I_p) = 1$ and $\sqrt{I} = p$. As R/I is unmixed, $I = p$. Therefore $(I, x) = (p, x) = (m^3, x)$. We next calculate $p^{(2)}$.

A straightforward calculation shows there exist e, f, g such that

$$xe = 2ad - bd + 4d^2x, \tag{3.8}$$

$$xf = 2c^2 + ab + 2bdx, \quad \text{and} \tag{3.9}$$

$$xg = b^2 + 4cd. \tag{3.10}$$

These equations show that $e, g, f \in p^{(2)}$. Furthermore,

$$e \equiv -y^4z \bmod (x), \tag{3.11}$$

$$f \equiv -2y^2z^3 + y^5 \bmod (x), \quad \text{and} \tag{3.12}$$

$$g \equiv 4z^5 - 4y^3z^2 \bmod (x). \tag{3.13}$$

In fact, explicitly,

$$\begin{aligned} e = {} & 4x^8 - 5x^4z^3 - 4x^5yz + 8xyz^4 - 4x^7 - 6x^3y^2z^2 \\ & + 6x^3z^3 + 4x^3y^3 - 4x^4yz - y^4z + 2xy^2z^2 - x^2z^3, \end{aligned}$$

$$f = 2x^7z + 2x^3z^4 - 15x^4yz^2 + 10x^5y^2 - 4x^6z + 10xy^2z^3 - 4x^2z^4 - 10x^2y^3z + 14x^3yz^2 - 2x^4y^2 + 2x^5z - 2y^2z^3 + 2xz^3 + 2xz^4 + y^5 - 2xy^3z + x^2yz^2, \quad \text{and}$$

$$g = 5x^5z^2 - 4xz^5 - 8xz^5 - 8x^6y + 4x^2yz^3 + 6x^3y^2z - 6x^4z^2 + 4z^5 - 4y^3z^2 + 4xyz^3 + xy^4 - 2x^2y^2z + x^3z^2.$$

From the equations (3.8)–(3.13) we obtain that $(p^{(2)}, x)$ contains

$$J = (m^6, x, y^4z, y^5 - 2y^2z^3, z^5 - y^3z^2).$$

Clearly $\lambda(R/J) = (1 + \cdots + 6) - 3 = 18$. However

$$\lambda(R/p^{(2)}, x) = \lambda(R/(p, x)) \cdot \lambda(R_p/p_p^2) = 6 \times 3 = 18,$$

so that $(p^{(2)}, x) = (J, x)$.

Next one checks that $bf^2 + 2ae^2 + geb \equiv 0 \bmod (x)$. Hence there is an element h such that $xh = bf^2 + 2ae^2 + geb$, so $h \in p^{(5)}$. A straightforward check shows that $h \equiv y^{12} - 36y^8z^5 + 72y^5z^8 \bmod (x)$. The leading form of $\bar{g}$ in $k[y, z] \simeq \mathrm{gr}_{m/(x)}(R/(x))$ is $4z^5 - 4y^3z^2$. Hence $(\bar{g})^*$ and $(\bar{h})^*$ are relatively prime so that $(\bar{g}^5)^* = ((\bar{g})^*)^5$ and $(\bar{h}^2)^* = ((\bar{h})^*)^2$ are also relatively prime. However $h^2, g^5 \in p^{(10)}$ and $\deg((\bar{h}^2)^*) = 24$, $\deg((\bar{g}^5)^*) = 25$. Hence $\lambda(R/(g^5, h^2, x)) = 24 \times 25 = 600 = 6 \times 10^2 = \lambda(R/(p, x)) \cdot 10^2$. The theorem implies now that $\bigoplus_{n \geq 0} p^{(n)}$ is Noetherian and the proof shows that $\sqrt{(g, h)} = p$.

Now we show the following.

THEOREM 3.14. *Let k be a field of characteristic $\neq 2$ and let p be the defining ideal in $k[[x, y, z]] = R$ of the curve $k[[t^4, t^n, t^m]]$, $n < m$. Then $\bigoplus_{n \geq 0} p^{(n)}$ is Noetherian.*

Proof. First we find a standard form for such curves. Consider $R/(p, x)$. This has length 4 so that its Hilbert function must be 1, 2, 1. If there is only a 1-dimensional socle, then R/p is Gorenstein and hence p is a complete intersection, in which case $p^n = p^{(n)}$ for all n and the theorem is trivially true. Hence there is a 1-form in the socle. (Note that (p, x) is an ideal generated by monomials so it is homogeneous in the polynomial ring $k[x, y, z]$.) Hence (p, x) must be equal to (y^2, yz, z^3, x) or (z^2, yz, y^3, x). However the first possibility forces $m < n$ which contradicts our assumption. Hence $(p, x) = (z^2, yz, y^3, x)$ can be assumed. Therefore the defining equations of p must be of the form (see [11])

$$\begin{aligned} & x^{p+q} - yz \\ (3.15) \qquad & y^3 - zx^q \\ & z^2 - y^2x^p \end{aligned}$$

which are the 2×2 minors of

$$\begin{pmatrix} z & y^2 & x^q \\ x^p & z & y \end{pmatrix}.$$

In this case $n = p + 2q$, $m = 2q + 3p$, and p must be odd else $(4, n, m) \neq 1$.

Choose $k \geq 0$ such that $kp < q \leq (k+1)p$ provided $q \geq p$. (We will treat $q<p$ at the end—this is an easy case.) By induction on j we claim there is an element $e_j \in p^{(j)}$ such that

$$e_j \equiv y^{2j+1} + (-1)^j x^{q-(j-1)p} z^{2j-1} \bmod (x^{q-(j-2)p}) \tag{3.16}$$

for $1 \leq j \leq k+1$.

For $j=1$, we have that $e_1 = y^3 - zx^q \in p$, which satisfies (3.16). Suppose e_j has been constructed and $j<k$. Then

$$e_j(z^2 - y^2x^p) \equiv y^{2j+1}z^2 + (-1)^j x^{q-(j-1)p} z^{2j+1} - y^{2j+3}x^p \bmod (x^{q-(j-2)p}).$$

(Note here that $p < q-(j-2)p$.) On the other hand,

$$(x^{p+q} - zy)^2(e_{j-1}) \equiv$$
$$(x^{2p+2q} - 2x^{p+q}zy + z^2y^2)(y^{2j-1} + (-1)^{j-1}x^{q-(j-2)p}z^{2j-3}) \bmod (x^{q-(j-3)p})$$

so

$$(x^{p+q} - zy)^2(e_{j-1}) \equiv y^{2j+1}z^2 + (-1)^{j-1}x^{q-(j-2)p}z^{2j+1}y^2 \bmod (x^{q-(j-3)p})$$
$$\equiv y^{2j+1}z^2 \bmod (x^{q-(j-2)p}).$$

Hence if we set

$$t = -e_j(z^2 - y^2x^p) + (x^{p+q} - zy)^2 e_{j-1} \tag{3.18}$$

then

$$t \equiv y^{2j+3}x^p + (-1)^{j+1}x^{q-(j-1)p}z^{2j+1} \bmod (x^{q-(j-2)p}).$$

As $t \in p^{(j+1)}$ it follows that if we define $e_{j+1} = t/x^p$ then $e_{j+1} \in p^{(j+1)}$ and

$$e_{j+1} \equiv y^{2j+3} + (-1)^{j+1}x^{q-jp}z^{2j+1} \bmod (x^{q-(j-1)p}),$$

which finishes the induction.

For $j=k+1$ we obtain an element $e_{k+1} \in p^{(k+1)}$ such that

$$e_{k+1} \equiv y^{2k+3} + (-1)^{k+1}x^{q-kp}z^{2k+1} \bmod (x^{q-(k-1)p}).$$

Now we apply the process as in (3.18) to obtain that

$$\begin{aligned} &-e_{k+1}(z^2 - y^2x^p) + (x^{p+q} - zy)^2 e_k \\ &\equiv y^{2k+5}x^p + (-1)^{k+2}x^{q-kp}z^{2k+3} \bmod (x^{q-(k-1)p}). \end{aligned} \tag{3.20}$$

However now $p \geq q - kp$. We consider two cases.

Case 1. $p = q - kp$. In this case we find an element e_{k+2} in $p^{(k+2)}$ such that

$$e_{k+2} \equiv y^{2k+5} + (-1)^{k+2}z^{2k+3} \bmod (x).$$

Also, $zy \in (p, x)$. Since

$$\lambda(R/(zy, e_{k+2}, x)) = (2k+5) + (2k+3) = 4(k+2) = 4(k+2)\cdot 1,$$

applying Theorem 3.1 gives the assertion.

Case 2. $p>q-kp$. In this case we find from (3.20) an element $e_{k+2}\in p^{(k+2)}$ such that

$$e_{k+2}\equiv y^{2k+5}x^{(k+1)p-q}+(-1)^{k+2}z^{2k+3} \bmod (x^p).$$

Hence, mod (x^p),

$$e_{k+1}e_{k+2}\equiv(-1)^{k+2}z^{2k+3}y^{2k+e}+y^{4k+8}x^{(k+1)p-q}z^{4k+4}x^{q-kp}. \tag{3.21}$$

Thus

$$e_{k+1}e_{k+2}+(x^{p+q}-zy)^{2k+3}(-1)^k\equiv y^{4k+8}x^{(k+1)p-q}-z^{4k+4}x^{q-kp} \bmod (x^p).$$

Depending on whether $q-kp\leq(k+1)p-q$ or not we obtain that in $p^{(2k+3)}$ there is an element f such that either

$$f\equiv y^{4k+8} \bmod (x), \tag{3.22}$$

$$f\equiv z^{4k+4} \bmod (x), \quad \text{or} \tag{3.23}$$

$$f\equiv y^{4k+8}-z^{4k+4} \bmod (x). \tag{3.24}$$

We do each case separately.

In the case of (3.22),

$$\lambda(R/f,e_{k+2},x)=(4k+8)(2k+3)=4(2k+2)(2k+3).$$

Since $f\in p^{(2k+3)}$ and $e_{k+2}\in p^{(k+2)}$, we are done by using Theorem 3.1.

In the case of (3.23),

$$\lambda(R/(f,e_{k+1},x))=(4k+4)(2k+3)=4(k+1)(2k+3).$$

Again, apply Theorem 3.1.

In the case of (3.24),

$$\lambda(R/(zy,f,x))=(4k+8)+(4k+4)=4(2k+3).$$

Since $zy\in(p,x)$, applying Theorem 3.1 finishes.

Finally we do the case where $q<p$. In this case we proceed exactly as in case 2 above with $k=0$. The proof obtained goes through without change. □

We end this section by proving a theorem which shows the difference between $\bigoplus_{n\geq 0}p^{(n)}$ being Noetherian and p being a set-theoretic complete intersection.

THEOREM 3.25. *Let R be a* 3*-dimensional regular local ring with infinite residue field and let p be a height-*2 *prime ideal of R. Then the following are equivalent.*

(i) $\bigoplus_{n\geq 0}p^{(n)}$ *is Noetherian.*

(ii) *There exist $f,g\in p$ such that $\sqrt{(f,g)}=p$ and* $\operatorname{ht}(f^*,g^*)=2$, *where f^* and g^* are the leading forms of $f/1$ and $g/1$ in* $\operatorname{gr}_{p_p}(R_p)=\bigoplus_{n\geq 0}p_p^n/p_p^{n+1}$.

Proof. Assume (i). Then Cowsik [4] showed that $\ell(p^{(k)})=2$ for some k. Hence $p^{(k)}$ is integral over (f,g) for some f and g in $p^{(k)}$. Hence $\sqrt{(f,g)}=p$. In R_p, $f/1$ and $g/1$ are a reduction of p_p^k, so $p_p^{kn}=(f,g)p_p^{kn-k}$. It follows that in $G=\operatorname{gr}_{p_p}(R_p)$ the leading forms of f and g must be relatively prime, which proves (ii).

Next assume (ii). Let r be the degree of the leading form of $f/1$ and s be the degree of the leading form of $g/1$ in $G=\mathrm{gr}_{p_p}(R_p)$. Then f^s and g^r have leading forms $(f^*)^s$ and $(g^*)^r$ of degree rs in G, and $(f^*)^s$ and $(g^*)^r$ are relatively prime. We claim that $(f^s,g^r)R_p=p_p^{rs}$. This follows at once from the fact that $(f^s)^*$ and $(g^r)^*$ are relatively prime of degree rs, together with Proposition 3, Appendix 5 of Zariski and Samuel [33].

Now consider $I=\overline{(f^s,g^r)}$ in R. We claim $I=p^{(k)}$. As $\ell(I)=2$ and $\bar{I}=I$, by McAdam [19, Proposition 4.1] $\mathrm{Ass}(R/I)$ all have height 2. Since $\sqrt{I}=\sqrt{(f,g)}=p$ it follows that $\mathrm{Ass}(R/I)=\{p\}$. Thus it suffices to show $I_p=p_p^{(k)}=p_p^k$. However, $I_p=(\overline{(f^s,g^r)})_p=\overline{(f^s,g^r)R_p}=p_p^k$. Hence $I=p^{(k)}$ and so $\ell(p^{(k)})=2$. Since $p^{(k)}$ is integrally closed and of analytic spread 2, and since R is regular, [9, Theorem 4.1] shows that $p^{(k)n}$ are integrally closed for all $n\geq 1$. Since $\ell(p^{(k)n})=2$, by McAdam [19, Proposition 4.1] it follows that $p^{(k)n}$ is unmixed for $n\geq 1$, and so $p^{(k)n}=p^{(kn)}$ for all $n\geq 1$. This is equivalent to $\bigoplus_{n\geq 0}p^{(n)}$ being Noetherian. (See [4].)

4. Integral closures and Hilbert functions. In this section we apply the same method as in Section 2 to study the Hilbert function $\bar{H}_I(n)=\lambda(R/\overline{I^n})$. Practically everything done in Section 2 goes through and even more since the formula $\overline{I^n}:(x,y)=\overline{I^{n-1}}$ is automatically satisfied.

We begin by doing the fundamental lemma for this case. Throughout we assume that R is a 2-dimensional C–M analytically unramified local ring with infinite residue field.

Accordingly, we let $\bar{P}_I(n)$ be the Hilbert polynomial such that $\bar{P}_I(n)=\lambda(R/\overline{I^n})$ for large n. Rees [25] proved such a polynomial exists. Let $\bar{H}_I(n)=\lambda(R/\overline{I^n})$ for every $n\geq 0$. If (x,y) is a reduction of I then $\overline{I^n}:(x,y)=\overline{I^{n-1}}$. (Check this on valuations.) The same proof as in Lemma 2.4 now shows the following.

LEMMA 4.1. *Let (R,m) be a 2-dimensional local C–M analytically unramified ring and I an m-primary ideal with minimal reduction (x,y). Then*

$$\lambda(\overline{I^{n+1}}/(x,y)\overline{I^n})=[\bar{P}_I(n+1)-\bar{H}_I(n+1)]$$
$$+[\bar{P}_I(n-1)-\bar{H}_I(n-1)]-2[\bar{P}_I(n)-\bar{H}_I(n)].$$

Proof. We briefly recall the proof. Use the exact sequence

$$0\to K\to(R/\overline{I^n})^2\to(x,y)/(x,y)\overline{I^n}\to 0$$

and we find that $K\simeq R/(\overline{I^n}:(x,y))=R/\overline{I^{n-1}}$. As in (2.1), $\lambda(R/(x,y))=\bar{e}_0$, where we write

$$\lambda(R/\overline{I^n})=\bar{e}_0\binom{n+1}{2}-\bar{e}_1 n+\bar{e}_2.$$

The rest follows as in (2.1). □

From Lemma 4.1 (just as from Lemma 2.1) we obtain the following remarks, where we set $\bar{v}_n=\lambda(\overline{I^{n+1}}/(x,y)\overline{I^n})$.

REMARK 4.2.

$$\lambda(R/\bar{I})-[\bar{e}_0-\bar{e}_1]=\sum_{n=1}^{\infty}\bar{v}_n.$$

REMARK 4.3.

$$\bar{e}_2 = \sum_{n=1}^{\infty} n\bar{v}_n.$$

Rees [26] obtained essentially these equations using a proof upon which the fundamental lemma is based.

Similarly, the same proof as in Theorem 2.11 shows the following.

THEOREM 4.4. *Let (R, m) be a 2-dimensional local C–M analytically unramified ring and $(x, y) = I$ an m-primary ideal. Then the following are equivalent.*

(i) $\bar{H}_I(n) = \bar{P}_I(n)$ *for all* $n \geq 1$.

(ii) $\overline{I^n} = (x, y)\overline{I^{n-1}}$ *for all* $n \geq 3$.

Proof. Again we briefly summarize the argument. Set $\bar{u}_n = \bar{P}_I(n) - \bar{H}_I(n)$. Now assume (i). Then

$$\lambda(\overline{I^{n+1}}/(x, y)\overline{I^n}) = \bar{u}_{n+1} + \bar{u}_{n-1} - 2\bar{u}_u = 0,$$

provided $n - 1 \geq 1$ (i.e., $n \geq 2$). Hence $\overline{I^n} = (x, y)\overline{I^{n-1}}$ for all $n \geq 3$.

Next assume (ii). Then, again by Lemma 4.1, $\bar{u}_{n+1} + \bar{u}_{n-1} - 2\bar{u}_n = 0$ for $n \geq 2$. Since $\bar{u}_N = 0$ for $N \gg 0$, it follows that $\bar{u}_1 = \bar{u}_2 = \cdots = \bar{u}_N = 0$, which is (i). □

Next we summarize the theory of the coefficients of the Hilbert polynomial $\bar{P}_I(n)$ and the difference between $\bar{P}_I(n)$ and $\bar{H}_I(n)$.

As usual, let $I \subset R$, R a 2-dimensional C–M analytically unramified local ring and I an m-primary ideal with minimal reduction (x, y). Write

$$\bar{P}_I(n) = \bar{e}_0\binom{n+1}{2} - \bar{e}_1 n + \bar{e}_2,$$

and let $\bar{u}_i = [\bar{P}_I(i) - \bar{H}_I(i)]$.

THEOREM 4.5 (cf. [20, Proposition 2.1]).

(i) $\bar{e}_2 \geq 0$.

(ii) $\lambda(R/\bar{I}) \geq \bar{e}_0 - \bar{e}_1$. $\lambda(R/\bar{I}) = \bar{e}_0 - \bar{e}_1$ *if and only if* $\bar{e}_2 = 0$.

(iii) $\bar{u}_i > \bar{u}_j$ *if* $i < j$ *unless* $\bar{u}_i = 0$. *If* $\bar{u}_i = 0$ *then* $\bar{u}_j = 0$ *for* $j \geq i$.

(iv) *The least n such that* $\overline{I^{n+1}} = \overline{I^n}(x, y)$ *is independent of the minimal reduction* (x, y).

Proof. Obviously $\bar{v}_n \geq 0$ for all n. Hence (i) and the first statement of (ii) follow immediately from Remarks 4.2 and 4.3. If $\lambda(R/\bar{I}) = \bar{e}_0 - \bar{e}_1$, then Remark 4.2 coupled with $\bar{v}_n \geq 0$ implies that $\bar{v}_n = 0$ for every $n \geq 1$. Then by Remark 4.3 we must have $\bar{e}_2 = 0$. Conversely, $\bar{e}_2 = 0$ implies $\bar{v}_n = 0$ for $n \geq 1$, which implies that $\lambda(R/\bar{I}) = \bar{e}_0 - \bar{e}_1$.

Since $\bar{u}_{n+1} + \bar{u}_{n-1} - 2\bar{u}_n \geq 0$ for every $n \geq 1$, we conclude if $\bar{u}_i \leq \bar{u}_{i+1}$ then (since $2\bar{u}_{i+1} \leq \bar{u}_{i+2} + \bar{u}_i \leq \bar{u}_{i+2} + \bar{u}_{i+1}$) we have $\bar{u}_{i+1} \leq \bar{u}_{i+2}$, with equality if and only if $\bar{u}_i = \bar{u}_{i+1}$. It follows that unless $\bar{u}_i = \bar{u}_{i+1} = \cdots = 0$ we must have $\bar{u}_i > \bar{u}_{i+1}$, which is the content of (iii).

Finally, (iv) is clear from the lemma since the right-hand side of the main formula does not depend on the reduction (x, y). □

Morales [20] showed the intriguing theorem that if $\bar{H}_I(n)=\bar{P}_I(n)$ for all $n\geq 1$ then $\bigoplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ is C–M in the case that R is a germ of a normal surface over $\mathbf{C}$. We will prove this in characteristic p.

In the context of Theorems 2.11 and 4.4 we identify what more is needed to conclude that either $\bigoplus_{n\geq 0} I^n/I^{n+1}$ or $\bigoplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ is C–M.

THEOREM 4.6. *Let* (R,m) *be a 2-dimensional C–M (and analytically unramified for* (ii)) *local ring with infinite residue field, and let I be an m-primary ideal with minimal reduction* (x,y).

(i) *If* $H_I(n)=P_I(n)$ *for* $n\geq 1$ *and* $I=\tilde{I}$ *then* $\mathrm{gr}_I(R)$ *is C–M if and only if* $I^2\cap(x,y)=(x,y)I$.

(ii) *If* $\bar{H}_I(n)=\bar{P}_I(n)$ *for* $n\geq 1$ *then* $\bigoplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ *is C–M if and only if* $\overline{I^2}\cap(x,y)=(x,y)\bar{I}$.

Proof. First we prove (i). Let u and v be the images of x and y in $I/I^2\subseteq \mathrm{gr}_I(R)=G$. We claim u and v are a G-sequence. By Theorem 2.11 we have that $I^{n+1}=I^n(x,y)$ for $n\geq 2$. Assume $I^2\cap(x,y)=(x,y)I$.

Suppose u is a zero divisor. Then there is an element $f\in I^n/I^{n+1}$ for some $n\geq 0$ such that $f\notin I^{n+1}$ and $f\cdot x\in I^{n+2}$. If $n=0$ then $f\notin I$, but $f\cdot x\in I^2$. Hence $fx\in I^2\cap(x,y)=I(x,y)$, so we may write $fx=xi_1+yi_2$ with $i_1,i_2\in I$. Then $x(f-i_1)=yi_2$ shows that $f-i_1\in(y)$ or $f\in I$, which contradicts our assumptions. Hence $n\geq 1$. Then $n+2\geq 3$ so $I^{n+2}=I^{n+1}(x,y)$ or $fx=xi_1+yi_2$, where $i_1,i_2\in I^{n+1}$. Then $x(f-i_1)=yi_2$ so there is a $t\in R$ with $xt=i_2$ and $f-i_1=yt$. We may assume f has been chosen of least degree. Then $xt\in I^{n+1}$ implies $t\in I^n$, and hence $f=i_1+yt\in I^{n+1}$, a contradiction. Thus u (and similarly v) are not zero divisors in G.

Now suppose there is a relation $r^*u+s^*v=0$ in G. We may assume r^* and s^* are homogeneous of the same degree (otherwise u or v would be a zero divisor).

Thus we may assume $r,s\in I^k\setminus I^{k+1}$ and $rx+sy\in I^{k+2}$, where r,s are liftings of r^* and s^* to R. We also assume k is chosen least so that there exist such elements with $s^*\notin(x^*)$. If $k=0$ then $rx+sy\in I^2\cap(x,y)=(x,y)I$, while if $k\geq 1$ then $I^{k+2}=I^{k+1}(x,y)$. In either case $rx+sy\in(x,y)I^{k+1}$, so that we may write $rx+sy=ax+by$ with $a,b\in I^{k+1}$. Thus $x(r-a)=y(b-s)$ and so there is a $t\in R$ with $xt=b-s$ and $yt=r-a$. Hence $xt\in I^k$ and $yt\in I^k$. Since x^* and y^* are not zero divisors in G, t must be in I^{k-1}. Hence in G we may write $x^*t^*=-s^*$ and $y^*t^*=r^*$; thus $s^*\in(x^*)$, which contradicts our assumption.

Conversely if G is C–M and if $rx+sy\in I^2$, then in G either $r^*x^*+x^*y^*=0$, $r^*x^*=0$, or $s^*y^*=0$. In any case r and s must be in I so that $rx+sy\in(x,y)I$.

The proof of (ii) is exactly the same, although slightly easier as in this case the images of x and y in $\bigoplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ are necessarily nonzero divisors without any assumptions. □

In light of the theorem of Morales in case (ii), $\bar{H}_I(n)=\bar{P}_I(n)$ should force $\overline{I^2}\cap(x,y)=(x,y)\bar{I}$. Since $\bar{H}_I(n)=\bar{P}_I(n)$ is equivalent to $\overline{I^n}=(x,y)\overline{I^{n-1}}$ for $n\geq 3$, one is led to suspect $\overline{I^2}\cap(x,y)=(x,y)\bar{I}$ with *no* assumptions on $\bar{H}_I(n)$. The next theorem proves this is the case if char $R=p>0$ and R is C–M (of arbitrary dimension). However we first recall the pseudo-valuation $\bar{v}_I(x)$ studied by Rees [27].

If I is an ideal and $x \in R$, set $v_I(x) = \sup\{n \mid x \in I^n\}$ and $\bar{v}_I(x) = \lim_{n\to\infty} v_I(x^n)/n$. This limit exists and is a rational number (see [27] and [19, Chapter 11]). Furthermore, when R is Noetherian $\bar{v}_I(x) \geq k$ if and only if $x \in \overline{I^k}$.

THEOREM 4.7. *Let R, m be a d-dimensional C–M local ring of characteristic $p > 0$. Let $x_1, \ldots, x_d$ be any system of parameters and set $I = (x_1, \ldots, x_d)$. Then*

$$\overline{I^n} \cap I^{n-1} = I^{n-1}\bar{I} \quad \textit{for all} \quad n \geq 2.$$

Proof. We will write x^α for $x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$. Suppose

$$t = \sum r_\alpha x^\alpha \in \overline{I^n},$$

where the sum ranges over all α with

$$|\alpha| = \sum_{i=1}^{d} \alpha_i = n-1.$$

Then $\bar{v}_I(t) \geq n$, so that $\lim_{m\to\infty} v_I(t^m)/m \geq n$. Set $v_I(t^{p^m}) = k_m$. Then

$$\lim_{m\to\infty} k_m/p^m \geq n.$$

We have that $t^{p^m} \in I^{k_m}$, so that

$$\sum r_\alpha^{p^m} x^{\alpha p^m} \in I^{k_m}.$$

We may assume m is chosen so that $k_m > p^m(n-1)$. As $x_1, \ldots, x_d$ is a regular sequence, all the relations on monomials in $x_1, \ldots, x_d$ are generated by the "mixed" Koszul relations: that is, if m_1 and m_2 are monomials in $x_1, \ldots, x_d$ (say, $m_1 = x_1^{i_1} \ldots x_d^{i_d}$ and $m_2 = x_1^{j_1} \ldots x_d^{j_d}$) then the "mixed" Koszul relation determined by m_1 and m_2 is $x_1^{f_1} \ldots x_d^{f_d} m_1 = x_1^{e_1} \ldots x_d^{e_d} m_2$, where

$$f_q = \max(i_q, j_q) - i_q, \quad e_q = \max(i_q, j_q) - j_q.$$

From (4.8) we see that $r_i^{p^m}$ is in the ideal generated by the mixed Koszul coefficients of $x^{\alpha p^m}$ with $\{x^{\beta p^m} \mid \beta \neq \alpha,\ |\beta| = n-1\}$ and all monomials of degree k_m. Hence

$$r_\alpha^{p^m} \in (x_1^{p^m}, \ldots, x_d^{p^m}) + (x_1, \ldots, x_d)^{k_m - p^m(n-1)}$$

so that $v_I(r_\alpha^{p^m}) \geq \min(p^m, k_m - p^m(n-1))$ and

$$\lim_{m\to\infty} v_I(r_\alpha^{p^m})/p^m \geq \min(1, \lim_{m\to\infty} (k_m - p^m(n-1))/p^m) = 1.$$

Therefore $\bar{v}_I(r_\alpha) \geq 1$ and so $r_\alpha \in \bar{I}$ for every α. □

COROLLARY 4.8. *Let R, m be a 2-dimensional C–M analytically unramified local ring of characteristic $p > 0$, and let I be a m-primary ideal. Assume $\bar{P}_I(n) = \bar{H}_I(n)$ for all $n \geq 1$. Then $\bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$ is C–M.*

Proof. We may assume R/m is infinite. Now the corollary follows at once from Theorem 4.7 and Theorem 4.6(ii). □

Finally, we apply the same techniques as in Theorem 4.7 to study F-pure rings. Recall if R is a local reduced ring of characteristic $p > 0$ then R is said to be F-pure if the Frobenius map is pure. In particular this implies that if we set $R^p = \{r^p \mid r \in R\}$ then, for $i_1, \ldots, i_n \in R$, $(i_1^p, \ldots, i_n^p)R \cap R^p = (i_1^p, \ldots, i_n^p)R^p$.

Hochster and Lipman [unpublished] showed if R is C–M and F-pure and if $x_1, \ldots, x_d$ is a system of parameters of R then $r^{d+1} \in (x_1, \ldots, x_d)$ if $r \in \overline{(x_1, \ldots, x_d)}$. (In fact they prove this even for F-pure type.) In Proposition 4.9 we will strengthen this.

If R is regular then in [17] and [16] algebraic proofs were given of the stronger result that $\overline{(x_1, \ldots, x_d)^d} \subseteq (x_1, \ldots, x_d)$. If R is 2-dimensional regular and contains a field then this containment together with Theorem 4.7 and the theorem of the appendix show that $\overline{(x_1, x_2)^2} \subseteq (x_1, x_2)$ so that $\overline{(x_1, x_2)^2} \cap (x_1, x_2) = \overline{(x_1, x_2)^2}$, but also this intersection is $(x_1, x_2)\overline{(x_1, x_2)}$. Hence $(x_1, x_2)\overline{(x_1, x_2)} = \overline{(x_1, x_2)^2}$, a result known and proved in [17] and [26] even for pseudo-rational singularities of dimension two.

PROPOSITION 4.9. *Let R, m be a C–M reduced d-dimensional local ring of characteristic $p > 0$ which is F-pure. If $x_1, \ldots, x_d$ is an s.o.p. for R and $I = (x_1, \ldots, x_d)$, then $\overline{I^{d+1}} \subseteq I$.*

Proof. Let $r \in \overline{I^{d+1}}$. Then $v_I(r) \geq d+1$. Choose $n \gg 0$ so that $v_I(r^n) \geq dn$. Then $r^n \in (x_1, \ldots, x_d)^{dn} \subseteq (x_1^n, \ldots, x_d^n)$. Choose m so that $p^m \geq n$. Then

$$r^{p^m} \in (x_1^{p^m}, \ldots, x_d^{p^m})R \cap R^{p^m} = (x_1^{p^m}, \ldots, x_d^{p^m})R^{p^m}$$

so that $r^{p^m} = \sum_{i=1}^d y_i^{p^m} x_i^{p^m}$. Then $(r - \sum_{i=1}^d y_i x_i)^{p^m} = 0$ and so $r = \sum_{i=1}^d y_i x_i \subseteq I$. □

Appendix. In this appendix we present a theorem, extending Theorem 4.7 to arbitrary characteristics, whose proof was shown to me by M. Hochster. The proof is the "standard" reduction to characteristic p; however, although standard I found it by no means obvious. Any mistakes or unclearness are due to this author. We prove the following.

THEOREM. *Let (R, m) be a d-dimensional C–M local ring containing* **Q**, *and let I be an m-primary ideal generated by $x_1, \ldots, x_d$. Then $\overline{I^n} \cap I^{n-1} = I^{n-1}\overline{I}$ for $n \geq 2$.*

REMARK. D. Rees communicated that he proved this theorem if $d = n = 2$ without assumption on the characteristic of R, using his theory of degree functions.

Proof of Theorem. First we replace R by $\hat{R}$, the completion of R. We claim that $\overline{I}\hat{R} = (I\hat{R})^-$ for any m-primary ideal I of R. For choose $N \gg 0$ so that $m^N \subset I$ and suppose $u \in (I\hat{R})^-$. Then in $\hat{R}$ there is an equation,

$$u^n + r_1 u^{n-1} + \cdots + r_n = 0,$$

with $r_i \in I^i\hat{R}$. Choose $\tilde{u}, \tilde{r}_1, \ldots, \tilde{r}_n \in R$ such that $u \equiv \tilde{u}$, $r_i \equiv \tilde{r}_i \mod m^{nN}\hat{R}$. Then $(\tilde{u})^n + \tilde{r}_1(\tilde{u})^{n-1} + \cdots + \tilde{r}_n = u^n + r_1 u^{n-1} + \cdots + r_n \mod m^{nN}\hat{R}$ so that

$$t = (\tilde{u})^n + \tilde{r}_1(\tilde{u})^{n-1} + \cdots + \tilde{r}_n \in m^{nN}\hat{R} \cap R = m^{nN}.$$

Thus $(\tilde{u})^n + \tilde{r}_1(\tilde{u})^{n-1} + \cdots + \tilde{r}_n - t = 0$ and $\tilde{r}_n - t \in I^n$, $\tilde{r}_i \in I^i$ (since $\tilde{r}_i - r_i \in m^{nN} \subseteq I^n$ implies $\tilde{r}_i \in I^i\hat{R} \cap R = I^i$) so that $\tilde{u} \in \overline{I}$. Now $u - \tilde{u} \in m^{nN}\hat{R} \subseteq I\hat{R}$ so $u \in \overline{I}\hat{R}$ as claimed.

Now suppose the theorem is false. Choose $u \in \overline{I^n} \cap I^{n-1}$, $u \notin I^{n-1}\bar{I}$. After passing to $\hat{R}$, $u \in (I^n\hat{R})^- \cap I^{n-1}\hat{R}$, and since $I^{n-1}(I\hat{R})^- = (I^{n-1}\bar{I})\hat{R}$ it follows that $u \notin (I^{n-1}\hat{R})(I\hat{R})^-$. Thus we assume R is complete. Let k be a coefficient field of R and set $A = k[[x_1, \ldots, x_d]] \subset R$. Then A is a regular local ring and R is a finite free A-module (as R is C–M). We claim we may replace k be $\bar{k}$, its algebraic closure. Let $A' = \bar{k}[[x_1, \ldots, x_d]]$, $R' = A' \otimes_A R$. By [12, (vi), (viii), and Lemma 2.4], if J is an ideal of R then $\bar{J}R' = (JR')^-$. Also $R \to R'$ is flat as A' is flat over A, so that $JR' \cap R = J$ if J is an ideal of R. It follows as above that we may assume k is algebraically closed, and we go back to our original notation.

Choose an A-basis $e_1, \ldots, e_m$ of R where we may assume $e_1 = 1$ and $e_2, \ldots, e_m \in m$, where m is the maximal ideal of R. (This is possible as $R/m \simeq A/M$, where $M = (x_1, \ldots, x_d)$ is the maximal ideal of A.) We may uniquely write

$$(A.1) \qquad e_i e_j = \sum_{k=1}^{m} r_{ijk} e_k \quad \text{with } r_{ijk} \in A.$$

We refer to $\{r_{ijk}\}$ as the "structure constants" of R over A. Since $Me_1 + Ae_2 + \cdots + Ae_m$ is the unique maximal ideal of R, we see that

$$(A.2) \qquad r_{ij1} \in M \quad \text{if } i \text{ or } j \neq 1,$$

and, since $e_1 = 1$, it follows that

$$(A.3) \qquad r_{1ik} = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

The commutative law shows that $r_{ijk} = r_{jik}$ while the associative law translates into quadratic equations in the r_{ijk} with ± 1 as coefficients. We fix these equations $F_1, \ldots, F_t$ where we put variables X_{ijk} in place of r_{ijk} with the convention that $X_{ijk} = X_{jik}$, and in place of X_{1ik} we put 1 if $k = i$ and 0 if $k \neq i$. For future use we note that if A is any commutative ring and $a_{ijk} \in A$ satisfying (A.2), (A.3), and the equations $F_1, \ldots, F_t$, then we may construct a commutative A-algebra $C = Ae_1' \oplus \cdots \oplus Ae_m'$, free as an A-module, by *defining* $e_i' \cdot e_j' = \sum_{k=1}^m a_{ijk} e_k'$ and extending this linearly.

Next consider the element $u \in R$ which is chosen so that $u \in \overline{I^n} \cap I^{n-1}$, $u \notin I^{n-1}\bar{I}$. Write

$$(A.4) \qquad u = \sum_{i=1}^{m} u_i e_i, \quad u_i \in A.$$

Since $u \in I^{n-1}$ it follows that $u_i \in M^{n-1}$. As $u \in \overline{I^n}$ there is an equation

$$u^p + b_1 u^{p-1} + \cdots + b_p = 0$$

with $b_i \in I^{ni}$. Write $b_i = \sum_{j=1}^m b_{ij} e_j$ with $b_{ij} \in M^{ni}$.

Since $u_i \in M^{n-1}$ we may write $u_i = \sum_{|\alpha| = n-1} u_{i\alpha} x^\alpha$, where $x^\alpha = x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}$. Given any elements $z = \sum_{i=1}^m (\sum_{|\alpha|=n-1} z_{i\alpha} x^\alpha) e_i$ and $b_j = \sum_{k=1}^m b_{jk} e_k$ with $z_{i\alpha}, b_{jk} \in A$, we may expand the expression

$$z^p + b_1 z^{p-1} + \cdots + b_p$$

using (A.1) to write it as $\sum_{\ell=1}^{m} G_\ell(z_{i\alpha}, b_{ij}, r_{ijk})e_\ell$. Let $G_\ell(Z_{i\alpha}, B_{jk}, R_{ijk})$ be the polynomial (with coefficients in $Z[x_1, \dots, x_d]$) obtained in the obvious way from the expression $G_\ell(z_{i\alpha}, b_{ij}, r_{ijk})$. Consider the system of equations

$$\text{(A.5)}\qquad \begin{aligned} F_1(R_{ijk}) = \cdots = F_t(R_{ijk}) = 0,\\ G_1(Z_{i\alpha}, B_{ij}, R_{ijk}) = \cdots = G_m(Z_{i\alpha}, B_{ij}, R_{ijk}) = 0,\\ R_{ijk} - R_{jik} = 0, \quad \text{and}\\ R_{1ik} - \delta_{ik} = 0. \end{aligned}$$

We have shown that $r_{ijk} = R_{ijk}$, $b_{ij} = B_{ij}$, and $u_{i\alpha} = Z_{i\alpha}$ satisfy these equations and (of course) $r_{ijk}, b_{ij}, u_{i\alpha} \in A = k[[x_1, \dots, x_d]]$.

By the Artin Approximation Theorem [1, Theorem 1.10] we may find solutions $\tilde{u}_{i\alpha}, \tilde{r}_{ijk}, \tilde{b}_{ij}$ to (A.5) in some etale neighborhood B of $D = k[X_1, \dots, X_d]_{(X_1, \dots, X_d)}$ (actually in D^h, the hensilization of D, but D^h is the direct limit of etale neighborhoods of D) such that the new solutions approximate our original solutions to any fixed power of M. In particular, choose such solutions such that $u_{i\alpha} \equiv \tilde{u}_{i\alpha}$, $r_{ijk} \equiv \tilde{r}_{ijk}$, and $b_{ij} \equiv \tilde{b}_{ij} \bmod M^\delta$ where $\delta = m + pn$.

Use $\tilde{r}_{ijk}$ as new structure constants to construct an algebra $S = Bf_1 \oplus \cdots \oplus Bf_m$, free of rank m as a B-module. Since $\tilde{r}_{ijk}$ satisfy $F_1, \dots, F_t$, $\tilde{r}_{ijk} = \tilde{r}_{jik}$, and $\tilde{r}_{1jk} = \delta_{jk}$, we see that S is a commutative B-algebra and f_1 is the unit. Consider the ideal $J = (x_1, \dots, x_d)S$. We claim that the element $\tilde{u} = \sum_{i=1}^{m} (\sum_\alpha \tilde{u}_{i\alpha} x^\alpha) f_i$ satisfies $\tilde{u} \in \overline{J^n} \cap J^{n-1}$, $\tilde{u} \notin J^{n-1}\bar{J}$. First we prove that $\tilde{u} \in \overline{J^n} \cap J^{n-1}$. Obviously $\tilde{u} \in J^{n-1}$. Since $\tilde{b}_{ij} - b_{ij} \in M^\delta \subseteq M^{ni}$, $\tilde{b}_{ij} \in M^{ni} \cap B = m_B^{ni}$. Hence if we let $\tilde{b}_i = \sum_{j=1}^{m} \tilde{b}_{ij} f_j$ then $\tilde{b}_i \in J^{ni}$, and furthermore $\tilde{u}$ satisfies the equation $(\tilde{u})^p + \tilde{b}_1(\tilde{u})^{p-1} + \cdots + \tilde{b}_p = 0$ as $\tilde{b}_{ij}$, $\tilde{r}_{ijk}$ and $\tilde{u}_{i\alpha}$ satisfy $G_1, \dots, G_m$. Thus $\tilde{u} \in \overline{J^n}$. It remains to see that $\tilde{u} \notin J^{n-1}\bar{J}$. To show this we require a lemma which is interesting in its own right.

LEMMA. *Let (R, m) be a regular local ring with quotient field K and let S be a torsion-free R-algebra, finitely generated as an R-module. Let $I = mS$. Suppose $y \in \overline{I^n}$. Then y satisfies an integral equation $y^p + a_1 y^{p-1} + \cdots + a_p = 0$ with $a_i \in I^{ni}$ where $p = \dim_K(K \otimes_R S)$.*

Proof. Let V be the m-adic prime divisor of R. Thus V is a DVR with uniformizing parameter t $(= x_1$, say) and $t^i V \cap R = m^i$. We know there is a polynomial,

$$y^p + a_1 y^{p-1} + \cdots + a_p = 0 \quad \text{with } a_i \in m^{in} S,$$

so we may write

$$y^p + t^n a_1' y^{p-1} + \cdots + t^{np} a_p' = 0$$

with $a_i' \in S[m/t]$. Thus, y/t^n is integral over $S[m/t]$ and hence over $R[m/t]$ (since S is integral over R) and finally over V as $V = R[m/t]_{(t)}$. Now consider the minimal polynomial $p(x)$ of y over K = quotient field of R. Since y is integral over R, the coefficients of $p(X)$ all lie in R [2]. Over $\bar{K}$, the algebraic closure of K, write $p(x) = \prod_{i=1}^{\ell} (X - \lambda_i)$. Note that $\ell \leq \dim_K(S \otimes_R K)$ (since y can be considered in $\mathrm{End}_K(S \otimes_R K)$). Let $f(X)$ be the monic polynomial giving the equation of integrality of y/t^n over V; write $f(X) = X^N + c_1 X^{N-1} + \cdots + c_N$, $c_i \in V$, and let

$g(X)=X^N+c_1t^nX^{N-1}+\cdots+c_Nt^{nN}$ so that $g(y)=0$. Hence over $K[X]$, $p(X)$ divides $g(X)$ and in particular $g(\lambda_i)=0$ for every $i=1,\ldots,\ell$, so that $f(\lambda_i/t^n)=0$ and hence λ_i/t^n are integral over V. Write $p(X)=X^\ell+p_1X^{\ell-1}+\cdots+p_\ell$. As $p_1,\ldots,p_\ell$ are symmetric functions in $\lambda_1,\ldots,\lambda_\ell$, we see that $p_1/t^n,\ldots,p_\ell/t^{\ell n}$ must all be integral over V and in K. Hence $p_i/t^{in}\in V$ so $p_i\in t^{in}V\cap R=m^{in}$. As $p(y)=0$ we have proved the lemma. □

Now suppose $\tilde{u}\in J^{n-1}\bar{J}$. As $\tilde{u}=\sum_\alpha(\sum_{i=1}^m \tilde{u}_{i\alpha}f_i)x^\alpha$, and since $x_1,\ldots,x_d$ are necessarily a regular S-sequence, it follows that $\sum_{i=1}^m \tilde{u}_{i\alpha}f_i\in\bar{J}$ for each α (as the relations on distinct monomials in $x_1,\ldots,x_d$ are contained in $(x_1,\ldots,x_d)$). On the other hand since $u=\sum_\alpha(\sum_i u_{i\alpha}e_i)x^\alpha$ and $u\notin I^{n-1}\bar{I}$, for some α, $\sum_i u_{i\alpha}e_i\notin\bar{I}$. By the lemma, $\tilde{v}=\sum_i\tilde{u}_{i\alpha}f_i$ satisfies an integral equation over J of degree m (as S is free of rank m over B and B is a regular local ring). Suppose this equation is $(\tilde{v})^m+\tilde{c}_1(\tilde{v})^{m-1}+\cdots+\tilde{c}_m=0$ with $\tilde{c}_i\in J^i$.

Write $\tilde{c}_i=\sum_{j=1}^m c_{ij}f_j$, so that $c_{ij}\in m_B^i$. Using the structure constants $\tilde{r}_{ijk}$ and the expression $\tilde{v}=\sum_i\tilde{u}_{i\alpha}f_i$, expand

$$(\tilde{v})^m+\tilde{c}_1(\tilde{v})^{m-1}+\cdots+\tilde{c}_m$$

to $\sum_{\ell=1}^m H_\ell(\tilde{u}_{i\alpha},\tilde{r}_{ijk},c_{ij})f_\ell$, so that H_ℓ is a polynomial expression in $\tilde{u}_{i\alpha},\tilde{r}_{ijk},c_{ij}$ and $H_\ell(\tilde{u}_{i\alpha},\tilde{r}_{ijk},c_{ij})=0$ for $\ell=1,\ldots,m$.

Expanding in a similar manner $v^m+c_1v^{m-1}+\cdots+c_m$ using (A.1) yields that $v^m+c_1v^{m-1}+\cdots+c_m=\sum_{\ell=1}^m H_\ell(u_{i\alpha},r_{ijk},c_{ij})e_\ell$. (Here $c_i=\sum_{j=1}^m c_{ij}e_j$.) Hence

$$v^m+c_1v^{m-1}+\cdots+c_m\equiv\sum_{\ell=1}^m H_\ell(\tilde{u}_{i\alpha},\tilde{r}_{ijk},c_{ij})e_\ell=0\ (\mathrm{mod}\ M^\delta).$$

Thus $v^m+c_1v^{m-1}+\cdots+c_m\in M^\delta$. Since $M^\delta\subset M^m$ by choice of δ, it follows that v is integral over $MR=I$ since $c_i=\sum_{j=1}^m c_{ij}e_j\in m_B^iR\subset I^i$. This contradicts our choice of v and proves that $\tilde{u}\notin J^{n-1}\bar{J}$.

We now fix our attention on S. S is semilocal and $J\subseteq Jab(S)$. Since $\tilde{u}\notin J^{n-1}\bar{J}$ this must also be true at some maximal ideal n of S. Then S_n is a local d-dimensional C–M ring which is a spot (that is, essentially of finite type) over an algebraically closed field k, $x_1,\ldots,x_d$ in S_n is a s.o.p. generating an ideal J, and there is a $\tilde{u}$ such that $\tilde{u}\in\overline{J^n}\cap J^{n-1}$, $\tilde{u}\notin J^{n-1}\bar{J}$.

Now we start anew. To simplify notation call $\tilde{u}$ just u. There are equations, $u^p+b_1u^{p-1}+\cdots+b_p=0$, $b_i\in J^{ni}$, and $u=\sum_{|\alpha|=n-1}u_\alpha x^\alpha$. Write each b_i as a combination of monomials of degree ni in $x_1,\ldots,x_d$ and collect all the coefficients —call these coefficients $c_1,\ldots,c_N$. Since S_n is a spot over k (and even localized at a maximal ideal of a finitely generated k-algebra), we can find a finitely generated k-algebra R and a maximal ideal m of R so that $S_n=R_m$. By localizing R at a single nonzero element $R\setminus m$ we can obtain a ring $R_f=T$, finitely generated over k such that

(i) $x_1,\ldots,x_d,c_1,\ldots,c_N,u_\alpha\in T$; and

(ii) $\sqrt{(x_1,\ldots,x_d)}=m_1$, the maximal ideal of T corresponding to m.

Hence $u\in T$, and if $I=(x_1,\ldots,x_d)T$ then $u\in\overline{I^n}\cap I^{n-1}$; but $u\notin I^{n-1}\bar{I}$ as $u\notin(I^{n-1}\bar{I})_{m_1}$ even.

Now we proceed to shrink k. Since $T/m_1\simeq k$ (as $k=\bar{k}$) we may find generators $z_1,\ldots,z_e$ for T over k such that $z_1,\ldots,z_e\in m_1$. There exists a homomorphism

$k[Z_1, \dots, Z_e]$ onto T sending Z_i to z_i. Let the kernel be generated by $f_1, \dots, f_r$. Choose a subfield L of k, finitely generated over $\mathbf{Q}$ such that

(i) $f_1, \dots, f_r \in L[Z_1, \dots, Z_e]$,
(ii) $x_1, \dots, x_d, c_1, \dots, c_N, u_\alpha \in L[z_1, \dots, z_e]$, and
(iii) $\sqrt{(x_1, \dots, x_d)} = (z_1, \dots, z_e)$.

(This is possible by writing equations over k showing that z_i are nilpotent over $(x_1, \dots, x_d)$ and adding the field coefficients of these equations to $\mathbf{Q}$.)

Put $A = L[z_1, \dots, z_e]$. Clearly $A \otimes_L k = T$. Thus $(z_1, \dots, z_e)A$ is a maximal ideal of A of height d. Furthermore $x_1, \dots, x_d$ are an A-sequence since they are a T-sequence and T is flat over A. Let $I_1 = (x_1, \dots, x_d)A$. Then $u \in \overline{I_1^n} \cap I_1^{n-1}$, but $u \notin I_1^{n-1}\overline{I}_1$ else $u \in I_1^{n-1}\overline{I}_1$ in T.

Next choose a finitely generated $\mathbf{Z}$-algebra C whose quotient field is L, and then localize C at finitely many elements so that the following conditions hold:

(i) $f_1, \dots, f_r \in C[Z_1, \dots, Z_e]$,
(ii) $x_1, \dots, x_d, c_1, \dots, c_N, u_\alpha \subseteq C[z_1, \dots, z_e]$,
(iii) $z_1, \dots, z_e$ are in $\operatorname{rad}((x_1, \dots, x_d)C[z_1, \dots, z_e])$,
(iv) $C[z_1, \dots, z_e]$ is free as a C-module, and
(v) if $J = (x_1, \dots, x_d)C[z_1, \dots, z_e]$ and $D = C[z_1, \dots, z_e]$ then D/J and D are free C-modules.

Clearly (i)–(iii) are possible by inverting the denominators of similar equations holding in $L[z_1, \dots, z_e]$. Conditions (iv) and (v) are possible by the theorem of generic flatness ([18, §22.A]).

As a C-module $D \simeq C \oplus m_2$, where $m_2 = \sqrt{J} = (z_1, \dots, z_e)$. It follows that $\operatorname{ht}(m_2) = d$. Since $x_1, \dots, x_d$ are a regular sequence over $L[z_1, \dots, z_e]$, by inverting possibly another element of C we may assume: (vi) $x_1, \dots, x_d$ are a D-sequence. (We will use the notation C and D even if we change C by inverting possibly finitely many elements of C, and change D accordingly. Notice that none of (i)–(vi) will be affected by inverting further elements of C.)

In D, clearly $u \in \overline{J^n} \cap J^{n-1}$ and $u \notin J^{n-1}\overline{J}$. Let P be a maximal ideal of C and set $D_p = D \otimes_C C/PC$; let m_p be the image of m_2 and use a "$'$" to denote the images of elements of D in D_p. Note that $P + m_2$ is a maximal ideal of D, since $D/(P+m_2) \simeq C/PC$ since $D \simeq C \oplus m_2$.

Suppose we can find such a P so that

(i) $x_1', \dots, x_d'$ are a $(D_p)_{m_p}$ sequence, and
(ii) $u' \notin (J')^{n-1}\overline{J'}$.

Then we are done. For clearly $u' \in \overline{(J')^n} \cap (J')^{n-1}$, as this is true even in D. Since necessarily $\sqrt{(x_1', \dots, x_d')} = m_p$ it follows that $\operatorname{Ass}(D_p/(J')^{n-1}\overline{(J')}) = \{m_p\}$ so that $u' \notin (J')^{n-1}\overline{(J')}$ implies $u' \notin (J'_{m_p})^{n-1}\overline{(J'_{m_p})}$. Then $(D_p)_{m_p}$ will be a d-dimensional C–M local ring of characteristic > 0 (since C/P is a finite field), and thus by Theorem 4.7 such a u' cannot exist.

First we observe that the Koszul complex $K.(x_1, \dots, x_d; D)$ is a free D-resolution of D/J. Since we obtained that D/J is C-free, it follows that

$$\operatorname{Tor}_i^C(C/P, D/J) = 0 \quad \text{for } i \geq 1$$

and hence $K.(x_1, \dots, x_d; D) \otimes_C C/P = K.(x_1', \dots, x_d'; D_p)$ is exact. Thus locally $x_1', \dots, x_d'$ are a regular sequence. Thus (i) holds for any maximal ideal P of C.

To prove (ii) we first invert an element of C so that $D/\bar{J}$ and $\mathrm{gr}_J(D)=D/J\oplus J/J^2\oplus\cdots$ are free C-modules. An easy induction using

$$0\to J^k/J^{k+1}\to D/J^{k+1}\to D/J^k\to 0$$

shows that D/J^k are then projective C-modules for all k. In addition, D/J^k is a finite projective C-module since $D/m_2\simeq C$ and some power of m_2 is contained in J.

Now, since $u\in J^{n-1}$ we may write $u=\sum_{|\alpha|=n-1}u_\alpha x^\alpha$. Then some $u_\alpha\notin\bar{J}$. Fix this α. If we show there is a P such that $u'_\alpha\notin\overline{J'}$, we then claim $u'\notin(J')^{n-1}\overline{(J')}$. For if $u'=\sum_{|\alpha|=n-1}v'_\alpha(x')^\alpha$ with $v'_\alpha\in\overline{J'}$, then the equation

$$\sum_{|\alpha|=n-1}v'_\alpha(x')^\alpha=\sum_{|\alpha|=n-1}u'_\alpha x'^\alpha$$

plus the fact that $x'_1,\ldots,x'_d$ are a regular sequence shows that

$$u'_\alpha\in(v'_\alpha,x'_1,\ldots,x'_i,\ldots,x'_d)\subset\overline{J'},$$

which contradicts our assumption. Put $u_\alpha=w$.

Consider D_p. This is a d-dimensional local ring containing a field. Suppose $w\in\bar{J}'$. We claim there is a fixed bound for the minimal degree of the integral equation satisfied by w over J', depending only on D and J. Note there is a fixed power of m_p lying on J, which shows that D/J is a *finite* free C-module of rank f, say. Then $\lambda(D_p/J')=f$ for every maximal ideal P of C.

Complete D_p. Then $\hat{D}_p$ is a free module of rank f over the regular local ring $F[[x_1,\ldots,x_d]]$ where F is a coefficient field of $\hat{D}_p$. Hence the lemma shows that w' satisfies an integral equation of degree f over $J\hat{D}_p$. By lifting the coefficients of this equation to D_p mod m_p^f we obtain an integral equation for w' over $JD_p=J'$ of degree f. Hence there is an equation

$$(*)\qquad (w')^f+a_1(w')^{f-1}+\cdots+a_f=0,\quad a_i\in(J')^i.$$

By induction we claim that $(w')^{f+n}\in(J')^{n+1}$ for all $n\geq 0$. For $n=0$, $(*)$ shows this. Suppose $(w')^{f+n-1}\in(J')^n$ and multiply $(*)$ by $(w')^n$. Then

$$(w')^{n+f}=-a(w')^{n+f-1}-a_2(w')^{n+f-2}-\cdots-a_f(w')^n\in(J')^{n+1}$$

as required, since $a_i(w')^{n+f-i}\in J'^i\cdot J'^{n-i+1}\subset J'^{n+1}$. (If $n\geq i$ this holds; if $n<i$ then $a^i\in(J')^i$ so that $a^i\in(J')^{n+1}$.)

We have shown that if $w'\in\bar{J}'$, then $(w')^{n+f}\in(J')^{n+1}$ and f *does not depend* on P. Thus if $w'\in\bar{J}'$ for some $P\in m-\mathrm{spec}(C)$ then $w^{n+f}\in(J^{n+1},PD)$. Suppose there is a $P\in m-\mathrm{spec}(C)$ and n such that $w^{n+f}\notin(J^{n+1},PD)$. Then (by the above) $w'\notin\bar{J}'$, which finishes the proof of the theorem. Hence we may assume $w^{n+f}\in(J^{n+1},PD)$ for all $P\in m-\mathrm{spec}(C)$, and all n. As D/J^{n+1} is projective over C this implies that $w^{n+f}\in J^{n+1}$. Hence

$$\lim_{n\to\infty}v_J(w^{n+f})/(n+f)\geq\lim_{n\to\infty}(n+1/n+f)=1,$$

which shows by [27] that $w\in\bar{J}$. This contradiction proves the theorem. □

COROLLARY (cf. [20]). *Let R, m be a 2-dimensional C–M local ring containing a field and I an m-primary ideal. If $\bar{P}_I(n)=\bar{H}_I(n)$ for all $n\geq 1$ then $\bigoplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ is C–M.*

Proof. This is immediate from the theorem of this appendix and Theorem 4.6(ii). □

We can generalize the theorem of the appendix and Theorem 4.7 as follows.

THEOREM. *Let R, m be a local C–M ring containing a field. If $I=(x_1,\dots,x_g)$ is generated by a regular sequence, then*

$$\overline{I^n}\cap I^{n-1}=I^{n-1}\overline{I}.$$

Proof. Clearly $I^{n-1}\overline{I}\subset\overline{I^n}\cap I^{n-1}$, so it suffices to prove equality at all $p\in \operatorname{Ass}(R/I^{n-1}\overline{I})$. We claim $\operatorname{Ass}(R/I^{n-1}\overline{I})=\operatorname{Ass}(R/I)$. To see this suppose that $r\cdot s\in I^{n-1}\overline{I}$ and r is a nonzero divisor modulo I. Then r is also a nonzero divisor modulo $\overline{I}$ since I is generated by a regular sequence (see [19, Proposition 4.1]). Since $rs\in I^{n-1}$ we get $s\in I^{n-1}$ as $\operatorname{Ass}(R/I^{n-1})=\operatorname{Ass}(R/I)$, so that $s=\sum_{|\alpha|=n-1}s_\alpha x^\alpha$. Since $rs\in I^{n-1}\overline{I}$, we may write

$$\sum_{|\alpha|=n-1} rs_\alpha x^\alpha = rs = \sum_{|\alpha|=n-1} t_\alpha x^\alpha \quad \text{with } t_\alpha\in\overline{I}.$$

Since $x_1,\dots,x_g$ is a regular sequence, this equation shows that $rs_\alpha - t_\alpha\in(x_1,\dots,x_g)$ so that $rs_\alpha\in\overline{I}$, which implies $s_\alpha\in\overline{I}$ so that $s\in I^{n-1}\overline{I}$. This proves our claim. Now if $p\in\operatorname{Ass}(R/I)$ then I_p is generated by an s.o.p. of R_p which is a C–M local ring containing a field. Hence Theorem 4.7 and the theorem of this appendix finish the proof. □

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Department of Mathematics
Purdue University
West Lafayette, IN 47907