HOLOMORPHIC VECTOR FIELDS ON COMPLEX MANIFOLDS

Leon Karp

1. Introduction. Let M^n be a compact complex manifold of complex dimension n, and let $\mathrm{Hol}(M)$ denote the biholomorphisms of M^n. It is known [2] that $\mathrm{Hol}(M)$ is a Lie group which, in general, is not compact. The identity component $\mathrm{Hol}(M)^0$ is generated by vector fields which, when "transferred" to M^n, are holomorphic—that is, locally of the form $Z=\sum a_k(\partial/\partial z_k)$, where the a_k are holomorphic functions. More precisely, if ψ_t is a one-parameter subgroup of $\mathrm{Hol}(M)$ then its generator X is an infinitesimal automorphism of the complex structure, and Z is holomorphic exactly when $Z=X-iJX$ for such a real vector field X. Here J denotes the complex structure. If the subgroup ψ_t has no fixed points then X (and hence Z) is nonsingular.

It was shown by Matsushima [11], using Blanchard's theorem [1] on projective embeddings, that if M^n is projective algebraic with first Betti number $b_1(M)=0$ then M^n admits no nonsingular holomorphic vector field. This result was extended to all compact Kähler manifolds by Carrell and Lieberman [4], and by Sommese [12]. In this paper we extend the theorem to more general complex manifolds, and prove an analogue for all compact complex manifolds that is similar to a result of Bott [3].

To formulate the simplest of our results we recall the definition of the Hodge numbers $h^{p,q}(M)=\dim_{\mathbf{C}} H^q(M,\Omega^p)$. It is known [5] that the Euler characteristic

$$\begin{aligned}\chi(M^n) &= \sum_{0\le p,q\le n} (-1)^{p+q} h^{p,q}(M)\\ &= 2+\sum_{0<p+q=\text{even}<2n} h^{p,q}(M)-\sum_{p+q=\text{odd}} h^{p,q}(M).\end{aligned}$$

Consequently, if $\sum_{p+q=\text{odd}} h^{p,q}=0$ then, according to the well-known theorem of Poincaré–Hopf, M^n admits no nonsingular vector field. For holomorphic vector fields we have the following refinement.

THEOREM A. *Let M^n be an n-dimensional compact complex manifold. If $\sum_{0\le p\le n-1} h^{p,p+1}(M)=0$ then M^n admits no nonsingular holomorphic vector field.*

In Section 2 we prove Theorem A and in Section 3 we present two generalizations of the theorem of Carrell–Lieberman and Sommese. Section 4 treats some related results and points out some open problems.

Most of the results of this paper are contained in part of the author's thesis [9]. It is a pleasure to thank L. Nirenberg for his encouragement and support.

Received September 4, 1985.
Research partially supported by NSF Grant MCS 81-02051.
Michigan Math. J. 34 (1987).

2. Preliminaries and the proof of Theorem A. Throughout the paper we will let Z denote a vector field of type $(1,0)$, that is, of the form $X-iJX$ for some real vector field X. Let g be an hermitian metric and, if Z is nonsingular, set

$$\zeta=\frac{1}{\|Z\|^2}g(\cdot,\bar{Z}). \tag{2.1}$$

Thus ζ is a form of type $(1,0)$ and $i_Z\zeta=\zeta(Z)\equiv 1$, where i_Z denotes contraction with Z.

We will make repeated use of the fact that the operator

$$\bar{\partial}\circ i_Z+i_Z\circ\bar{\partial}\equiv 0 \tag{2.2}$$

if Z is holomorphic. This fact is well known and may be proved by noting that $\bar{\partial}\circ i_Z+i_Z\circ\bar{\partial}$ is a derivation of degree zero on the bundle of smooth forms, and then checking that $(\bar{\partial}\circ i_Z+i\circ\bar{\partial})\alpha=0$ if $\alpha=f\in C^\infty$, dz^k, or $d\bar{z}^k$.

LEMMA 2.1. *If Z is a nonsingular holomorphic vector field on a complex manifold M then $i_Z\bar{\partial}\zeta=0$, where ζ is given by* (2.1).

Proof. Using (2.1) and (2.2) we have $i_Z\bar{\partial}\zeta=-\bar{\partial}i_Z\zeta=-\bar{\partial}(1)=0$. □

LEMMA 2.2. *Let the compact complex manifold M^n admit a holomorphic vector field Z. If $\sum_{0\le p\le n-1}h^{p,p+1}=0$ then there exists a sequence of differential forms α_q of type (q,q) such that $\bar{\partial}\alpha_q=i_Z\alpha_{q+1}$ for $0\le q\le n-1$.*

Proof. Let α_n be a volume form for M^n (which exists since M^n is orientable). Then $\bar{\partial}\alpha_n=0$ and so $\bar{\partial}i_Z\alpha_n=-i_Z\bar{\partial}\alpha_n=0$. If $h^{n-1,n}=0$, then (by Dolbeault's theorem, cf. [15]) $\exists\alpha_{n-1}$ of type $(n-1,n-1)$ such that $\bar{\partial}\alpha_{n-1}=i_Z\alpha_n$. It follows that $\bar{\partial}i_Z\alpha_{n-1}=-i_Z\bar{\partial}\alpha_{n-1}=-i_Z^2\alpha_n=0$, and so $\exists\alpha_{n-2}$ such that $\bar{\partial}\alpha_{n-2}=i\alpha_{n-1}$. In general, if there exist $\{\alpha_n,\alpha_{n-1},\dots,\alpha_{q+1}\}$ such that $\bar{\partial}\alpha_{r+1}=i\alpha_{r+2}$, $q\le r\le n-2$, then $\bar{\partial}i_Z\alpha_{q+1}=-i_Z\bar{\partial}\alpha_{q+1}=-i_Z^2\alpha_{q+2}=0$ and so, if $h^{q,q+1}=0$, there exists α_q of type (q,q) such that $\bar{\partial}\alpha_q=i_Z\alpha_{q+1}$. This completes the proof. □

We can now turn to the

Proof of Theorem A. If Z is a nonsingular holomorphic vector field we can define the $(1,0)$ form ζ as in (2.1), and then the operator

$$T\stackrel{\text{def}}{=}\ell_\zeta\circ i_\zeta+i_\zeta\circ\ell_\zeta,$$

where ℓ_ζ denotes left exterior multiplication by ζ, is just multiplication by $\zeta(Z)\equiv 1$ (i.e., the identity operator) on forms. Let $\omega=\alpha_n$ be a volume form and let $\{\alpha_q\}$, $0\le q\le n-1$, be the forms of type (q,q) described in Lemma 2.2. Then, since $\zeta\wedge\Omega=0$ if Ω is of type (n,n),

$$\begin{aligned}\omega=T\omega&=\zeta\wedge i_Z\omega=\zeta\wedge\bar{\partial}\alpha_{n-1}\\&=\bar{\partial}\zeta\wedge\alpha_{n-1}+\text{an exact form}\\&=T^2\omega=\zeta\wedge\bar{\partial}\zeta\wedge i_Z\alpha_{n-1}+\text{an exact form}\\&=(\bar{\partial}\zeta)^2\wedge\alpha_{n-2}+\text{an exact form,}\end{aligned}$$

where we have used Lemma 2.1 in deriving the third line. Continuing in this fashion we have

$$\omega = (\bar{\partial}\zeta)^k \wedge \alpha_{n-k} + \text{an exact form} \quad (k \geq 1)$$

and, finally,

$$\omega = \alpha_0 \cdot (\bar{\partial}\zeta)^n + \text{an exact form},$$

where $\alpha_0 \in C^\infty$. Applying T one more time, we obtain

$$\omega = \text{an exact form},$$

since $i_Z(\bar{\partial}\zeta)^n = 0$. This contradicts the assumption that $\int_M \omega > 0$ and completes the proof. □

Theorem A may be applied to derive a number of interesting results concerning group actions. We mention only the following.

COROLLARY A.1. *Let M^n be a compact complex n-manifold with*

$$\sum_{0 \leq p \leq n-1} h^{p,p+1} = 0.$$

If M^n is homogeneous (i.e., if $\mathrm{Hol}(M)^0$ *acts transitively on M^n), then* $\mathrm{Hol}(M)^0$ *is not nilpotent.*

Proof. If $G = \mathrm{Hol}(M)^0$ is nilpotent then the center C of the Lie algebra of G is nontrivial (cf. [7]). Since the action is transitive it follows that every $Z' \in C$ induces a nonsingular holomorphic vector field on M^n. The desired conclusion now follows from the theorem. □

REMARK. According to a result of Bott [3], if a compact complex n-manifold admits a nonsingular holomorphic vector field then its Chern numbers vanish. This result neither includes nor is included in Theorem A.

3. Generalizations of the theorem of Carrell–Lieberman and Sommese. In order to formulate our results in this section in a simple manner, we introduce the following generalization of the notion of a Kähler manifold.

DEFINITION. A compact complex manifold M with $\dim_{\mathbf{C}} M = n$ is of *type* K_p if there exists a family $\{\mu_j\}$ of $\bar{\partial}$-closed forms μ_j of degree $\leq 2+p$ and positive integers $\{r_j\}$ such that the exterior product $\prod \mu_j^{r_j}$ is of type (n, n) and $\int_M \prod \mu_j^{r_j} \neq 0$.
M^n is of *type* K_p' if there exists a $\bar{\partial}$-closed form μ of type $(2+p, 2+p)$ such that $n = r(2+p)$ and $\int \mu^r \neq 0$.

REMARK. If M^n is of type K_p' then it is certainly of type K_p, and every compact n-dimensional Kähler manifold is of type K_0' as is every compact complex manifold with Chern number $[C_1^n] \neq 0$.

We can now formulate the main result of this section.

THEOREM B. *Let M^n be a compact complex n-manifold of type K_0. If M^n admits a nonsingular holomorphic vector field then $h^{0,1}(M^n) + h^{1,0}(M^n) \neq 0$.*

Since for compact Kähler manifolds we have $h^{1,0}=h^{0,1}=\frac{1}{2}b_1(M)$, it is evident that we have the following.

COROLLARY B.1 (Matsushima [11], Carrell–Liebermann [4], Sommese [12]). *If M^n is an n-dimensional compact Kähler (esp. projective algebraic) manifold which admits a nonsingular holomorphic vector field, then $b_1(M)\neq 0$.*

Arguing as in Section 2, we have the following analogue of Corollary A.2.

COROLLARY B.2. *If M^n is an homogeneous compact complex n-manifold of type K_0 then* $\mathrm{Hol}(M^n)^0$ *is not nilpotent.*

Proof of Theorem B. Since M^n is of type K_0 we have forms $\{\mu_j\}$, each of degree ≤ 2, and integers $\{r_j\}$ with $\int\prod\mu_j^{r_j}\neq 0$. We may decompose the product $\prod\mu_j^{r_j}$ as

$$\prod\mu_j^{r_j}=\prod_k F^{p_k,q_k}=F^{0,1}\wedge F^{1,0}\wedge F^{1,1}\wedge F^{0,2}\wedge F^{2,0},$$

where F^{p_k,q_k} is a product of $\bar\partial$-closed forms, each of type (p_k,q_k), and

$$0\leq p_k+q_k\leq 2.$$

We may assume that the factor $F^{0,1}$ is absent, for otherwise there exist $\bar\partial$-closed forms μ of type $(0,1)$ which are not exact (because $\int\prod\mu_j^{r_j}\neq 0$) and so

$$h^{0,1}(M^n)\neq 0.$$

Similarly, we may assume that the factor $F^{1,0}$ is absent. Thus

$$\int\mu_j^{r_j}=F^{1,1}\wedge F^{2,0}\wedge F^{0,2}.$$

Suppose now that Z is a nonsingular holomorphic vector field. Since Z is of type $(1,0)$, $i_ZF^{0,2}=0$. On the other hand, we may also assume that $i_ZF^{2,0}=0$. In fact, $F^{2,0}=\prod\beta_j$ for some holomorphic 2-forms β_j, $i_ZF^{2,0}=\sum_j i_Z\beta_j\prod_{k\neq j}\beta_k$, and the holomorphic 1-forms $\{i_Z\beta_j\}$ must vanish if $h^{1,0}=0$. Set $F=F^{1,1}$, $R=F^{2,0}\wedge F^{0,2}$ (the factors annihilated by i_Z) and factor F as $F=\prod_{j=1}^m\alpha_j$, where the α_j's are $\bar\partial$-closed $(1,1)$ forms. As in the proof of Theorem A, set $T=\ell_\zeta\circ i_Z+i_Z\circ\ell_\zeta$, with ζ as in (2.1). With $P=F\wedge R$ and $i=i_Z$ we have

$$\begin{aligned}P=TP&=\zeta\wedge iP=\zeta\wedge i(F)\wedge R\\&=\zeta\wedge\left[\sum_{k=1}^m i(\alpha_k)\prod_{j\neq k}\alpha_j\right]\wedge R.\end{aligned}$$

Hence, for some k, $i(\alpha_k)\not\equiv 0$. Now $\bar\partial i(\alpha_k)=-i\bar\partial(\alpha_k)=0$, so $i(\alpha_k)$ is $\bar\partial$-closed. To show $h^{0,1}\neq 0$ it suffices to show that there do not exist $\{f_k\}_{k=1}^m$, $f_k\in C^\infty$, such that $\bar\partial f_k=i(\alpha_k)$ for all k. Suppose such f_k did exist. Let

$$\Pi(p_1,\dots,p_k)\stackrel{\text{def}}{\equiv}\alpha_1\wedge\cdots\wedge\hat\alpha_{p_1}\wedge\cdots\wedge\hat\alpha_{p_2}\wedge\cdots\wedge\hat\alpha_{p_k}\wedge\cdots\wedge\alpha_m,$$

where the caret ˆ indicates omission. Extend the definition of Π so that $\Pi(p_1,\dots,p_k)$ is symmetric in $(p_1,\dots,p_k)$. Let

$$\Sigma_k^{1,1} = \sum_{1 \le p_1 < \cdots < p_k \le m} f_{p_1} \cdots f_{p_k} \Pi(p_1, \ldots, p_k), \quad 1 \le k \le m,$$

and let $\Sigma_0 = F = \prod_{j=1}^m \alpha_j$. We claim that

$$(*) \qquad i(\Sigma_k^{1,1}) = \bar\partial(\Sigma_{k+1}^{1,1}).$$

We postpone the proof of (*) until the proof of the theorem is completed. Now

$$\begin{aligned} F \wedge R = T(F \wedge R) &= \zeta \wedge i(F) \wedge R = \zeta \wedge i(\Sigma_0^{1,1}) \wedge R \\ &= \zeta \wedge \bar\partial(\Sigma_1^{1,1}) \wedge R = -\bar\partial(\zeta \wedge \Sigma_1^{1,1} \wedge R) + \bar\partial\zeta \wedge \Sigma_1^{1,1} \wedge R \\ &= \bar\partial\zeta \wedge \Sigma_1^{1,1} \wedge R + \bar\partial\text{-exact form.} \end{aligned}$$

Applying T again we have $F \wedge R = T^2(F \wedge R) = \zeta \wedge i(\bar\partial\zeta \wedge \Sigma_1^{1,1}) \wedge R + \bar\partial$-exact form $= \zeta \wedge \bar\partial\zeta \wedge i(\Sigma_1) \wedge R + \bar\partial$-exact form $= (\bar\partial\zeta)^2 \wedge \Sigma_2^{1,1} \wedge R + \bar\partial$-exact form. It is easy to see that continuing in this fashion we have, for $1 \le k \le m$,

$$F \wedge R = \bar\partial\omega_k + (\bar\partial\zeta)^k \wedge \Sigma_k \wedge R, \quad \text{where } \omega_k \in \Lambda^{n,n-1}.$$

Choosing $k = m$ in this last expression we have:

$$\begin{aligned} P = F \wedge R &= \bar\partial\omega_m + (\bar\partial\zeta)^m \prod_{j=1}^m f_j \wedge R \\ &= d\omega_m + \prod_{j=1}^m f_j (\bar\partial\zeta)^m \wedge R, \end{aligned}$$

since $d\omega = \bar\partial\omega$ if $\omega \in \Lambda^{n,n-1}$. Applying Stokes' theorem we find that

$$\int P = \int \prod_{j=1}^m f_j (\bar\partial\zeta)^m \wedge R \neq 0.$$

However, this is absurd, since

$$\begin{aligned} \prod_{j=1}^m f_j (\bar\partial\zeta)^m \wedge R &= T\left[\prod_{j=1}^m f_j (\bar\partial\zeta)^m \wedge R\right] \\ &= \prod_{j=1}^m f_j \, \zeta \wedge [i(\bar\partial\zeta)^m \wedge R + (\bar\partial\zeta)^m \wedge iR] = 0. \end{aligned}$$

To complete the proof it only remains to prove (*):

$$\begin{aligned} i(\Sigma_k^{1,1}) &= \sum_{1 \le p_1 < \cdots < p_k \le m} f_{p_1} \cdots f_{p_k} i(\Pi(p_1, \ldots, p_k)) \\ &= \sum_{p_1 < p_2 < \cdots < p_k} f_{p_1} \cdots f_{p_k} \left[\sum_{q=1}^m \bar\partial f_q \Pi(p_1, \ldots, p_k, q)\right] \\ &= \sum_{q=1}^n \sum_{\substack{1 \le p_1 < \cdots < p_k \le m \\ q \ne p_\ell}} f_{p_1} \cdots f_{p_k} \bar\partial f_q \Pi(p_1, \ldots, p_k, q) \\ &= \sum_{\substack{p_j\text{'s and } q \text{ in} \\ \text{ascending order}}} \bar\partial(f_{p_1} \cdots f_{p_k} f_q) \Pi(p_1, \ldots, p_k, q) = \end{aligned}$$

$$= \sum_{1 \le p_1 < \cdots < p_{k+1} \le m} \bar\partial(f_{p_1} \cdots f_{p_{k+1}}) \Pi(p_1, \ldots, p_{k+1})$$
$$= \bar\partial \Sigma f_{p_1} \cdots f_{p_{k+1}} \Pi(p_1, \ldots, p_{k+1}) \quad (\text{since } \bar\partial \alpha_j = 0)$$
$$= \bar\partial \Sigma^{1,1}_{k+1}.$$

This completes the proof of the theorem. □

THEOREM C. *If M^n is a compact complex manifold of type K_0' which admits a nonsingular holomorphic vector field, then $h^{0,1}(M^n) \neq 0$.*

The proof is exactly the same as the proof of Theorem C; however, the consideration of $h^{1,0}$ is no longer necessary since no factors of type $(1,0)$ or $(2,0)$ can occur. Theorem C also generalizes the result of Matsushima, Carrell–Lieberman, and Sommese. □

REMARKS. (1) According to the result of Bott [3] cited at the end of Section 2, if M^n admits a nonsingular holomorphic vector field then $\int_M C_1^n = 0$, where $[C_1]$ is the first Chern class. It follows from Theorem C that $\int \omega^n = 0$ for every closed $(1,1)$ form on M^n if, in addition, $h^{0,1}(M^n) = 0$.

(2) The theorems of this section have real analogues which will be described elsewhere [10].

4. Related results and open questions. When M^n is Kähler it is possible to obtain certain refinements of the preceding results for group actions even when b_1 is not necessarily zero. For this purpose we let $\widetilde{\mathrm{Hol}}(M)^0$ denote the connected subgroup of $\mathrm{Hol}(M^n)^0$ with Lie algebra $[L, L]$, where L denotes the Lie algebra of $\mathrm{Hol}(M)$ (i.e., the holomorphic vector fields).

THEOREM D. *If M^n is a compact Kähler manifold then $\widetilde{\mathrm{Hol}}(M)^0$ cannot act freely on M^n (unless, of course, $\widetilde{\mathrm{Hol}}(M)^0 = \{\mathrm{id}\}$). Thus, if G is a connected subgroup of $\mathrm{Hol}(M)^0$ that acts freely on the compact Kähler manifold M^n, then G is abelian.*

Proof. Suppose that $\widetilde{\mathrm{Hol}}(M^n)^0 \neq \{\mathrm{id}\}$ and that $\widetilde{\mathrm{Hol}}(M^n)^0$ acts freely on M^n. Let Z be a nonsingular holomorphic vector field in $[L, L]$ that generates a flow in $\widetilde{\mathrm{Hol}}(M^n)^0$. It follows that there exist holomorphic vector fields $\{Z_i'\}$ and $\{Z_i''\}$ and constants $\{C_i\}$ such that $Z = \sum_i C_i [Z_i', Z_i'']$. We claim that, this being the case, there exists a smooth function $f \in C^\infty$ such that $i_Z \omega = \bar\partial f$, where ω is the Kähler form, even if $b_1 \neq 0$. Assuming for a moment the validity of this claim and applying the arguments of the proof of Theorems B and C to the product form $P = \omega^n$, we are immediately led to contradictory conclusions. On one hand $\int \omega^n \neq 0$ while, on the other,

$$\omega^n = T\omega^n = \zeta \wedge n i_Z \omega \wedge \omega^{n-1}$$
$$= nf \bar\partial \zeta \wedge \omega^{n-1} + \text{an exact form},$$

and so (applying T n more times)

$$\begin{aligned}\omega^n &= T^{n+1}\omega^n \\ &= n!\, f^n T(\bar\partial\zeta^n) + \text{an exact form} \\ &= \text{an exact form.}\end{aligned}$$

To prove the claim we recall that, for real vector fields X and Y,

$$[d\circ i_X + i_X\circ d, i_Y] = i_{[X,Y]}$$

(cf. [6, p. 93]). It is immediate that this formula extends (by linearity) without change to the complex case: $X = A+\sqrt{-1}B$, $Y = C+\sqrt{-1}D$. Now if Z' is a holomorphic vector field then $d\circ i_{Z'} + i_{Z'}\circ d = \partial\circ i_{Z'} + i_{Z'}\circ\partial$, since $\bar\partial\circ i_{Z'} + i_{Z'}\circ\bar\partial = 0$. It follows that if Z' and Z'' are holomorphic then

$$\begin{aligned} i_{[Z',Z'']}\omega &= [\partial i_{Z'} + i_{Z'}\partial, i_{Z''}]\omega \\ &= \partial i_{Z'} i_{Z''}\omega - i_{Z''}\partial i_{Z'}\omega + i_{Z'}\partial i_{Z''}\omega - i_{Z''} i_{Z'}\partial\omega \qquad (4.1)\\ &= i_{Z'}\partial i_{Z''}\omega - i_{Z''}\partial i_{Z'}\omega, \end{aligned}$$

since $\partial\omega = 0$ and $i_{Z'}i_{Z''}\omega = 0$. Now $\bar\partial(i_{Z''}\omega) = -i_{Z''}\bar\partial\omega = 0$, and so it follows from the theory of harmonic integrals on Kähler manifolds that $i_{Z''}\omega = \alpha_{0,1} + \bar\partial g$, for some $\bar\partial$-harmonic (0, 1) form $\alpha_{0,1}$ and some $g \in C^\infty$. Consequently,

$$\begin{aligned} i_{Z'}\partial i_{Z''}\omega &= i_{Z'}\partial(\alpha_{0,1} + \bar\partial g) = i_{Z'}\partial\bar\partial g \\ &= \bar\partial(i_{Z'}\partial g), \end{aligned}$$

where we have used the fact that $\partial\alpha_{0,1} = 0$ (since the metric is Kähler). Since a similar argument shows that $i_{Z''}\partial i_{Z'}\omega = \bar\partial(i_{Z''}\partial\tilde g)$ for some $\tilde g \in C^\infty$, the $\bar\partial$-exactness of $i_{[Z',Z'']}\omega$ now follows from (4.1). The claimed $\bar\partial$-exactness of $i_Z\omega$ therefore follows from the relation $Z = \sum C_i[Z_i', Z_i'']$, and the proof is complete. □

COROLLARY. *If M^n is a compact Kähler manifold with geometric genus $h^{n,0} \neq 0$ then* $\mathrm{Hol}(M^n)^0$ *is abelian.*

Proof. It suffices to show that the Lie algebra L of $\mathrm{Hol}(M)$ is abelian, $[L, L] = \{0\}$. Now if $[L, L] \neq \{0\}$ then there exist holomorphic vector fields Z' and Z'' such that the commutator $Z = [Z', Z''] \not\equiv 0$. Since $h^{n,0} \neq 0$ it follows from a result of Howard [8] that Z must be nonsingular. The argument in the proof of Theorem D then shows that no such Z can exist. □

REMARKS. (1) Observe that Riemann surfaces ($n = 1$) with $h^{1,0} = 1$ admit free abelian holomorphic actions while, for $h^{1,0} \geq 2$, $\mathrm{Hol}(M^1)^0 = \{\mathrm{id}\}$, as is well known.

(2) Although we have treated only manifolds of type K_0 and K_0' in Section 3, the methods of this paper allow one to prove analogous results for manifolds of type K_p ($p \geq 0$), when the assumption $h^{1,0}(M^n) + h^{0,1}(M^n) \neq 0$ is replaced by an analogous condition on the Hodge numbers $h^{r,s}(M)$ for $r+s \leq p+1$. We leave the exact formulation and the proof to the interest reader.

(3) Sommese has shown ([12], [13]) that if M^n is Kähler and $b_1(M) = 0$ then any solvable subgroup of $\mathrm{Hol}(M^n)$ has a fixed point. It would be interesting to know

whether or not Sommese's theorem extends to the more general situation considered in Sections 2 and 3 of this paper.

(4) It is reasonable to ask whether or not the fixed point theorems for 1-parameter subgroups of $\mathrm{Hol}(M^n)$ discussed here extend to single biholomorphisms $\psi \in \mathrm{Hol}(M^n)^0$, at least when ψ is sufficiently close to the identity. This is the case for the analogous situation of symplectic transformations (close to the identity) on a symplectic manifold (cf. [14]).

REFERENCES

1. A. Blanchard, *Sur les variétés analytiques complexes,* Ann. Sci. École Norm. Sup. (3) 73 (1956), 157–202.
2. S. Bochner and D. Montgomery, *Groups on analytic manifolds,* Ann. of Math. (2) 48 (1947), 659–669.
3. R. Bott, *Vector fields and characteristic numbers,* Michigan Math. J. 14 (1967), 231–244.
4. J. Carrell and D. Lieberman, *Holomorphic vector fields and Kähler manifolds,* Inv. Math. 21 (1973), 303–309.
5. A. Fröhlicher, *Relations between the cohomology groups of Dolbeault and topological invariants,* Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 641–644.
6. C. Godbillon, *Géométrie différentielle et mécanique analytique,* Hermann, Paris, 1969.
7. S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.
8. A. Howard, *Holomorphic vector fields on algebraic manifolds,* Amer. J. Math. 94 (1972), 1282–1290.
9. L. Karp, *Vector fields on manifolds,* Thesis, New York University–Courant Institute, 1976.
10. ———, *Killing fields and cohomology,* to appear.
11. Y. Matsushima, *Holomorphic vector fields and the first Chern class of a Hodge manifold,* J. Differential Geom. 3 (1969), 477–480.
12. A. Sommese, *Borel's fixed point theorem for Kähler manifolds and an application,* Proc. Amer. Math. Soc. 41 (1973), 51–54.
13. ———, *Holomorphic vector fields on compact Kähler manifolds,* Math. Ann. 210 (1974), 75–82.
14. A. Weinstein, *Lectures on symplectic manifolds,* Amer. Math. Soc., Providence, R.I., 1977.
15. R. O. Wells, Jr., *Differential analysis on complex manifolds,* Prentice-Hall, Englewood Cliffs, N.J., 1973.

Department of Mathematics
and Computer Science
The City University of New York
H. H. Lehman College
Bronx, New York 10468